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COMPUTING WITH THE STRATIFICATION OF THE NULLCONE FOR spin_{15}

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In loving memory of Nodar Berikashvili, precious teacher and friend

Abstract. In this paper we use the GAP computer algebra system [12], predominantly the GAP package SLA [10], to find representatives of generic orbits in each Hesselink stratum of the nullcone for the representation of a simple Lie algebra of type B7 on the space of spinors in 15 dimensions.

The computations would not be possible without the code kindly provided to us by Willem de Graaf [9].

1. Introduction

1.1. **Hesselink strata.** Everywhere in the paper G denotes a reductive algebraic group over a characteristic zero algebraically closed field.

Recall that for a representation ϱ of G acting on a vector space V, the *nullcone* $\mathfrak{N}_GV$ of the action is defined as

$$\mathfrak{N}_GV = \left\{v \in V \mid 0 \in \overline{Gv}\right\}.$$

That is, the nullcone is the union of all those G-orbits whose closures contain the zero of V. In fact, it is a Zariski closed subset of V, i. e. an affine algebraic variety, being the vanishing locus of all G-invariant polynomial functions on V. Occasionally we will just write $\mathfrak{N}V$ when G is clear from the context.

For example, in the adjoint representation of G on its Lie algebra $\mathfrak{g}$ the nullcone $\mathfrak{N}\mathfrak{g}$ consists precisely of all nilpotent elements of $\mathfrak{g}$. For this reason, also for arbitrary representations elements of the nullcone are frequently referred to as nilpotent elements, and their orbits as nilpotent orbits.

It is well known that for G reductive, there are only finitely many nilpotent G-orbits in $\mathfrak{g}$. More generally, it follows from the celebrated theory of *θ-groups* by Vinberg [29] that for any automorphism θ of G, the action of the group G^θ of fixed points of θ on each θ-eigenspace of $\mathfrak{g}$ has only finitely many nilpotent orbits; in this case too these orbits consist of certain elements of $\mathfrak{g}$ which are nilpotent in the sense of the Lie algebra structure of $\mathfrak{g}$.

Representations with the above property, i. e. with the property that $\mathfrak{N}_GV$ consists of only finitely many G-orbits, are called *visible*. Visible representations for ϱ irreducible have been classified by Kac in [16]. The closely related class of representations is that of *polar* representations studied by Dadok and Kac [7]. There are some polar representations that are not visible, but, unlike many visible representations, all polar representations admit a treatment very similar to that of the θ-groups. It must be noted that the precursor for the study of polar representations was the pioneering paper [13].

In 1978 several people [3, 14, 17, 28] started using a still more sophisticated approach to the study of nullcones. In particular, in [14] Hesselink found a method to decompose the nullcone into a finite number of *strata* $\mathbf{H} \subseteq \mathfrak{N}_GV$ for any representation V. Each stratum is a union of, in general, infinitely many orbits. The Hesselink strata have very useful geometric description: Hesselink proved that each stratum $\mathbf{H}$ is an irreducible algebraic subvariety in $\mathfrak{N}_GV$, open in its closure $\overline{\mathbf{H}}$. Moreover, to each $\mathbf{H}$ corresponds a parabolic subgroup $P_{\mathbf{H}} \subseteq G$ and an embedding $\iota_{\mathbf{H}} : \mathbf{H} \hookrightarrow E_{\mathbf{H}}$ identifying $\mathbf{H}$ with a Zariski open subset of the total space of an algebraic vector bundle $E_{\mathbf{H}} \to G/P_{\mathbf{H}}$ over the generalized flag manifold $G/P_{\mathbf{H}}$; in particular each $\mathbf{H}$ is a smooth rational variety. This $E_{\mathbf{H}}$ is the associated bundle for the principal $P_{\mathbf{H}}$-bundle $G \to G/P_{\mathbf{H}}$ with respect to the restriction of the action of G on V to the action of $P_{\mathbf{H}}$ on a certain subspace $V^{(\geqslant 2)}$ of V (see below). Moreover there is an onto map $\pi_{\mathbf{H}} : E_{\mathbf{H}} \to \overline{\mathbf{H}}$ which turns $E_{\mathbf{H}}$ into a resolution of singularities for $\overline{\mathbf{H}}$, and $\pi_{\mathbf{H}}^{-1}(\mathbf{H}) = i_{\mathbf{H}}(\mathbf{H})$.

In fact each stratum $\mathbf{H}$ is uniquely determined by its *characteristic* $h = h_{\mathbf{H}}$, which is a semisimple element of the Lie algebra $\mathfrak{g}$ of G, uniquely determined for each choice of a Borel subalgebra in $\mathfrak{g}$. Namely, with respect to the grading $\mathfrak{g}_h^{(*)}$ of $\mathfrak{g}$ by eigenvalues of ad_h, the Lie algebra $\mathfrak{p}_{\mathbf{H}}$ of $P_{\mathbf{H}}$ coincides with the nonnegative part $\mathfrak{g}_h^{(\geqslant 0)}$ of this grading. In particular, the Lie algebra of the Levi subgroup $G_h^{(0)} \subseteq P_{\mathbf{H}}$ is the centralizer $\mathfrak{g}_h^{(0)} = \mathfrak{z}_{\mathfrak{g}}(h)$ of h in $\mathfrak{g}$.

Moreover, V also acquires a grading $V_h^{(*)}$ by eigenvalues of $\varrho(h)$, and the fibre of $E_{\mathbf{H}} \to G/P_{\mathbf{H}}$ can be identified with the subspace $V_h^{(\geqslant 2)} \subseteq V$ spanned by all homogeneous elements of degrees $\geqslant 2$ with respect to this grading.

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To define the strata and their characteristics, let us choose a maximal torus T of G, let $\mathfrak{h} \subseteq \mathfrak{g}$ be its tangent algebra, and let $\mathfrak{X}(T)$ be the group of characters of T. Moreover let

$$\mathfrak{h}(\mathbf{Q}) = \{h \in \mathfrak{h} \mid \chi(h) \in \mathbf{Q} \text{ for all } \chi \in \mathfrak{X}(T)\},$$

and let us fix an invariant nonsingular symmetric bilinear form $\langle -, - \rangle$ on $\mathfrak{g}$ which is positive definite on $\mathfrak{h}(\mathbf{Q})$. In general, the structure of the stratification depends on the choice of such form. Note however that if G is simple, then such form is proportional to the Killing form.

Definition 1. For $v \in \mathfrak{N}_G V$, its *characteristic* is a $\langle -, - \rangle$-shortest h conjugate to an element of $\mathfrak{h}(\mathbf{Q})$ such that $v \in V_h^{(\geqslant 2)}$.

The stratum $\mathbf{H}$ of v consists of all vectors whose characteristics are conjugate to the characteristic of v.

In fact, for a stratum $\mathbf{H}$ corresponding to the characteristic h there is an open subset $U_h \subseteq V_h^{(\geqslant 2)}$ such that the stratum $\mathbf{H}$ is equal to GU_h. Namely, U_h consists of those $v \in V_h^{(\geqslant 2)}$ whose degree 2 component $v^{(2)} \in V_h^{(2)}$ does not lie in the nullcone $\mathfrak{N}_{\tilde{G}_h^{(0)}} V_h^{(2)}$, where $\tilde{G}_h^{(0)} \subseteq G_h^{(0)}$ is the subgroup whose Lie algebra $\tilde{\mathfrak{g}}_h^{(0)} \subseteq \mathfrak{g}_h^{(0)}$ is the $\langle -, - \rangle$-orthogonal complement of h in $\mathfrak{g}_h^{(0)}$. To simplify notation, in what follows we will occasionally drop the subscript $_h$ when the characteristic is uniquely recoverable from the context.

In subsequent works by several people, various alternative descriptions of the Hesselink strata have been obtained. For example, according to [23] the strata are basins of attraction for a Morse function on V determined by the norm functional with respect to a Hermitian form on V invariant with respect to the action of a maximal compact subgroup of G.

An explicit algorithm for computing Hesselink stratifications for nullcones of (finite-dimensional) representations of reductive groups has been described by Popov in [26]. A realization of this algorithm for semisimple groups was accomplished by A'Campo [1].

In the present paper, we assume the field of coefficients to be algebraically closed of characteristic zero, although by now the subject has been developed in positive characteristic too, see e. g. the work [6] based on the approach by Lusztig in [19–22]. There are also some works over non-algebraically closed fields, e. g. some results over the field of real numbers can be found in [4, 5].

1.2. **Spin representations.** Let us recall the construction of the spin representations. The detailed description can be found in [11, Lecture 20].

For a nonsingular quadratic form q on a complex vector space V of dimension n, this is a representation of the simple complex Lie algebra $\mathfrak{so}(q)$ as follows. Let $\mathrm{Cl}(q)$ be the Clifford algebra of the form q, and let $W \subseteq V$ be a maximal q-isotropic subspace of V. Then q identifies the dual space W^* with another maximal isotropic subspace, having zero intersection with W. The exterior algebras $\Lambda^*(W)$ and $\Lambda^*(W^*)$ can then be both identified with subalgebras in $\mathrm{Cl}(q)$. Moreover there are $\mathrm{Cl}(q)$-module structures on these exterior algebras that can be used to identify $\mathrm{Cl}(q)$ with either $\mathrm{End}(\Lambda^*(W))$ or $\mathrm{End}(\Lambda^*(W^*))$ for n even, and with $\mathrm{End}(\Lambda^*(W)) \oplus \mathrm{End}(\Lambda^*(W^*))$ for n odd, with each of $\mathrm{End}(\Lambda^*(W))$, $\mathrm{End}(\Lambda^*(W^*))$ being modules over the even part $\mathrm{Cl}^{\mathrm{even}}(q)$ of $\mathrm{Cl}(q)$.

As for $\mathfrak{so}(q)$, it can be identified with the Lie subalgebra of elements in $\mathrm{Cl}^{\mathrm{even}}(q)$ of the form $v_1v_2 - v_2v_1$ with $v_1, v_2 \in V$, with respect to the commutator Lie algebra structure on $\mathrm{Cl}(q)$. Under this identification, the element $a \in \mathrm{Cl}(q)$ corresponding to $A \in \mathfrak{so}(q)$ is characterized by $A(v) = av - va$ for all $v \in V$.

The spin representation $\mathrm{spin}(q)$ of $\mathfrak{so}(q)$ is the restriction to $\mathfrak{so}(q) \subset \mathrm{Cl}^{\mathrm{even}}(q)$ of the above action of $\mathrm{Cl}^{\mathrm{even}}(q)$ on $\Lambda^*(W)$. These representations integrate to representations of the simply connected simple complex algebraic group $\mathrm{Spin}(q)$, a two-sheeted covering of $\mathrm{SO}(q)$. The representation space of $\mathrm{spin}(q)$ is thus $\Lambda^*(W)$, so that dimension of this representation is 2^r, where r is the dimension of W. That is, $r = \left[\frac{n}{2}\right]$, which is equal to the rank of $\mathfrak{so}(q)$. The spin representation is irreducible for n odd; for even n it is a sum of two non-isomorphic irreducible representations called semispin representations, each of dimension 2^{r-1}, on the subspaces $\Lambda^{\mathrm{even}}(W)$ and $\Lambda^{\mathrm{odd}}(W)$ of even and odd degree multivectors of W respectively. These representations are dual to each other for r odd and both self-dual for r even. They are carried into each other by an exterior automorphism of $\mathrm{Spin}(q)$, induced by the permutation of two short legs on the Dynkin diagram D_r.

All possible forms q as above are equivalent up to a nonsingular linear transformation. We will choose for q the form on $V = \mathbf{C}^n$ given by

$$q(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i x_{n+1-i},$$

and denote the corresponding entities as $\mathfrak{so}_n$, Cl_n, SO_n and Spin_n. We will denote by spin_n the spin representation of Spin_n on $\Lambda^*(\mathbf{C}^{\frac{n-1}{2}})$ for n odd, and the semispin representation of Spin_n on $\Lambda^{\mathrm{odd}}(\mathbf{C}^{\frac{n}{2}})$ for n even.

Classification of orbits of spin_n for $n \leqslant 12$ has been achieved by Igusa in [15], for $n = 13$ by Kac and Vinberg in [13], for $n = 14$ by Popov in [24] and for $n = 16$ by Antonyan and Elashvili in [2]. For $n \leqslant 10$ and $n = 12, 14, 16$ these representations are θ-groups in the sense of [29] and are polar for $n = 11, 13$. It is known that in all these cases the nullcone consists of finitely many orbits; moreover in the Hesselink stratification each stratum consists of a single orbit.

Thus, as has been already established in [16], $\mathfrak{N}\mathrm{spin}_n$ consists of finitely many orbits iff $n \leqslant 14$ or $n = 16$, while $n = 15$ is the smallest n for which the nullcone consists of infinitely many orbits. This fact has been one of the motivations for us to investigate Hesselink strata of this particular representation.

1.3. **Generic orbits of strata in $\mathfrak{N}\mathrm{spin}_{15}$.** We have aimed at describing the Hesselink stratification for $\mathfrak{N}\mathrm{spin}_{15}$. We have used [1] successfully for some representations of relatively small dimensions, but all our attempts to apply that program to spin_{15} exhausted all computational resources that we possessed. We managed to achieve our goal using the excellent calculational tool [9] provided to us by Willem de Graaf. His program provides another realization of the algorithm described in [26] with several simplifications incorporated. The implementation for irreducible modules over semisimple algebras has been incorporated before in [10] as the routine `CharacteristicsOfStrata`. A detailed description of his version of the algorithm is in [8, (7.3.4)]. The new version he shared with us works with arbitrary representations of reductive groups, which has been indispensable for the more fine-grained analysis of nullcones for various subgroups $\tilde{G}^{(0)}$ involved in the strata. It gives straight away the set of characteristics, one for each stratum, together with dimensions of strata. Our own contribution consists mainly in describing, for each stratum, a representative with possibly minimal *support*, i. e. given by a sum of possibly few weight vectors in spin_{15}, which is *generic* in a certain sense.

This notion of genericity is defined using an identification of the representation space of spin_{15} with that of spin_{16}. Namely, the group Spin_{15} embeds in Spin_{16} as the set of fixed points for the aforementioned outer diagram automorphism of order 2, and the representation spin_{15} can be realized as the restriction of spin_{16} along the resulting embedding $\mathrm{Spin}_{15} \hookrightarrow \mathrm{Spin}_{16}$. This realization then provides an embedding of nullcones $\mathfrak{N}\mathrm{spin}_{15} \hookrightarrow \mathfrak{N}\mathrm{spin}_{16}$. We will use the latter embedding to identify $\mathfrak{N}\mathrm{spin}_{15}$ with a subset in $\mathfrak{N}\mathrm{spin}_{16}$.

Moreover, spin_{16} is an instance of a θ-group: in a group of type E_8 there is an element τ of order 2 with the centralizer isomorphic to Spin_{16}, and the action of the latter on the (-1)-eigenspace of τ in the adjoint representation $\mathfrak{e}_8$ of E_8 is a realization of spin_{16}. Thus in its turn $\mathfrak{N}\mathrm{spin}_{16}$ can be identified with a subset of $\mathfrak{N}\mathfrak{e}_8$, with each $\mathfrak{N}\mathrm{spin}_{16}$-orbit contained in a (unique) $\mathfrak{N}\mathfrak{e}_8$-orbit. Note also that as $\mathfrak{so}_{15}$-modules, $\mathfrak{so}_{16}$ is a direct sum of the adjoint representation and the 15-dimensional standard representation, while $\mathfrak{e}_8$ is the sum of the adjoint, standard and spin representations.

With the above identifications, we then have

Proposition 1. *For each stratum* $\mathbf{H}$ *of* $\mathfrak{N}\mathrm{spin}_{15}$ *there is a unique orbit* $\mathcal{O}_1$ *of* $\mathfrak{N}\mathrm{spin}_{16}$ *and unique orbit* $\mathcal{O}_2 \supseteq \mathcal{O}_1$ *of* $\mathfrak{N}\mathfrak{e}_8$ *with the property that both* $\mathbf{H} \cap \mathcal{O}_1$ *and* $\mathbf{H} \cap \mathcal{O}_2$ *are Zariski dense open subsets of* $\mathbf{H}$.

Proof. For a stratum $\mathbf{H}$ let Σ be the set of all those orbits $\mathcal{O}$ of $\mathfrak{N}\mathrm{spin}_{16}$ with $\mathbf{H} \cap \mathcal{O} \neq \varnothing$. Clearly $\mathbf{H}$ is the disjoint union of the latter intersections; a fortiori $\mathbf{H}$ is the union of its closed subsets $\mathbf{H} \cap \overline{\mathcal{O}}$, $\mathcal{O} \in \Sigma$.

Since $\mathbf{H}$ is irreducible, it follows that there is an $\mathcal{O} \in \Sigma$ with $\mathbf{H} \subseteq \overline{\mathcal{O}}$. Now each orbit of $\mathfrak{N}\mathrm{spin}_{16}$ is open in its closure, so there is a Zariski open $U \subseteq \mathfrak{N}\mathrm{spin}_{16}$ with $\mathcal{O} = \overline{\mathcal{O}} \cap U$. Consequently $\mathbf{H} \cap \mathcal{O} = \mathbf{H} \cap \overline{\mathcal{O}} \cap U = \mathbf{H} \cap U$, i. e. $\mathbf{H} \cap \mathcal{O}$ is open in $\mathbf{H}$.

Since all spin_{16}-orbits are pairwise disjoint while any two nonempty Zariski opens of an irreducible variety have nonempty intersection, it follows that an $\mathcal{O}$ with $\mathbf{H} \cap \mathcal{O}$ dense in $\mathbf{H}$ is in fact unique.

The statement about $\mathfrak{e}_8$ clearly follows from that about spin_{16}. □

Note that, each $\mathbf{H}$ being an irreducible variety, the union of all those spin_{15}-orbits from $\mathbf{H}$ which have largest possible dimension also contains a Zariski dense open subset of $\mathbf{H}$. With this, it makes sense to define generic elements of strata, as follows.

Definition 2. Call a vector $v \in \mathbf{H}$ in a stratum $\mathbf{H}$ of spin_{15} *generic*, if vectors equivalent to v under the action of Spin_{16} (hence also under the action of E_8) are dense in $\mathbf{H}$, and moreover dimension of its spin_{15}-orbit is largest possible for any spin_{15}-orbit in $\mathbf{H}$.

1.3.1. *Remark.* Thus (1) guarantees that each stratum contains generic elements, and they all are equivalent under the action of Spin_{16} (hence also under E_8). Note however that for a generic element v there might exist a v' which is Spin_{16}-equivalent to it but is not generic, i. e. dimension of $\mathrm{Spin}_{15}v'$ is strictly less than dimension of $\mathrm{Spin}_{15}v$. One example of such elements is given in 2.2.1 below.

On the other hand, in a stratum there might exist non-generic vectors with Spin_{15}-orbit of largest possible dimension. For example, in the stratum numbered 2, with characteristic 4404040, for the generic representative v

of this stratum given by us in the table "Representatives of generic orbits" below, the Spin_{15}-orbit of its projection $v^{(2)}$ has the same dimension as that of $\mathrm{Spin}_{15}v$, namely 104, but while the type of the E_8-orbit of v is $\mathrm{E}_7+\mathrm{A}_1$, that of $v^{(2)}$ is $\mathrm{E}_6+\mathrm{A}_1$, so that not just Spin_{16}-orbits but even E_8-orbits of v and $v^{(2)}$ are different.

1.3.2. *Remark.* More invariant and natural notion of genericity would be possible if we would know that representation of Spin_{15} on each stratum possesses *stabilizers in general position* (sgp, see e. g. [27, §7]. By definition this means existence of a Zariski open set of vectors with stabilizers belonging to the same conjugacy class of subgroups in Spin_{15}. This would readily follow if the strata would be affine varieties, but in general they are not, and we do not know whether it is so.

We must note however that there is some evidence for existence of sgp for each stratum. We took at random elements from $V^{(\geqslant 2)}$ and computed isomorphism type of the reductive part, as well as dimension sequences for the lower and upper central series of the nilradical, for the Lie algebras of their Spin_{15}-stabilizers (that is, their annihilators in $\mathfrak{so}_{15}$).

For each stratum we made ten attempts of this kind, choosing linear combinations with coefficients at all weight vectors from $V^{(\geqslant 2)}$ nonzero, and obtained identical data for these annihilators in all these ten attempts. This is as it should be since there is a Zariski open set of vectors whose Spin_{15}-stabilizers have conjugate reductive parts and equal dimensions of central series of nilradicals, the latter numbers being discrete invariants of the orbits.

So, although we do not know how to find out whether these are indeed the Lie algebras of the sgp in the strata, still we ensured that each of our representatives has the same data for its $\mathfrak{so}_{15}$-annihilator. Our representatives thus obey yet another notion of genericity, substantiated only by empirical evidence.

Let us also note that we encountered vectors generic in this sense but not in the sense of (2), as well as the other way round.

2. Description of computations

2.1. **Computing characteristics and dimensions of strata.** The list of characteristics, together with dimensions of their strata, is provided by the aforementioned code `strata` by Willem de Graaf [9]. In fact, the latter dimension can be readily read off the characteristic: as explained in [26], one has

$$\dim(\mathbf{H}_h)=\dim(\mathfrak{g}_h^{(<0)})+\dim(V_h^{(\geqslant 2)}). \tag{2.1}$$

Altogether, the program `strata` produces 169 strata, listed in the table named "Strata" below. The meaning of columns in this table is as follows.

The first column just describes a certain numbering of strata, for further reference. This numbering goes from higher to lower dimension of the strata, and from higher to lower largest possible dimension of orbits in the stratum for strata of the same dimensions. For strata with both of these dimensions the same the numbering is more or less arbitrary.

The second column, labeled "characteristic", gives characteristics of strata, in terms of values of simple roots on them. Thus, each characteristic h is given by a 7-tuple of numbers $(\alpha_1(h),\dots,\alpha_7(h))$ where $\alpha_1,\ \dots,\ \alpha_7$ are the simple roots for the root system B_7. If the latter list contains fractions, their common denominator is taken out – thus for example the record $(4,2,0,2,6,0,2)/3$ (for the 95th stratum) means $(\frac{4}{3},\frac{2}{3},0,\frac{2}{3},2,0,\frac{2}{3})$.

In the third column we give the dimension of the stratum, as described above in (2.1).

The fourth column gives the quantity d for the stratum, defined as the transcendence degree of the field of invariant rational functions on the stratum. It is equal to the difference between dimension of the stratum and largest possible dimension of an orbit in the stratum (we use the notion as used in [30]; see [25] for an alternative, more advanced version).

For convenience, here is the information about the numbers of strata according to d:

d	0	1	2	3	4	5	6	7	8	9
number of strata	57	29	21	8	14	12	12	7	7	2

In the fifth column we record the information whether for the given stratum its $V^{(2)}$ contains a vector which is generic in the sense of 2 above.

Finally the last two columns give the information about $\tilde{G}^{(0)}$ as a linear group action on $V^{(2)}$: the sixth column describes the isomorphism type of $\tilde{\mathfrak{g}}^{(0)}$ with the kernel of the action on $V^{(2)}$ (if any) factored out, and in the last column is the list of dimensions of all irreducible $\tilde{\mathfrak{g}}^{(0)}$-module components of $V^{(2)}$.

2.2. **Determining whether a given vector belongs to a given stratum.** In [27, Theorem 5.4] (attributed to [18, 23]; see also [26]), a criterion for a vector to belong to a stratum is given, as follows.

Let G be a reductive group with Lie algebra $\mathfrak{g}$, and let us choose a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$. Moreover, if $\mathfrak{h} \neq \mathfrak{g}$, let us also choose the set of positive roots for $\mathfrak{g}$. Reductivity means that $\mathfrak{g}$ splits into direct sum of a semisimple algebra $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ (the commutator subalgebra of $\mathfrak{g}$) and the center $\mathfrak{z}(\mathfrak{g}) \subseteq \mathfrak{h}$ of $\mathfrak{g}$. Then $\mathfrak{g}'$ and $\mathfrak{z}(\mathfrak{g})$ are mutually orthogonal that with respect to the nonsingular invariant symmetric bilinear form $\langle -, - \rangle$ on $\mathfrak{g}$ as chosen above. On each simple summand of $\mathfrak{g}'$ the latter is a nonlinear scalar multiple of the Killing form. In our case $\mathfrak{g}$ is usually realized as a subalgebra of some simple algebra, and we are just taking for $\langle -, - \rangle$ the restriction of the Killing form of that simple algebra.

Let ϱ be a representation of G on a vector space V, and let $\mathbf{H}$ be a stratum in $\mathfrak{N}_G V$ determined by its characteristic $h \in \mathfrak{h}$. In particular, each weight vector of V is an eigenvector of h. Let $\mathfrak{g}_h^{(0)} := \mathfrak{z}_{\mathfrak{g}}(h)$ be the centralizer of h in $\mathfrak{g}$, and let $\tilde{\mathfrak{g}}_h^{(0)} \subset \mathfrak{g}_h^{(0)}$ be the orthogonal complement (with respect to the above bilinear form) of h in $\mathfrak{g}_h^{(0)}$. Let also $G_h^{(0)}$, respectively $\tilde{G}_h^{(0)}$ be the subgroup of G with Lie algebra $\mathfrak{g}_h^{(0)}$, resp. $\tilde{\mathfrak{g}}_h^{(0)}$.

Then, every vector v in $\mathbf{H}$ is equivalent under the action of G to a vector in $V_h^{(\geqslant 2)}$, i. e. there is a $g \in G$ such that $gv = \sum_{d \geqslant 2} v^{(d)}$, with $v^{(d)}$ an eigenvector of $\varrho(h)$ with eigenvalue d; moreover, a vector $v = \sum_{d \geqslant 2} v^{(d)} \in V^{(\geqslant 2)}$ belongs to $\mathbf{H}$ if and only if $v^{(2)} \notin \mathfrak{N}_{\tilde{G}_h^{(0)}} V_h^{(2)}$. In other words,

$$\mathbf{H} \cap V^{(\geqslant 2)} = \pi^{-1}(V^{(2)} \setminus \mathfrak{N}_{\tilde{G}^{(0)}} V^{(2)}), \tag{2.2}$$

where $\pi : V^{(\geqslant 2)} \to V^{(2)}$ is the projection assigning to a vector its degree 2 homogeneous component. In particular, $\mathbf{H} \cap V^{(\geqslant 2)}$ is Zariski open in $V^{(\geqslant 2)}$.

Thus to determine representatives of various kinds of orbits in strata we primarily need means to decide that a vector $v \in V_h^{(2)}$ lies outside the nullcone $\mathfrak{N}_{\tilde{G}_h^{(0)}} V_h^{(2)}$.

We will use several tools for that. Let us list them here, in order of convenience for us.

Proposition 2. *For a stratum $\mathbf{H}$ with characteristic h, let $v \in V^{(\geqslant 2)}$ be a vector such that the $G_h^{(0)}$-orbit of the degree 2 homogeneous component $v^{(2)}$ of v is Zariski open in $V_h^{(2)}$. Then $v \in \mathbf{H}$.*

Proof. As explained above, $v \in \mathbf{H}$ iff $v^{(2)} \notin \mathfrak{N}_{\tilde{G}^{(0)}} V^{(2)}$. By assumption $G^{(0)} v^{(2)}$ is Zariski open in $V^{(2)}$. Now every element of $G^{(0)} v^{(2)}$ has form $\tilde{g} \exp(\lambda h) v^{(2)}$ for $\tilde{g} \in \tilde{G}^{(0)}$. Since $v^{(2)}$ is an eigenvector for $\rho(h)$, $\exp(\lambda h) v^{(2)}$ is a scalar multiple of $v^{(2)}$. So if $v^{(2)}$ would lie in $\mathfrak{N}_{\tilde{G}^{(0)}} V^{(2)}$ then any of its scalar multiples also would lie there, and then $\mathfrak{N} V^{(2)}$ would contain a (nonempty) Zariski open subset of $V^{(2)}$. Since nullcones are in fact Zariski closed, this would then imply that $\mathfrak{N} V^{(2)} = V^{(2)}$. This is impossible since $\mathbf{H}$ is not empty, so there are vectors with degree 2 components outside $\mathfrak{N} V^{(2)}$. □

In practice, it is very easy to find out whether the $G^{(0)}$-orbit of $v^{(2)}$ is open in $V^{(2)}$: this happens if and only if $\mathfrak{g}^{(0)} v^{(2)} = V^{(2)}$.

For spin_{15}, there are 120 strata possessing generic vectors v such that the $\tilde{G}_h^{(0)}$-orbit of $v^{(2)}$ is open in $V_h^{(2)}$ (where h is the characteristic of the stratum).

Our next tool is based on the following consideration.

Proposition 3. *Suppose that the characteristic h of a stratum $\mathbf{H}$ is also the characteristic for some spin_{16}-orbit $\mathcal{O}$. Then any $v \in V_h^{\geqslant 2}$ with $v^{(2)} \in \mathcal{O}$ belongs to $\mathbf{H}$.*

Proof. Denote by $G_{\mathcal{O}}^{(0)}$ the centralizer of h in Spin_{16} and by $\tilde{G}_{\mathcal{O}}^{(0)}$ the orthogonal complement of h in $G_{\mathcal{O}}^{(0)}$. Clearly $G_h^{(0)} \subseteq G_{\mathcal{O}}^{(0)}$ and $\tilde{G}_h^{(0)} \subseteq \tilde{G}_{\mathcal{O}}^{(0)}$.

Take any $v \in V_h^{(\geqslant 2)}$ with $v^{(2)} \in \mathcal{O}$. Then by an instance of (2.2) for Spin_{16} necessarily $v^{(2)} \notin \mathfrak{N}_{\tilde{G}_{\mathcal{O}}^{(0)}} V^{(2)}$. But $\tilde{G}_h^{(0)} \subseteq \tilde{G}_{\mathcal{O}}^{(0)}$ implies $\mathfrak{N}_{\tilde{G}_h^{(0)}} V^{(2)} \subseteq \mathfrak{N}_{\tilde{G}_{\mathcal{O}}^{(0)}} V^{(2)}$, hence also $v^{(2)} \notin \mathfrak{N}_{\tilde{G}_h^{(0)}} V^{(2)}$. Again by (2.2) this implies $v \in \mathbf{H}$. □

There are 50 strata satisfying the condition of 3, and for 38 of these there is no open $G^{(0)}$-orbit in $V^{(2)}$, so 2 is not applicable. In all these cases, to ensure that $v \in V^{(\geqslant 2)}$ belongs to $\mathbf{H}$ it suffices to check that the characteristic of $v^{(2)}$ viewed as an element of spin_{16} coincides with h.

2.2.1. *Remark.* Note that for any v as in 3, the projection $v^{(2)}$ also belongs to $\mathbf{H}$. However it might happen that no Spin_{15}-orbit intersecting $V^{(2)}$ reaches largest possible dimension of orbits from $\mathbf{H}$. This indeed happens for 10 strata. For example, in the stratum with characteristic 0402020 there are 99-dimensional Spin_{15}-orbits, while dimensions of orbits of elements from $V^{(2)}$ do not exceed 97.

There still remain 11 strata satisfying neither (2) nor (3). In these cases we used the fact that $v^{(2)} \notin \mathfrak{N}_{\tilde{G}^{(0)}} V^{(2)}$ if and only if $f(v^{(2)}) \neq 0$ for some $\tilde{G}^{(0)}$-invariant polynomial function f on $V^{(2)}$. In seven of these 11 cases we managed to compute the generators of the ring of invariant polynomials, but for strata numbered 24, 35, 58 and 133 we were unable to do it. In all of these cases except one (number 58) it turned out that there are generic representatives with the $V^{(2)}$ part satisfying the Dadok–Kac criterion, that is, such that no pair of $\tilde{\mathfrak{g}}^{(0)}$-weights from their support has difference equal to a root of $\tilde{\mathfrak{g}}^{(0)}$ (see [7, Proposition 1.2]).

For the stratum with number 58 we only have empirical evidence that our representative belongs to the stratum. Namely, we computed the nullcone for the action of $\tilde{\mathfrak{g}}^{(0)}$ on $V^{(2)}$ and, taking for each stratum vectors from $V^{(\geqslant 2)}$ at random (with nonzero coefficients at each weight vector) found largest dimensions of Spin_{15}-orbits of these vectors. In each case after 10 such attempts we obtained dimension not exceeding 98, while dimension of the orbit for our representative is 99. In the corresponding table, this representative is marked by a question mark.

2.3. **Finding representatives of generic orbits in strata.** The principal goal of this paper is to find generic elements in each stratum, with as small support as possible, i. e. those which are linear combinations of as few weight vectors of spin_{15} as possible. In particular, we find largest possible dimensions of orbits in each stratum. This dimension can be determined by several methods.

First, if we can exhibit an orbit of dimension equal to the dimension of the stratum, then trivially the largest possible dimension of the stratum coincides with the dimension of the stratum. This happens for 57 of the 169 strata. These are precisely the strata with $d = 0$, in the sense of 2.1 above.

Next, when a generic orbit in a stratum with characteristic h has a representative in $V^{(2)}$, an upper bound for this dimension is provided by dimension considerations. Note that for each homogeneous $v \in V^{(2)}$ of degree 2, actions of elements of $\mathfrak{g}$ on v decomposes into a family of maps

$$\cdot v : \mathfrak{g}_h^{(k)} \to V_h^{(k+2)},$$

where k runs over the union of sets of eigenvalues of $\mathrm{ad}(h)$ on $\mathfrak{g}$ and of $\rho(h)$ on V. It thus follows that dimensions of all G-orbits with representatives in $V_h^{(2)}$ are bounded above by the number $\sum_k \min\left(\dim(\mathfrak{g}_h^{(k)}), \dim(V_h^{(k+2)})\right)$. There are in fact 83 strata with largest possible dimension of the orbit attained on orbits with representatives in $V^{(2)}$, 42 of these having nonzero d.

In all cases, largest possible dimension of the stratum, as well as the spin_{16}-characteristic of the Spin_{16}-orbit dense in it, can be guessed empirically by general position considerations. Since the needed vectors form a set Zariski dense in the stratum, and each orbit of the stratum has a representative in $V^{(\geqslant 2)}$, clearly the intersection of the set of generic vectors is also Zariski dense in $V^{(\geqslant 2)}$. Thus if one takes a vector with support equal to the set of all weight vectors in $V^{(\geqslant 2)}$, with random integer coefficients, then this vector will be generic with probability very close to 1. So we compute orbit dimension and spin_{16} characteristic of a linear combination of all weight vectors from $V^{(\geqslant 2)}$ with coefficient at each weight vector a randomly chosen nonzero integer in the interval from -100 to 100. We took up to 10 attempts for each stratum, every time obtaining the same dimension and the same $\mathfrak{spin}_{16}$ orbit. After that, we were seeking vectors with the same dimension and spin_{16} orbit but with possibly small support, ensuring that they are in the stratum with the aid of the considerations from 2.2 above.

2.4. **About (non-generic) vectors with small support in a stratum.** There is, in a sense, an opposite question: what is the smallest number of weight vectors in the support of an element of a stratum? This number is only known to us for 133 strata for which we were able to compute invariants of the action of $\tilde{\mathfrak{g}}^{(0)}$ on $V^{(2)}$. In addition, for those strata with representatives obeying the Dadok–Kac criterion mentioned above we could exhibit representatives with quite small support, although we could not establish whether it is smallest possible. There remained only three cases (strata numbered 25, 58 and 94) where we could neither compute invariants nor apply the Dadok–Kac criterion.

In all these cases we found in $V^{(2)}$ representatives with support size not exceeding 7 (rank of Spin_{15}). It is not clear to us whether support of size not exceeding rank (which is 7 in this case) always exists. Note that, although for each characteristic h from the standard Cartan subalgebra there exists a vector v with support not exceeding rank which lies in $V^{(\geqslant 2)}$ and such that h is of smallest possible length with this property, it might happen that the $V^{(2)}$-component of v is in the nullcone of the $\tilde{\mathfrak{g}}^{(0)}$-action, while the actual characteristic for v does not belong to the standard Cartan subalgebra.

2.5. **Description of the tables.**

2.5.1. The table "spin weights dictionary" lists all 128 weights of spin_{15} in parallel with those for spin_{16} and the roots in the -1-eigenspace of τ (see 1.3). Notation for these weights is described in 3.2.

2.5.2. In the table named "spin_{15} spindle of weights" we give the weights of spin_{15} with the indication of which ones differ from each other by a simple root of B_7. A line joining two weights means that the lower one is obtained from the upper one by subtracting a simple root; moreover parallel lines mean subtraction of the same root.

2.5.3. The content of the table “Strata” is described in 2.1 above.

2.5.4. In the last table “Representatives of generic orbits” we exhibit generic orbit representatives (several, for some of the strata). In this table, the first column gives our numbering for the stratum.

In the second column we list the characteristic of the spin_{16}-orbit dense in the stratum.

The third column gives the Dynkin notation for the type of the $\mathfrak{e}_8$-orbit dense in the stratum. Let us recall that this notation gives the type of the minimal regular semisimple subalgebra containing our representative in which it is distinguished, i. e. its centralizer in this subalgebra is nilpotent. When there are several such subalgebras of different types, we choose the one whose root system is a subsystem of our chosen root system for B_7.

The fourth column depicts the Dynkin scheme of the representative. This is a graph whose nodes are the weights of its support, i. e. our representative is a linear combination of weight vectors with these weights. A number in parentheses in the superscript for a weight indicates the degree of this weight with respect to the characteristic h of the stratum, that is, this weight is an eigenvector for h with the indicated eigenvalue; if no such superscript is present this means that the corresponding h-eigenvalue is equal to 2 (i. e. the weight vector belongs to $V_h^{(2)}$). Two weights are connected by a straight line if the angle between them in the root lattice of E_8 (or also D_8, but not B_7) is $2\pi/3$, and by a dotted line if this angle is $\pi/3$. When there is no line between the weights this means they are orthogonal. In some cases, a node is encircled, and the symbol $\times 2$ is added next to the circle; this means that to obtain the correct representative not all coefficients at weight vectors must be equal. Specifically, taking the coefficient 2 at the indicated weight vector and 1 at all others gives the correct representative.

Finally in the last column we give a description of the generic centralizer, that is, the centralizer of the representative in $\mathfrak{so}_{15}$. Here, the notation $n_1, n_2, \dots, n_k$ means a nilpotent Lie algebra with n_i equal to dimensions of the associated graded of its lower central series. Thus n_1 is the number of minimal generators, n_2 is the dimension of the subspace spanned by single brackets of minimal generators, n_3 that of the subspace spanned by brackets of the form $[[x, y], z]$ for all possible minimal generators x, y, z, etc. The notation of type $\mathfrak{g} \ltimes (V_1, \dots, V_k)$ indicates a Lie algebra with reductive part $\mathfrak{g}$ and nilradical with V_i formed by elements in the ith filtration of the lower central series of the nilradical that do not belong to the $i+1$st filtration. Each V_i is described as a $\mathfrak{g}$-module; if $\mathfrak{g}$ is semisimple, dimensions of irreducible summands are given, while if $\mathfrak{g}$ contains a central torus then the relative weights of the torus actions are indicated by integers in parentheses, with weight multiplicities given in exponents; if dimension of the central torus exceeds 1 then the corresponding weights are given by columns of integers.

2.5.5. For convenience, in the end we also provide closure diagrams for nilpotent orbits in $\mathfrak{e}_8$ and spin_{16}.

3. Various notational conventions

3.1. **Embeddings $\mathfrak{so}_{15} \hookrightarrow \mathfrak{so}_{16} \hookrightarrow \mathfrak{e}_8$.** We will choose the orthonormal bases $\varepsilon_1, \dots, \varepsilon_r$ of the weight spaces for B_r and D_r; the roots of D_r are $\pm(\varepsilon_i - \varepsilon_j)$, $\pm(\varepsilon_i + \varepsilon_j)$ for $1 \leqslant i < j \leqslant r$, while those of B_r are as above and also $\pm\varepsilon_i$, $i = 1, \dots, r$. The corresponding root vectors will be denoted by $e_{i\bar{j}}$, $e_{\bar{i}j}$, e_{ij}, $e_{\overline{ij}}$, e_i and $e_{\bar{i}}$, respectively. Coroots in the standard Cartan subalgebras of these algebras will be denoted by $h_\alpha := [e_\alpha, e_{-\alpha}]$, for α any root; thus the standard simple coroot basis consists of $h_{1\bar{2}}$, $h_{2\bar{3}}, \dots,$ $h_{(r-1)\bar{r}}$, and one more element: for D_r it is $h_{r-1,r}$ while for B_r it is h_r. Thus

$$h_{i,\overline{i+1}} = [e_{i,\overline{i+1}}, e_{\bar{i},i+1}],$$
$$h_{r-1,r} = [e_{r-1,r}, e_{\overline{r-1,r}}]$$

and

$$h_r = [e_r, e_{\bar{r}}].$$

We will need the embedding I_{BD} of the simple Lie algebra of type B_7 into the simple Lie algebra of type D_8 as the subalgebra of fixed points of the diagram automorphism of D_8. Explicitly, it is given by

$$\begin{aligned}
I_{\mathrm{BD}}(e_{i\bar{j}}) &= e_{i\bar{j}},\\
I_{\mathrm{BD}}(e_{\bar{i}j}) &= e_{\bar{i}j},\\
I_{\mathrm{BD}}(e_{ij}) &= e_{ij},\\
I_{\mathrm{BD}}(e_{\overline{ij}}) &= e_{\overline{ij}},\\
I_{\mathrm{BD}}(e_i) &= e_{i\bar{8}} + e_{i8},\\
I_{\mathrm{BD}}(e_{\bar{i}}) &= e_{\bar{i}8} + e_{\overline{i8}},\\
I_{\mathrm{BD}}(h_{i,\overline{i+1}}) &= h_{i,\overline{i+1}},\\
I_{\mathrm{BD}}(h_7) &= h_{7\bar{8}} + h_{78}.
\end{aligned}$$

Moreover we will also need the embedding I_{DE} of the simple Lie algebra of type D_8 into the simple Lie algebra of type E_8 as the subalgebra of fixed points of an involutive (inner) automorphism of E_8.

We will use the realization of E_8 as the algebra (see e. g. [31])

$$\Lambda^3(\mathbf{C}^9)^* \oplus \mathfrak{sl}_9 \oplus \Lambda^3(\mathbf{C}^9).$$

Note that there is another realization of E_8 seemingly more suitable here, as the direct sum of $\mathfrak{so}_{16}$ and its semispin representation coming from the θ-group structure corresponding to an involutive automorphism of E_8. We found however that the realization from [31] has its advantages in bookkeeping, as we will see below.

Accordingly, the weight space of E_8 will be realized as spanned by $\varepsilon_1, \dots, \varepsilon_9$, with

$$\varepsilon_i = \tilde{\varepsilon}_i - \frac{1}{9}\sum_{j=1}^{9} \tilde{\varepsilon}_j,$$

where the $\tilde{\varepsilon}_j$ form an orthonormal basis in a 9-dimensional space. Thus $\varepsilon_1 + \dots + \varepsilon_9 = 0$, and

$$\langle \varepsilon_i, \varepsilon_j \rangle = \begin{cases} \frac{8}{9}, & i = j, \\ -\frac{1}{9}, & i \neq j. \end{cases}$$

The roots of E_8 are $\pm(\varepsilon_i - \varepsilon_j)$ and $\pm(\varepsilon_i + \varepsilon_j + \varepsilon_k)$, $1 \leqslant i < j < k \leqslant 9$. The corresponding root vectors will be denoted by $e_{i\bar{j}}$, $e_{\bar{i}j}$, e_{ijk} and $e_{\overline{ijk}}$ respectively. Coroots in the standard Cartan subalgebra of E_8 are $h_\alpha := [e_\alpha, e_{-\alpha}]$ as above, for α any root; thus the standard simple coroot basis consists of $h_{1\bar{2}}$, $h_{2\bar{3}}, \dots,$ $h_{7\bar{8}}$, and h_{678} (with the latter Killing orthogonal to all previous ones except $h_{5\bar{6}}$).

Explicitly, the embedding I_{DE} is given by

$$\begin{aligned}
I_{\mathrm{DE}}(e_{i\bar{j}}) &= e_{i\bar{j}},\\
I_{\mathrm{DE}}(e_{\bar{i}j}) &= e_{\bar{i}j},\\
I_{\mathrm{DE}}(e_{ij}) &= e_{ij9},\\
I_{\mathrm{DE}}(e_{\overline{ij}}) &= e_{\overline{ij9}},\\
I_{\mathrm{DE}}(h_{i,\overline{i+1}}) &= h_{i,\overline{i+1}},\ i < 8,\\
I_{\mathrm{DE}}(h_{78}) &= h_{789}\\
&\qquad (= -h_{1\bar{2}} - 2h_{2\bar{3}} - 3h_{3\bar{4}} - 4h_{4\bar{5}} - 5h_{5\bar{6}} - 4h_{6\bar{7}} - 2h_{7\bar{8}} - 2h_{678}\,).
\end{aligned}$$

Then composing I_{BD} and I_{DE} gives an embedding I_{BE} acting as follows:

$$\begin{aligned}
I_{\mathrm{BE}}(e_{i\bar{j}}) &= e_{i\bar{j}},\\
I_{\mathrm{BE}}(e_{\bar{i}j}) &= e_{\bar{i}j},\\
I_{\mathrm{BE}}(e_{ij}) &= e_{ij9},\\
I_{\mathrm{BE}}(e_{\overline{ij}}) &= e_{\overline{ij9}},\\
I_{\mathrm{BE}}(e_i) &= e_{i\bar{8}} + e_{i89},\\
I_{\mathrm{BE}}(e_{\bar{i}}) &= e_{\bar{i}8} + e_{\overline{i89}},\\
I_{\mathrm{BE}}(h_{i,\overline{i+1}}) &= h_{i,\overline{i+1}},\\
I_{\mathrm{BE}}(h_7) &= h_{7\bar{8}} + h_{789}.
\end{aligned}$$

3.2. **Embeddings** $\mathrm{spin}_{15} \hookrightarrow \mathrm{spin}_{16} \hookrightarrow \mathfrak{e}_8^-$**.** Next let us describe the correspondence between weight vectors of the spin_{15} representation of B_7, those of the spin_{16} representation of D_8, and the odd part of E_8 in the sense of the aforementioned involutive automorphism.

We mean the correspondence which respects the module structures obtained through restricting along I_{BD} and I_{DE}.

The 128 weights of spin_{15} are of the form $\mathbf{w} := \frac{1}{2}(\pm\varepsilon_1 \pm \dots \pm \varepsilon_7)$, with all possible combinations of + and − signs. It will be convenient for us to introduce a special notation for the corresponding weights and weight vectors. If the number of + signs is smaller than the number of − signs, we denote the weight for $\mathbf{w}$ by s, where s is the set of these indices i for which the sign at ε_i in $\mathbf{w}$ is +. Whereas if the number of + signs is larger than the number of − signs, then the corresponding weight will be denoted by $\bar{s}$, where now s is the set of those i for which the sign at ε_i in $\mathbf{w}$ is −. Note in particular that the highest weight of spin_{15}, i. e. the weight $\frac{1}{2}(\varepsilon_1 + \dots + \varepsilon_7)$ will be denoted by $\bar{\varnothing}$, and the lowest weight by $\varnothing$. The weight vector in spin_{15} corresponding to the weight denoted by s, respectively $\bar{s}$, will be denoted by u_s, resp. $u_{\bar{s}}$.

For spin_{16} we choose that of the two semispin representations of D_8 whose 128 weights have form $\frac{1}{2}(\pm\varepsilon_1\pm\cdots\pm\varepsilon_8)$, with odd number of + signs. Similarly to the above, the weight with less + signs will be denoted by s where s is the set of indices with + signs, and the weight with less − signs by $\bar{s}$, where s is the set of indices with − signs. The corresponding weight vectors will be denoted by v_s, resp. $v_{\bar{s}}$.

Finally, the roots of the odd part of E_8, i. e. those corresponding to the −1 eigenvalue of the aforementioned involutive automorphism, are $\pm(\varepsilon_i-\varepsilon_9)$ and $\pm(\varepsilon_i+\varepsilon_j+\varepsilon_k)$ with $i,j,k<9$.

The isomorphism J_{BD} from spin_{15} to the restriction of spin_{16} along I_{BD}, i. e. the linear isomorphism satisfying

$$J_{\mathrm{BD}}(g\cdot x)=I_{\mathrm{BD}}(g)\cdot J_{\mathrm{BD}}(x)$$

for all g from B_7 and x from spin_{15} is explicitly given by

$$J_{\mathrm{BD}}(u_s)=\begin{cases}v_s & \text{if cardinality of } s \text{ is odd,}\\ v_{s\cup\{8\}} & \text{if cardinality of } s \text{ is even.}\end{cases}$$

and similarly

$$J_{\mathrm{BD}}(u_{\bar{s}})=\begin{cases}v_{\bar{s}} & \text{if cardinality of } s \text{ is odd,}\\ v_{\overline{s\cup\{8\}}} & \text{if cardinality of } s \text{ is even.}\end{cases}$$

In more detail,

$$\begin{aligned}J_{\mathrm{BD}}(u_{\bar{\varnothing}})&=v_{\bar{8}},\\ J_{\mathrm{BD}}(u_{\bar{i}})&=v_{\bar{i}},\\ J_{\mathrm{BD}}(u_{\overline{ij}})&=v_{\overline{ij8}},\\ J_{\mathrm{BD}}(u_{\overline{ijk}})&=v_{\overline{ijk}},\\ J_{\mathrm{BD}}(u_{ijk})&=v_{ijk},\\ J_{\mathrm{BD}}(u_{ij})&=v_{ij8},\\ J_{\mathrm{BD}}(u_i)&=v_i,\\ J_{\mathrm{BD}}(u_{\varnothing})&=v_8.\end{aligned}$$

As for the isomorphism J_{DE} from spin_{16} to the odd part of E_8, it is explicitly given by

$$\begin{aligned}J_{\mathrm{DE}}(v_{\bar{i}})&=e_{\bar{i}9},\\ J_{\mathrm{DE}}(v_{\overline{ijk}})&=e_{\overline{ijk}},\\ J_{\mathrm{DE}}(v_{ijk})&=e_{ijk},\\ J_{\mathrm{DE}}(v_i)&=e_{i\bar{9}}.\end{aligned}$$

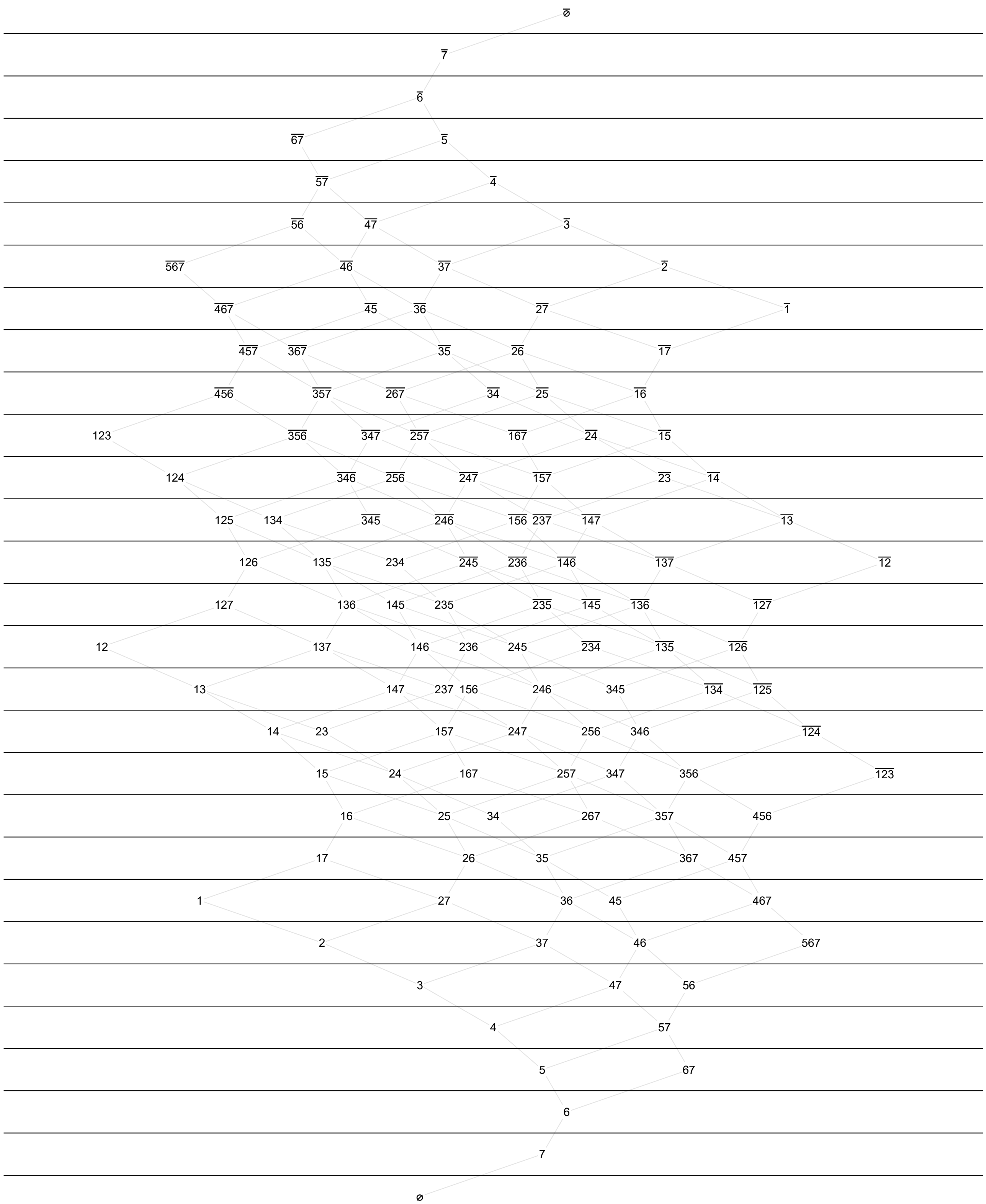
Finally, the isomorphism J_{BE} from spin_{15} to the odd part of E_8 is just the composite of J_{BD} and J_{DE}, so it is given by

$$\begin{aligned}J_{\mathrm{BE}}(u_{\bar{\varnothing}})&=e_{\bar{8}9},\\ J_{\mathrm{BE}}(u_{\bar{i}})&=e_{\bar{i}9},\\ J_{\mathrm{BE}}(u_{\overline{ij}})&=e_{\overline{ij8}},\\ J_{\mathrm{BE}}(u_{\overline{ijk}})&=e_{\overline{ijk}},\\ J_{\mathrm{BE}}(u_{ijk})&=e_{ijk},\\ J_{\mathrm{BE}}(u_{ij})&=e_{ij8},\\ J_{\mathrm{BE}}(u_i)&=e_{i\bar{9}},\\ J_{\mathrm{BE}}(u_{\varnothing})&=e_{8\bar{9}}.\end{aligned}$$

For readers convenience, we collect in a table all these correspondences between weights of spin_{15} and spin_{16} and roots of the −1-part of E_8 together.

spin weights dictionary

spin_{15}	spin_{16}	E_8	spin_{15}	spin_{16}	E_8	spin_{15}	spin_{16}	E_8	spin_{15}	spin_{16}	E_8
$\bar{\varnothing}$	$\bar{8}$	$\bar{8}9$	$\overline{167}$	$\overline{167}$	$\overline{167}$	235	235	235	167	167	167
$\bar{7}$	$\bar{7}$	$\bar{7}9$	$\overline{257}$	$\overline{257}$	$\overline{257}$	145	145	145	15	158	158
$\bar{6}$	$\bar{6}$	$\bar{6}9$	$\overline{347}$	$\overline{347}$	$\overline{347}$	136	136	136	456	456	456
$\bar{5}$	$\bar{5}$	$\bar{5}9$	$\overline{356}$	$\overline{356}$	$\overline{356}$	127	127	127	357	357	357
$\overline{67}$	$\overline{678}$	$\overline{678}$	123	123	123	$\overline{126}$	$\overline{126}$	$\overline{126}$	34	348	348
$\bar{4}$	$\bar{4}$	$\bar{4}9$	$\overline{14}$	$\overline{148}$	$\overline{148}$	$\overline{135}$	$\overline{135}$	$\overline{135}$	267	267	267
$\overline{57}$	$\overline{578}$	$\overline{578}$	$\overline{23}$	$\overline{238}$	$\overline{238}$	245	245	245	25	258	258
$\bar{3}$	$\bar{3}$	$\bar{3}9$	$\overline{157}$	$\overline{157}$	$\overline{157}$	$\overline{234}$	$\overline{234}$	$\overline{234}$	16	168	168
$\overline{47}$	$\overline{478}$	$\overline{478}$	$\overline{247}$	$\overline{247}$	$\overline{247}$	236	236	236	457	457	457
$\overline{56}$	$\overline{568}$	$\overline{568}$	$\overline{256}$	$\overline{256}$	$\overline{256}$	146	146	146	367	367	367
$\bar{2}$	$\bar{2}$	$\bar{2}9$	$\overline{346}$	$\overline{346}$	$\overline{346}$	137	137	137	35	358	358
$\overline{37}$	$\overline{378}$	$\overline{378}$	124	124	124	12	128	128	26	268	268
$\overline{46}$	$\overline{468}$	$\overline{468}$	$\overline{13}$	$\overline{138}$	$\overline{138}$	$\overline{125}$	$\overline{125}$	$\overline{125}$	17	178	178
$\overline{567}$	$\overline{567}$	$\overline{567}$	$\overline{147}$	$\overline{147}$	$\overline{147}$	345	345	345	467	467	467
$\bar{1}$	$\bar{1}$	$\bar{1}9$	$\overline{237}$	$\overline{237}$	$\overline{237}$	$\overline{134}$	$\overline{134}$	$\overline{134}$	45	458	458
$\overline{27}$	$\overline{278}$	$\overline{278}$	$\overline{156}$	$\overline{156}$	$\overline{156}$	246	246	246	36	368	368
$\overline{36}$	$\overline{368}$	$\overline{368}$	$\overline{246}$	$\overline{246}$	$\overline{246}$	156	156	156	27	278	278
$\overline{45}$	$\overline{458}$	$\overline{458}$	$\overline{345}$	$\overline{345}$	$\overline{345}$	237	237	237	1	1	$1\bar{9}$
$\overline{467}$	$\overline{467}$	$\overline{467}$	134	134	134	147	147	147	567	567	567
$\overline{17}$	$\overline{178}$	$\overline{178}$	125	125	125	13	138	138	46	468	468
$\overline{26}$	$\overline{268}$	$\overline{268}$	$\overline{12}$	$\overline{128}$	$\overline{128}$	$\overline{124}$	$\overline{124}$	$\overline{124}$	37	378	378
$\overline{35}$	$\overline{358}$	$\overline{358}$	$\overline{137}$	$\overline{137}$	$\overline{137}$	346	346	346	2	2	$2\bar{9}$
$\overline{367}$	$\overline{367}$	$\overline{367}$	$\overline{146}$	$\overline{146}$	$\overline{146}$	256	256	256	56	568	568
$\overline{457}$	$\overline{457}$	$\overline{457}$	$\overline{236}$	$\overline{236}$	$\overline{236}$	247	247	247	47	478	478
$\overline{16}$	$\overline{168}$	$\overline{168}$	$\overline{245}$	$\overline{245}$	$\overline{245}$	157	157	157	3	3	$3\bar{9}$
$\overline{25}$	$\overline{258}$	$\overline{258}$	234	234	234	23	238	238	57	578	578
$\overline{34}$	$\overline{348}$	$\overline{348}$	135	135	135	14	148	148	4	4	$4\bar{9}$
$\overline{267}$	$\overline{267}$	$\overline{267}$	126	126	126	$\overline{123}$	$\overline{123}$	$\overline{123}$	67	678	678
$\overline{357}$	$\overline{357}$	$\overline{357}$	$\overline{127}$	$\overline{127}$	$\overline{127}$	356	356	356	5	5	$5\bar{9}$
$\overline{456}$	$\overline{456}$	$\overline{456}$	$\overline{136}$	$\overline{136}$	$\overline{136}$	347	347	347	6	6	$6\bar{9}$
$\overline{15}$	$\overline{158}$	$\overline{158}$	$\overline{145}$	$\overline{145}$	$\overline{145}$	257	257	257	7	7	$7\bar{9}$
$\overline{24}$	$\overline{248}$	$\overline{248}$	$\overline{235}$	$\overline{235}$	$\overline{235}$	24	248	248	$\varnothing$	8	$8\bar{9}$



spin_{15} spindle of weights

Strata

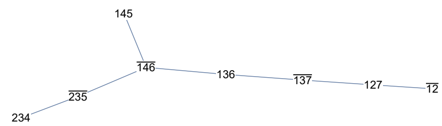
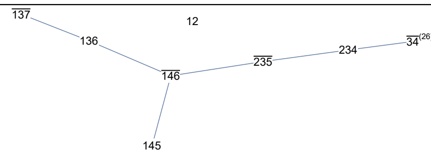
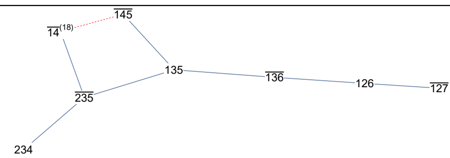
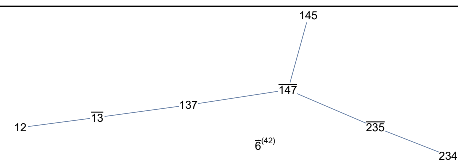
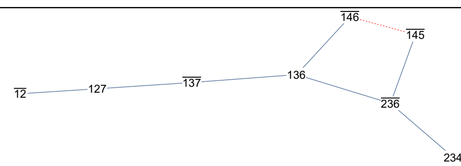
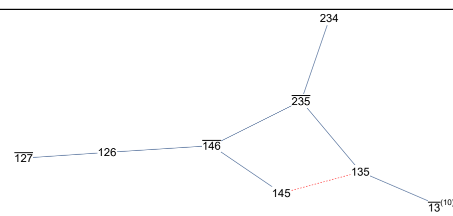
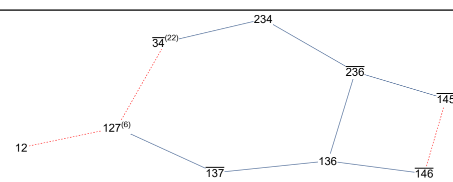
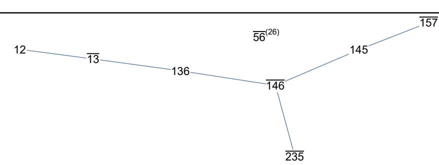
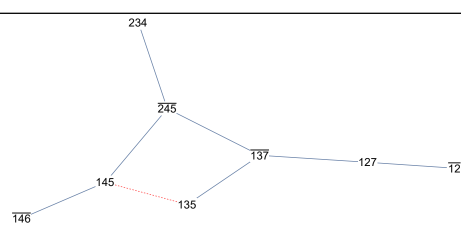
#	characteristic	dim(stratum)	d(stratum)	generic $V^{(2)}$-orbit	$\tilde{\mathfrak{g}}^{(0)}$	$V^{(2)}$
1	$(8,4,4,4,4,4,4)$	113	8	yes	T^6	$\mathbf{1}^8$
2	$(8,8,4,4,4,4,4)$	113	9	no	T^6	$\mathbf{1}^7$
3	$(4,4,4,4,4,4,8)$	113	9	no	T^6	$\mathbf{1}^7$
4	$(8,4,4,4,4,0,4)$	112	8	no	$\mathrm{A}_1+\mathrm{T}^5$	$\mathbf{1}^5\oplus\mathbf{2}^2$
5	$(4,4,4,4,0,4,4)$	112	8	yes	$\mathrm{A}_1+\mathrm{T}^5$	$\mathbf{1}^4\oplus\mathbf{2}^3$
6	$(4,4,0,4,4,4,4)$	112	8	no	$\mathrm{A}_1+\mathrm{T}^5$	$\mathbf{1}^3\oplus\mathbf{2}^3$
7	$(4,8,4,4,0,4,4)$	112	8	no	$\mathrm{A}_1+\mathrm{T}^5$	$\mathbf{1}^3\oplus\mathbf{2}^3$
8	$(8,4,4,0,4,0,4)$	111	7	no	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^3\oplus\mathbf{2}^2\oplus\mathbf{4}$
9	$(4,4,0,4,4,0,4)$	111	7	yes	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^2\oplus\mathbf{2}^3\oplus\mathbf{4}$
10	$(4,4,4,0,4,4,0)$	111	7	no	$\mathrm{A}_1+\mathrm{B}_1+\mathrm{T}^4$	$\mathbf{2}^4\oplus\mathbf{4}$
11	$(8,0,4,4,4,0,4)$	111	7	no	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^3\oplus\mathbf{2}^2\oplus\mathbf{4}$
12	$(4,0,4,4,0,4,4)$	111	8	no	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}\oplus\mathbf{2}^3\oplus\mathbf{4}$
13	$(8,0,4,0,4,0,4)$	110	6	no	$3\mathrm{A}_1+\mathrm{T}^3$	$\mathbf{1}\oplus\mathbf{2}^2\oplus\mathbf{4}^2$
14	$(4,4,0,4,0,4,0)$	110	6	yes	$2\mathrm{A}_1+\mathrm{B}_1+\mathrm{T}^3$	$\mathbf{2}^2\oplus\mathbf{4}\oplus\mathbf{8}$
15	$(4,4,4,4,0,0,4)$	110	6	no	$\mathrm{A}_2+\mathrm{T}^4$	$\mathbf{1}^3\oplus\mathbf{3}^3$
16	$(2,6,2,2,2,2,2)$	110	6	no	T^6	$\mathbf{1}^7$
17	$(0,4,4,0,4,0,4)$	110	7	no	$3\mathrm{A}_1+\mathrm{T}^3$	$\mathbf{1}\oplus\mathbf{2}^2\oplus\mathbf{4}^2$
18	$(4,0,4,0,4,4,0)$	110	7	no	$2\mathrm{A}_1+\mathrm{B}_1+\mathrm{T}^3$	$\mathbf{2}\oplus\mathbf{4}^3$
19	$(0,4,0,4,0,4,4)$	110	8	no	$3\mathrm{A}_1+\mathrm{T}^3$	$\mathbf{1}\oplus\mathbf{2}^2\oplus\mathbf{8}$
20	$(2,2,2,2,2,2,2)$	109	6	yes	T^6	$\mathbf{1}^8$
21	$(2,4,2,2,2,2,2)$	109	6	no	T^6	$\mathbf{1}^7$
22	$(4,0,4,4,0,0,4)$	109	6	no	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{T}^3$	$\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{3}^2\oplus\mathbf{6}$
23	$(4,4,0,0,4,0,4)$	109	7	no	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{T}^3$	$\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{3}^2\oplus\mathbf{6}$
24	$(0,4,4,0,0,4,0)$	108	5	no	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{B}_1+\mathrm{T}^2$	$\mathbf{2}\oplus\mathbf{4}\oplus\mathbf{12}$
25	$(0,4,0,4,0,0,4)$	108	6	yes	$2\mathrm{A}_1+\mathrm{A}_2+\mathrm{T}^2$	$\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{3}\oplus\mathbf{12}$
26	$(4,0,0,4,0,4,0)$	108	6	yes	$2\mathrm{A}_2+\mathrm{A}_1+\mathrm{B}_1+\mathrm{T}^2$	$\mathbf{2}\oplus\mathbf{6}\oplus\mathbf{12}$
27	$(2,0,2,2,2,2,2)$	108	6	no	$\mathrm{A}_1+\mathrm{T}^5$	$\mathbf{1}^3\oplus\mathbf{2}^3$
28	$(4,4,4,0,0,0,4)$	107	4	no	$\mathrm{A}_3+\mathrm{T}^3$	$\mathbf{1}^2\oplus\mathbf{4}^2\oplus\mathbf{6}$
29	$(4,4,0,4,2,0,2)$	107	4	no	$\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^2\oplus\mathbf{2}^3$
30	$(6,0,2,4,2,0,2)$	107	4	no	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^4\oplus\mathbf{4}$
31	$(2,2,2,2,2,0,2)$	107	5	yes	$\mathrm{A}_1+\mathrm{T}^5$	$\mathbf{1}^4\oplus\mathbf{2}^3$
32	$(8,2,0,2,2,0,2)$	107	5	no	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^4\oplus\mathbf{4}$
33	$(4,4,2,0,4,0,2)$	107	5	no	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^4\oplus\mathbf{4}$
34	$(1,2,1,3,1,1,3)$	107	6	no	T^6	$\mathbf{1}^6$
35	$(4,4,0,0,0,4,0)$	106	4	no	$\mathrm{A}_3+\mathrm{B}_1+\mathrm{T}^2$	$\mathbf{2}\oplus\mathbf{8}\oplus\mathbf{12}$
36	$(3,1,1,3,1,2,1)$	106	4	no	T^6	$\mathbf{1}^7$
37	$(2,4,2,0,2,0,2)$	106	4	no	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^3\oplus\mathbf{2}^2\oplus\mathbf{4}$
38	$(4,0,4,0,0,0,4)$	106	4	no	$\mathrm{A}_1+\mathrm{A}_3+\mathrm{T}^2$	$\mathbf{2}\oplus\mathbf{4}\oplus\mathbf{6}\oplus\mathbf{8}$
39	$(0,0,4,0,0,4,0)$	106	5	yes	$2\mathrm{A}_2+\mathrm{B}_1+\mathrm{T}^1$	$\mathbf{6}\oplus\mathbf{18}$
40	$(2,0,2,0,2,2,2)$	106	5	no	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}\oplus\mathbf{2}^3\oplus\mathbf{4}$
41	$(4,0,4,2,0,2,0)$	106	5	no	$\mathrm{A}_1+\mathrm{B}_1+\mathrm{T}^3$	$\mathbf{2}^3\oplus\mathbf{4}$
42	$(0,2,2,0,4,0,2)$	106	5	no	$3\mathrm{A}_1+\mathrm{T}^3$	$\mathbf{1}^2\oplus\mathbf{4}^2$
43	$(2,2,2,0,2,0,2)$	105	4	yes	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^2\oplus\mathbf{2}^3\oplus\mathbf{4}$
44	$(2,2,2,2,0,2,0)$	105	4	no	$\mathrm{A}_1+\mathrm{B}_1+\mathrm{T}^4$	$\mathbf{2}^4\oplus\mathbf{4}$
45	$(4,3,1,1,0,2,1)$	105	4	no	$\mathrm{A}_1+\mathrm{T}^5$	$\mathbf{1}^2\oplus\mathbf{2}^3$
46	$(2,2,0,2,0,2,2)$	105	5	no	$2\mathrm{A}_1+\mathrm{T}^4$	$\mathbf{1}^3\oplus\mathbf{2}^2\oplus\mathbf{4}$
47	$(0,4,0,0,4,0,0)$	105	5	yes	$\mathrm{A}_1+\mathrm{A}_2+\mathrm{B}_2+\mathrm{T}^1$	$\mathbf{4}\oplus\mathbf{24}$

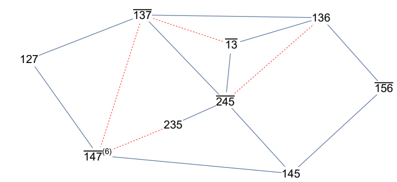
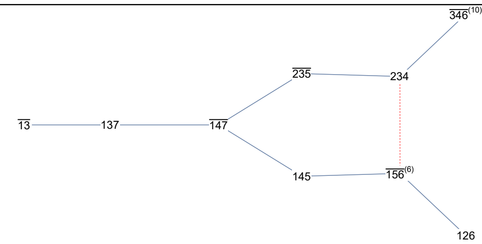
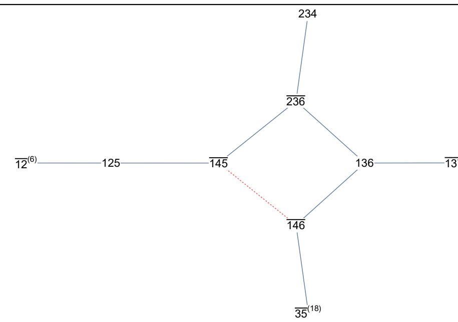
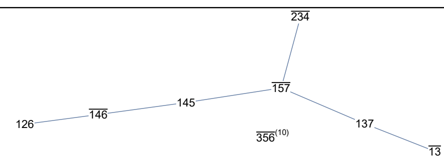
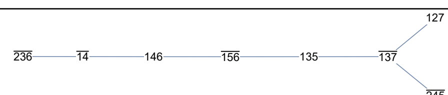
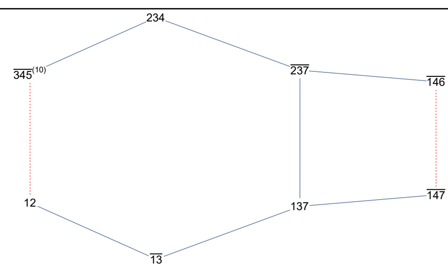
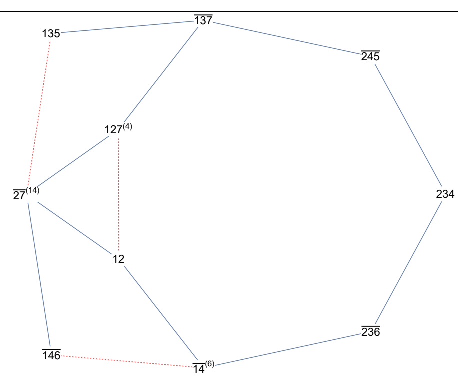
#	characteristic	dim(stratum)	d(stratum)	generic $V^{(2)}$-orbit	$\tilde{\mathfrak{g}}^{(0)}$	$V^{(2)}$
48	$(6,0,2,0,2,0,2)$	104	3	no	$3A_1+T^3$	$\mathbf{1}^2\oplus\mathbf{2}^2\oplus\mathbf{4}$
49	$(0,2,0,2,2,0,2)$	104	4	no	$3A_1+T^3$	$\mathbf{1}^3\oplus\mathbf{2}\oplus\mathbf{8}$
50	$(1,1,3,0,1,2,1)$	104	4	no	A_1+T^5	$\mathbf{1}^4\oplus\mathbf{2}^2$
51	$(3,3,5,3,2,1,2)/2$	104	5	no	T^6	$\mathbf{1}^6$
52	$(0,0,0,4,0,0,4)$	104	6	no	$A_2+A_3+T^1$	$\mathbf{4}\oplus\mathbf{18}$
53	$(4,4,0,0,2,0,2)$	103	2	no	A_2+T^3	$\mathbf{1}^2\oplus\mathbf{3}^3$
54	$(2,2,2,2,1,0,1)$	103	2	no	T^5	$\mathbf{1}^6$
55	$(8,4,4,4,4,0,4)/3$	103	3	yes	A_1+T^5	$\mathbf{1}^3\oplus\mathbf{2}^3$
56	$(2,0,2,0,2,0,2)$	103	3	yes	$3A_1+T^3$	$\mathbf{2}^3\oplus\mathbf{4}^2$
57	$(1,2,1,1,1,1,1)$	102	2	no	T^6	$\mathbf{1}^7$
58	$(4,0,0,0,0,4,0)$	102	3	yes	$A_4+B_1+T^1$	$\mathbf{10}\oplus\mathbf{20}$
59	$(1,2,1,0,3,0,1)$	102	3	no	$2A_1+T^4$	$\mathbf{1}^2\oplus\mathbf{2}^2\oplus\mathbf{4}$
60	$(2,0,2,2,0,2,0)$	102	4	no	$2A_1+B_1+T^3$	$\mathbf{2}^3\oplus\mathbf{8}$
61	$(1,1,1,1,1,1,1)$	101	2	yes	T^6	$\mathbf{1}^8$
62	$(4,0,0,4,0,0,0)$	101	2	yes	$A_2+B_3+T^1$	$\mathbf{8}\oplus\mathbf{24}$
63	$(3,1,0,1,0,2,1)$	101	2	no	$2A_1+T^4$	$\mathbf{1}\oplus\mathbf{2}^3\oplus\mathbf{4}$
64	$(0,4,0,2,0,2,0)$	101	2	no	$2A_1+B_1+T^2$	$\mathbf{2}\oplus\mathbf{4}\oplus\mathbf{8}$
65	$(0,2,0,0,2,0,2)$	100	2	no	$2A_1+A_2+T^2$	$\mathbf{1}^2\oplus\mathbf{3}\oplus\mathbf{12}$
66	$(6,2,2,2,2,6,2)/3$	100	2	no	T^6	$\mathbf{1}^6$
67	$(2,0,4,0,0,0,2)$	100	4	no	$A_1+A_3+T^2$	$\mathbf{1}\oplus\mathbf{6}\oplus\mathbf{8}$
68	$(4,4,0,4,4,0,4)/3$	99	1	yes	$2A_1+T^4$	$\mathbf{1}^3\oplus\mathbf{2}^2\oplus\mathbf{4}$
69	$(0,2,0,2,0,2,0)$	99	2	yes	$3A_1+B_1+T^2$	$\mathbf{4}^2\oplus\mathbf{8}$
70	$(20,8,4,0,4,12,8)/7$	99	2	no	T^5	$\mathbf{1}^5$
71	$(12,4,12,16,4,0,8)/7$	99	2	no	T^5	$\mathbf{1}^5$
72	$(4,4,8,4,4,4,4)/5$	99	3	yes	T^6	$\mathbf{1}^7$
73	$(0,0,0,0,4,0,0)$	99	5	yes	A_4+B_2	$\mathbf{40}$
74	$(4,0,0,0,2,0,2)$	98	1	no	A_3+T^2	$\mathbf{1}\oplus\mathbf{4}^2\oplus\mathbf{6}$
75	$(0,4,2,0,0,0,2)$	98	1	no	$A_1+A_3+T^2$	$\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{12}$
76	$(1,1,1,0,1,1,1)$	98	2	no	A_1+T^5	$\mathbf{1}^3\oplus\mathbf{2}^3$
77	$(4,2,0,0,0,2,0)$	98	2	no	$A_3+B_1+T^2$	$\mathbf{2}^2\oplus\mathbf{12}$
78	$(1,0,1,1,0,1,2)$	98	2	no	$2A_1+T^4$	$\mathbf{1}^4\oplus\mathbf{4}$
79	$(2,0,2,0,0,2,0)$	98	3	yes	$A_1+A_2+B_1+T^2$	$\mathbf{2}\oplus\mathbf{6}\oplus\mathbf{12}$
80	$(2,0,0,2,0,0,2)$	97	1	no	$2A_2+T^2$	$\mathbf{3}^3\oplus\mathbf{9}$
81	$(0,4,4,0,4,0,4)/3$	97	2	no	$3A_1+T^3$	$\mathbf{1}\oplus\mathbf{2}^2\oplus\mathbf{8}$
82	$(7,2,3,2,0,3,2)/3$	96	1	no	A_1+T^5	$\mathbf{1}^2\oplus\mathbf{2}^3$
83	$(1,0,1,0,2,1,0)$	96	2	no	$2A_1+B_1+T^3$	$\mathbf{2}\oplus\mathbf{4}\oplus\mathbf{8}$
84	$(12,6,0,2,0,0,4)/3$	95	0	no	T^3	$\mathbf{1}^4$
85	$(2,4,2,2,2,2,2)/3$	95	1	no	T^6	$\mathbf{1}^7$
86	$(1,0,1,0,5,1,1)/2$	95	1	no	$2A_1+T^4$	$\mathbf{1}^2\oplus\mathbf{2}^2\oplus\mathbf{4}$
87	$(0,1,1,1,8,1,1)/3$	95	1	no	A_1+T^5	$\mathbf{1}^2\oplus\mathbf{2}^3$
88	$(2,1,0,1,1,0,1)$	95	1	no	$2A_1+T^4$	$\mathbf{1}^5\oplus\mathbf{4}$
89	$(2,0,2,0,1,0,1)$	95	1	no	$2A_1+T^3$	$\mathbf{1}^2\oplus\mathbf{2}^2\oplus\mathbf{4}$
90	$(1,0,1,1,1,0,1)$	94	0	no	$2A_1+T^4$	$\mathbf{1}^2\oplus\mathbf{2}^2\oplus\mathbf{4}$
91	$(1,2,1,1,3,1,1)/2$	94	0	no	T^6	$\mathbf{1}^6$
92	$(1,1,0,1,0,1,1)$	94	1	yes	$2A_1+T^4$	$\mathbf{1}\oplus\mathbf{2}^5$
93	$(4,4,0,4,0,4,0)/3$	94	1	yes	$2A_1+B_1+T^3$	$\mathbf{2}\oplus\mathbf{4}^3$
94	$(2,2,0,0,2,0,0)$	94	1	no	$A_2+B_2+T^2$	$\mathbf{4}^2\oplus\mathbf{12}$
95	$(4,2,0,2,6,0,2)/3$	94	2	no	$2A_1+T^4$	$\mathbf{1}^3\oplus\mathbf{4}$
96	$(0,0,4,0,0,0,0)$	94	3	yes	A_2+B_4	$\mathbf{48}$

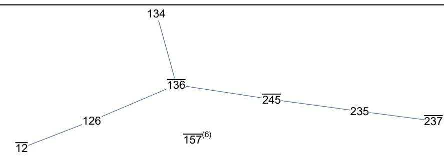
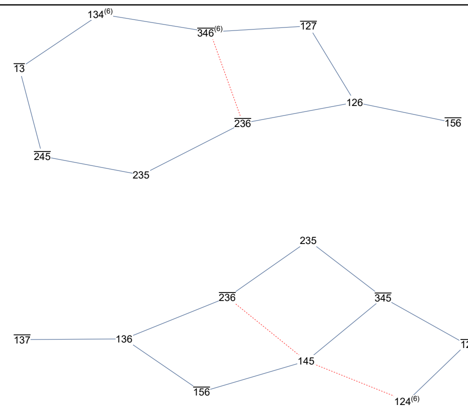
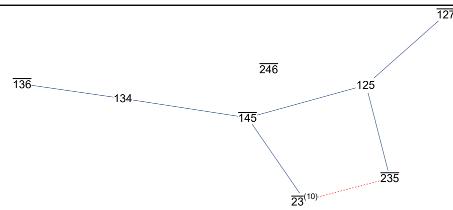
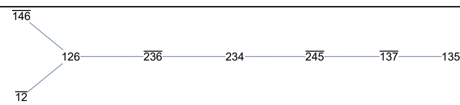
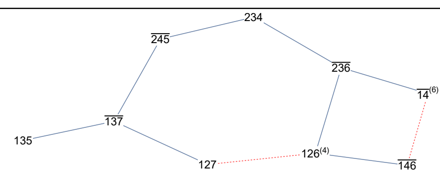
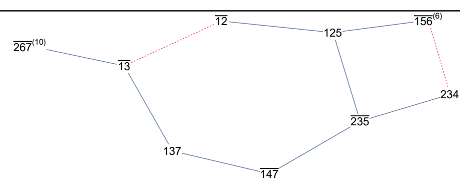
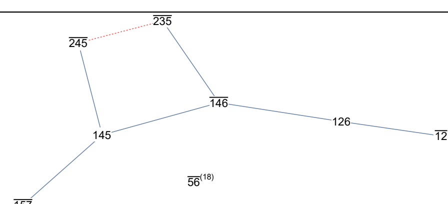
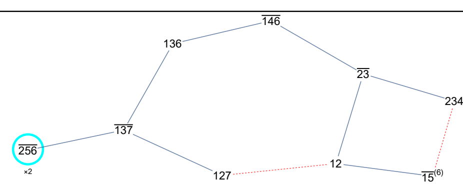
#	characteristic	dim(stratum)	d(stratum)	generic $V^{(2)}$-orbit	$\tilde{\mathfrak{g}}^{(0)}$	$V^{(2)}$
97	$(3,3,0,2,1,2,0)/2$	93	0	no	$A_1+B_1+T^4$	$\mathbf{2}^3\oplus\mathbf{4}$
98	$(0,2,0,1,1,0,1)$	93	0	no	$3A_1+T^3$	$\mathbf{1}^2\oplus\mathbf{4}^2$
99	$(0,0,0,2,0,2,0)$	93	0	no	$A_3+B_1+T^1$	$\mathbf{8}\oplus\mathbf{12}$
100	$(1,1,1,0,1,0,1)$	93	1	yes	$2A_1+T^4$	$\mathbf{1}^3\oplus\mathbf{2}^2\oplus\mathbf{4}$
101	$(4,0,4,12,0,8,4)/7$	92	0	no	A_1+T^4	$\mathbf{1}^3\oplus\mathbf{2}^2$
102	$(2,2,0,1,0,1,0)$	92	0	no	$A_1+B_1+T^3$	$\mathbf{2}^3\oplus\mathbf{4}$
103	$(0,8,0,4,8,0,4)/5$	92	0	no	$3A_1+T^3$	$\mathbf{1}\oplus\mathbf{4}^2$
104	$(0,0,2,0,0,0,2)$	92	1	yes	$A_2+A_3+T^1$	$\mathbf{1}\oplus\mathbf{3}\oplus\mathbf{18}$
105	$(6,0,0,0,0,0,2)$	92	1	no	A_5+T^1	$\mathbf{1}\oplus\mathbf{15}$
106	$(0,0,0,0,0,0,4)$	92	2	no	A_6	$\mathbf{35}$
107	$(8,4,4,4,4,4,4)/7$	91	0	yes	T^6	$\mathbf{1}^7$
108	$(8,2,0,2,2,0,2)/3$	91	0	yes	$2A_1+T^4$	$\mathbf{1}^4\oplus\mathbf{4}$
109	$(4,4,4,0,0,0,4)/3$	91	0	no	A_3+T^3	$\mathbf{1}\oplus\mathbf{4}^2\oplus\mathbf{6}$
110	$(0,2,0,0,0,2,0)$	91	1	yes	$A_1+A_3+B_1+T^1$	$\mathbf{8}\oplus\mathbf{16}$
111	$(2,1,0,1,0,1,0)$	90	0	yes	$2A_1+B_1+T^3$	$\mathbf{2}^2\oplus\mathbf{4}^2$
112	$(4,2,24,2,2,2,2)/9$	90	0	no	T^6	$\mathbf{1}^6$
113	$(0,1,2,0,0,1,0)$	90	2	yes	$A_1+A_2+B_1+T^2$	$\mathbf{6}\oplus\mathbf{12}$
114	$(4,2,4,2,0,2,0)/3$	89	0	no	$A_1+B_1+T^4$	$\mathbf{2}^3\oplus\mathbf{4}$
115	$(1,0,1,0,1,0,1)$	89	1	yes	$3A_1+T^3$	$\mathbf{1}^3\oplus\mathbf{2}\oplus\mathbf{8}$
116	$(24,20,28,12,8,4,8)/23$	88	0	no	T^6	$\mathbf{1}^5$
117	$(1,0,1,1,0,1,0)$	88	1	yes	$2A_1+B_1+T^3$	$\mathbf{4}^3$
118	$(2,0,6,0,2,0,2)/3$	88	1	no	$3A_1+T^3$	$\mathbf{1}\oplus\mathbf{2}\oplus\mathbf{8}$
119	$(2,2,1,0,0,0,1)$	88	1	no	A_3+T^3	$\mathbf{1}^3\oplus\mathbf{6}$
120	$(32,16,28,16,8,0,8)/23$	87	0	no	T^5	$\mathbf{1}^5$
121	$(0,8,12,0,4,0,4)/7$	87	0	no	$2A_1+T^3$	$\mathbf{1}\oplus\mathbf{2}^2\oplus\mathbf{4}$
122	$(2,0,0,0,2,0,0)$	87	2	yes	$A_3+B_2+T^1$	$\mathbf{4}\oplus\mathbf{24}$
123	$(1,1,1,1,1,1,1)/2$	86	0	yes	T^6	$\mathbf{1}^7$
124	$(0,2,0,2,0,0,0)$	86	1	yes	$A_1+B_3+T^1$	$\mathbf{8}\oplus\mathbf{16}$
125	$(6,6,2,6,6,6,6)/11$	85	0	yes	T^6	$\mathbf{1}^6$
126	$(2,3,0,5,1,0,1)/3$	85	1	no	T^4	$\mathbf{1}^4$
127	$(12,4,4,4,8,4,4)/11$	84	0	no	T^6	$\mathbf{1}^6$
128	$(12,4,0,4,0,8,0)/7$	84	1	yes	$A_1+B_1+T^3$	$\mathbf{2}\oplus\mathbf{4}^2$
129	$(0,1,0,1,0,1,0)$	83	0	yes	$3A_1+B_1+T^2$	$\mathbf{2}\oplus\mathbf{4}\oplus\mathbf{8}$
130	$(2,2,2,0,2,0,2)/3$	82	0	yes	$2A_1+T^4$	$\mathbf{1}^2\oplus\mathbf{2}^2\oplus\mathbf{4}$
131	$(8,0,4,0,4,4,4)/7$	82	0	no	$2A_1+T^4$	$\mathbf{1}^3\oplus\mathbf{4}$
132	$(2,0,1,0,0,0,1)$	82	1	yes	$A_1+A_3+T^2$	$\mathbf{1}^3\oplus\mathbf{12}$
133	$(0,0,4,0,0,4,0)/3$	82	1	yes	$2A_2+B_1+T^1$	$\mathbf{2}\oplus\mathbf{18}$
134	$(1,0,2,1,0,1,1)/2$	81	0	no	$2A_1+T^4$	$\mathbf{1}\oplus\mathbf{2}^2\oplus\mathbf{4}$
135	$(4,4,0,4,0,4,0)/5$	80	0	yes	$2A_1+B_1+T^3$	$\mathbf{2}^2\oplus\mathbf{4}^2$
136	$(6,0,2,0,0,4,0)/3$	80	0	no	B_1+T^2	$\mathbf{2}^3$
137	$(1,0,0,1,0,0,1)$	79	0	yes	$2A_2+T^2$	$\mathbf{3}^2\oplus\mathbf{9}$
138	$(0,1,1,0,0,0,1)$	79	0	yes	$A_1+A_3+T^2$	$\mathbf{1}\oplus\mathbf{6}\oplus\mathbf{8}$
139	$(1,0,1,0,0,1,0)$	78	0	yes	$A_1+A_2+B_1+T^2$	$\mathbf{2}^2\oplus\mathbf{12}$
140	$(0,2,6,0,2,0,4)/5$	78	0	no	$2A_1+T^3$	$\mathbf{1}^2\oplus\mathbf{4}$
141	$(2,4,2,0,4,2,0)/5$	77	0	yes	$A_1+B_1+T^4$	$\mathbf{2}^3\oplus\mathbf{4}$
142	$(0,0,0,0,0,2,0)$	77	0	yes	A_5+B_1	$\mathbf{30}$
143	$(4,0,0,0,0,0,0)$	77	1	yes	B_6	$\mathbf{64}$
144	$(8,0,0,0,0,0,4)/3$	77	1	no	A_5+T^1	$\mathbf{1}\oplus\mathbf{20}$
145	$(2,1,0,1,1,0,1)/2$	76	0	yes	$2A_1+T^4$	$\mathbf{1}^4\oplus\mathbf{4}$

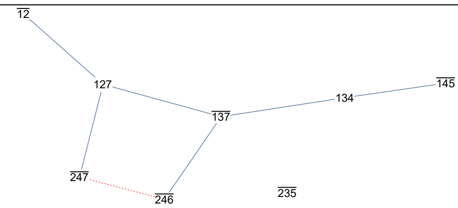
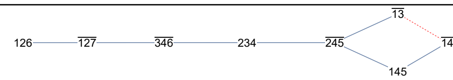
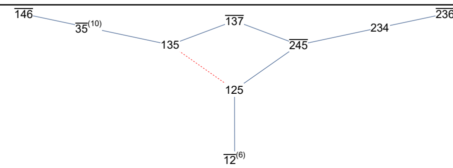
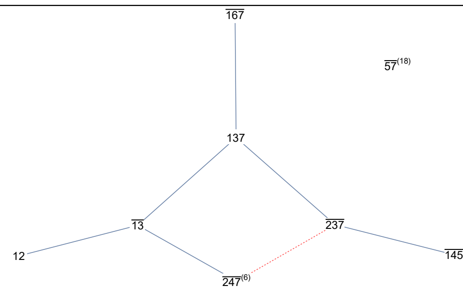
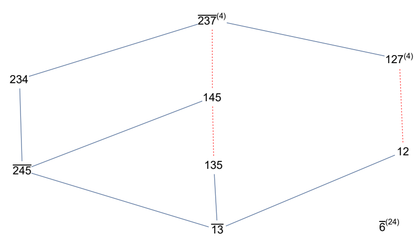
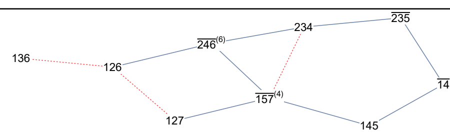
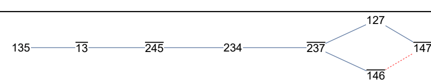
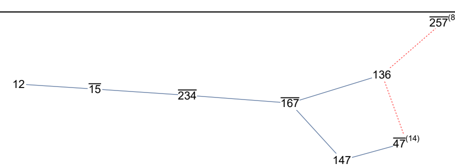
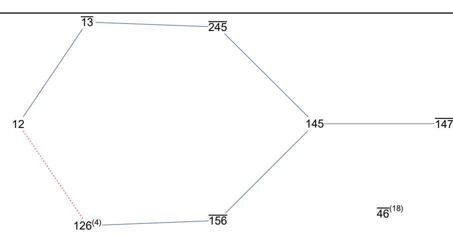
#	characteristic	dim(stratum)	d(stratum)	generic $V^{(2)}$-orbit	$\tilde{\mathfrak{g}}^{(0)}$	$V^{(2)}$
146	$(0,4,0,0,0,4,4)/5$	76	0	yes	$A_1+A_3+T^2$	$\mathbf{6}\oplus\mathbf{8}$
147	$(4,12,4,8,0,4,4)/13$	75	0	yes	T^5	$\mathbf{1}^5$
148	$(32,2,0,2,2,0,2)/11$	75	0	yes	$2A_1+T^4$	$\mathbf{1}^3\oplus\mathbf{4}$
149	$(8,8,0,4,8,0,4)/11$	75	0	yes	$2A_1+T^4$	$\mathbf{1}^3\oplus\mathbf{4}$
150	$(0,12,0,8,4,0,8)/13$	75	0	yes	$3A_1+T^3$	$\mathbf{1}\oplus\mathbf{4}^2$
151	$(0,0,0,2,0,0,0)$	74	1	yes	A_3+B_3	$\mathbf{32}$
152	$(6,2,2,8,2,4,2)/11$	72	0	yes	T^6	$\mathbf{1}^5$
153	$(16,4,2,0,0,0,2)/7$	71	0	yes	A_3+T^3	$\mathbf{1}^2\oplus\mathbf{6}$
154	$(0,0,0,2,0,0,4)/3$	70	0	yes	A_3+T^1	$\mathbf{1}\oplus\mathbf{6}$
155	$(0,2,4,0,6,0,2)/7$	69	0	yes	$2A_1+T^3$	$\mathbf{1}^2\oplus\mathbf{4}$
156	$(2,0,0,1,0,0,0)$	69	0	yes	A_2+B_3	$\mathbf{24}$
157	$(0,1,0,0,0,1,0)$	67	0	yes	$A_1+A_3+B_1+T^1$	$\mathbf{4}\oplus\mathbf{12}$
158	$(4,2,0,0,0,2,0)/3$	66	0	yes	$A_3+B_1+T^2$	$\mathbf{2}\oplus\mathbf{12}$
159	$(2,0,0,2,0,0,2)/3$	64	0	yes	$2A_2+T^2$	$\mathbf{1}^2\oplus\mathbf{9}$
160	$(20,12,0,4,0,0,8)/17$	63	0	yes	T^3	$\mathbf{1}^3$
161	$(3,0,0,1,0,0,1)/2$	63	0	yes	$2A_2+T^2$	$\mathbf{1}\oplus\mathbf{9}$
162	$(1,0,0,0,1,0,0)$	59	0	yes	A_3+B_2	$\mathbf{16}$
163	$(1,0,0,0,0,0,1)$	57	0	yes	A_5+T^1	$\mathbf{1}\oplus\mathbf{15}$
164	$(8,0,0,0,0,0,4)/5$	56	0	yes	A_5	$\mathbf{15}$
165	$(0,2,0,0,0,0,0)$	55	0	yes	B_5	$\mathbf{32}$
166	$(0,4,2,0,0,0,2)/5$	54	0	yes	A_3+T^2	$\mathbf{1}\oplus\mathbf{6}$
167	$(0,0,4,0,0,0,0)/3$	46	0	yes	B_4	$\mathbf{16}$
168	$(0,0,0,1,0,0,0)$	42	0	yes	B_3	$\mathbf{8}$
169	$(0,0,0,0,0,0,4)/7$	29	0	yes	(0)	$\mathbf{1}$

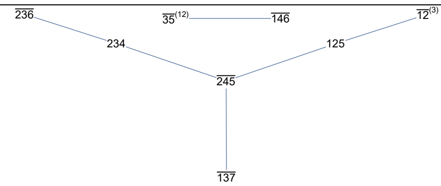
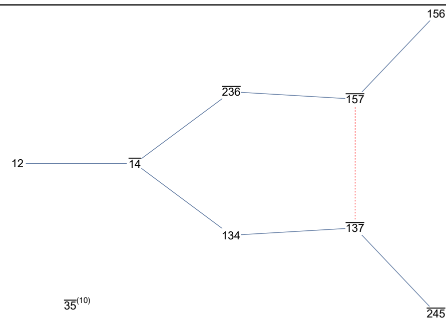
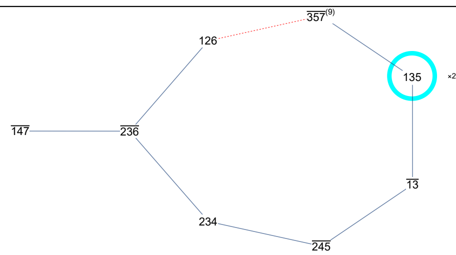
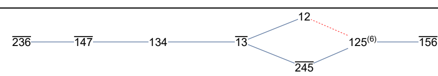
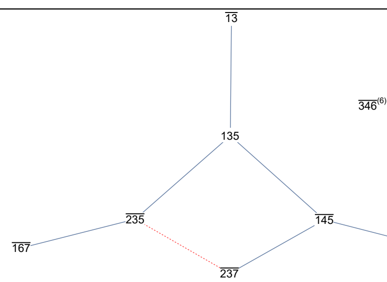
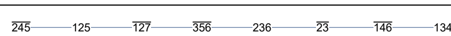
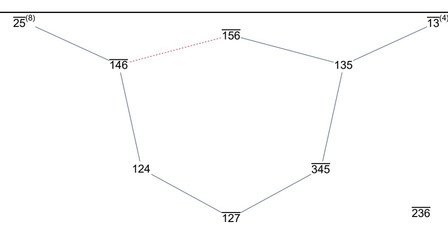
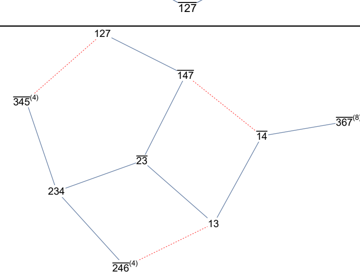
Representatives of generic orbits

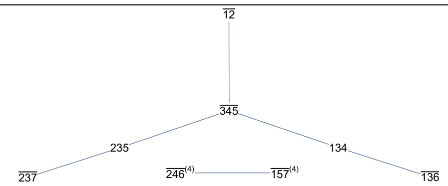
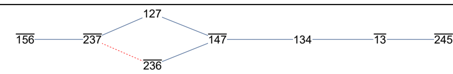
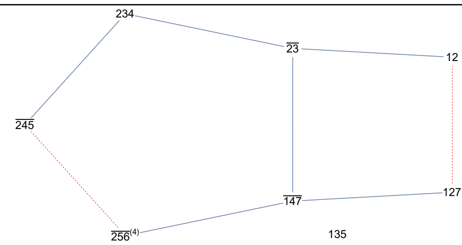
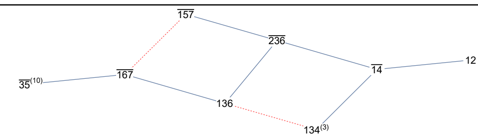
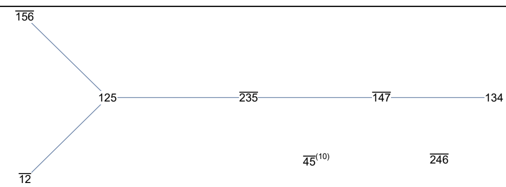
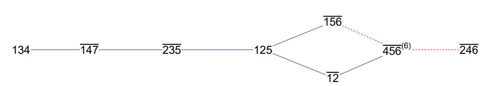
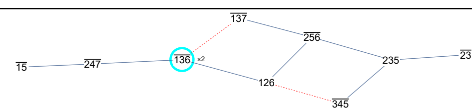
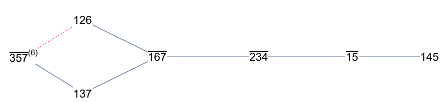
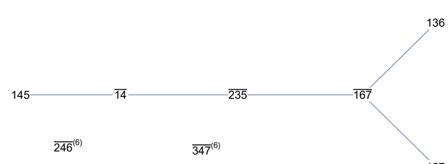
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
1	84444444	E_8		(0)
2	44040404	$E_7 + A_1$		1
3	44440444	$E_8(a_1)$		1
4	44040404	$E_7 + A_1$		1
5	44440444	$E_8(a_1)$		1
6	44044044	$E_8(a_2)$		1
7	44040400	D_8		1
8	44040404	$E_7 + A_1$		1
9	44044044	$E_8(a_2)$		1

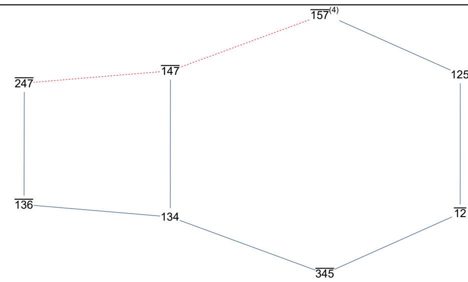
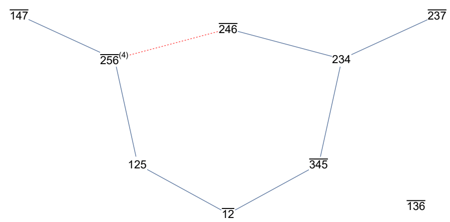
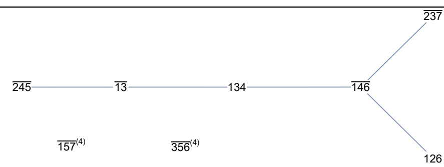
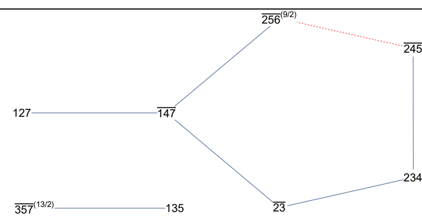
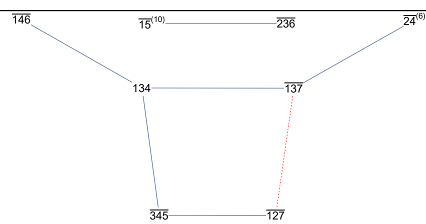
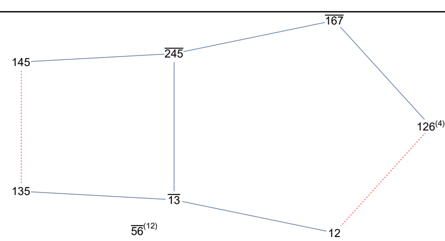
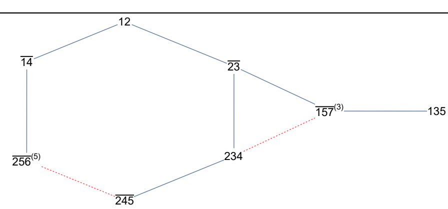
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
10	44040400	D_8		1
11	44040404	$E_7 + A_1$		1
12	24222224	$E_7 + A_1$		2
13	44040404	$E_7 + A_1$		1
14	44040400	D_8		1
15	44040400	D_8		1
16	22222022	$D_8(a_1)$		1

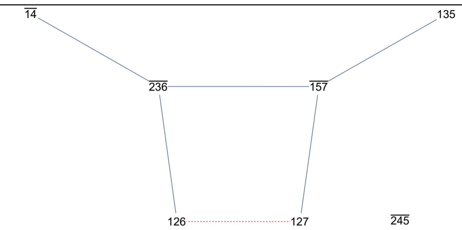
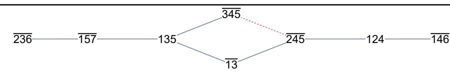
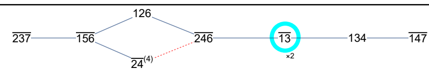
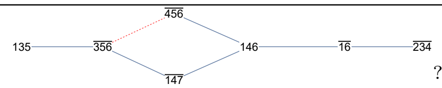
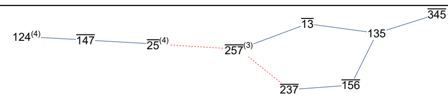
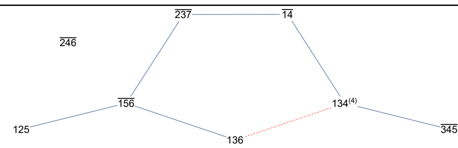
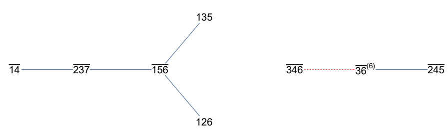
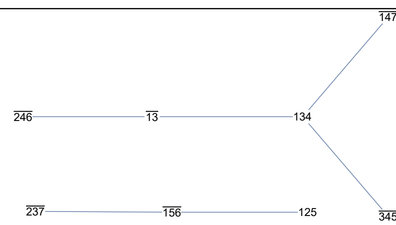
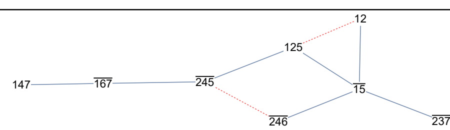
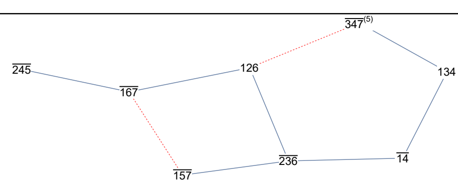
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
17	24222224	$E_7 + A_1$		2
18	22222222	D_8		2
19	04040044	$E_7(a_1) + A_1$		3
20	22222222	D_8		2
21	22222022	$D_8(a_1)$		2
22	22222024	$E_7(a_1) + A_1$		2
23	04040044	$E_7(a_1) + A_1$		3
24	22222022	$D_8(a_1)$		2

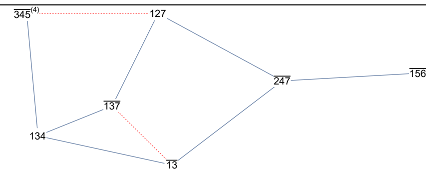
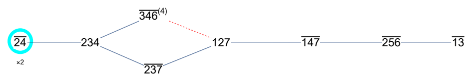
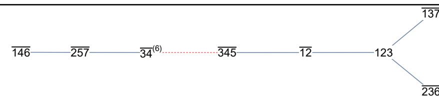
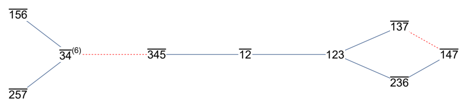
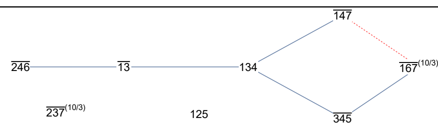
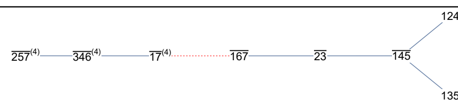
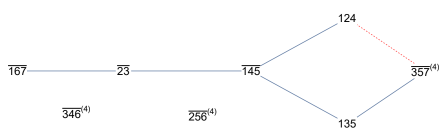
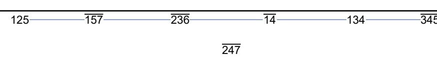
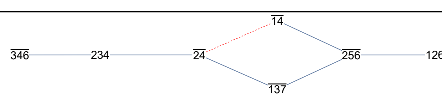
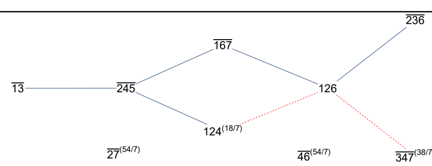
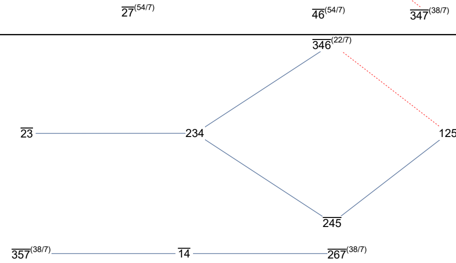
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
25	04040044	$E_7(a_1)+A_1$		3
26	40040400	$D_8(a_1)$		3
27	04004004	E_6+A_2		3
28	22202024	$E_7(a_2)+A_1$		2
29	40040004	D_6+2A_1		2
30	40040004	$D_7(a_1)$		2
31	22222022	$D_8(a_1)$		3
32	40040004	$D_7(a_1)$		3
33	22202024	$E_7(a_2)+A_1$		3

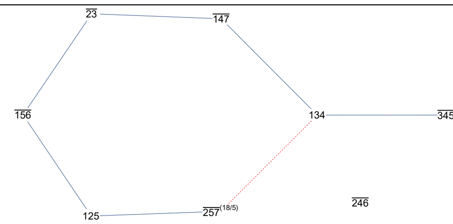
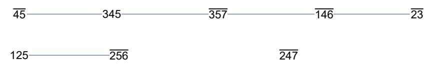
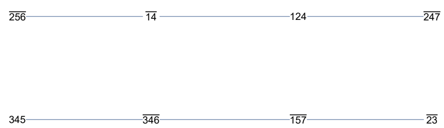
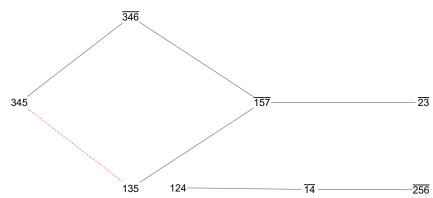
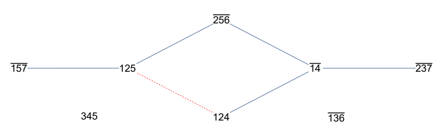
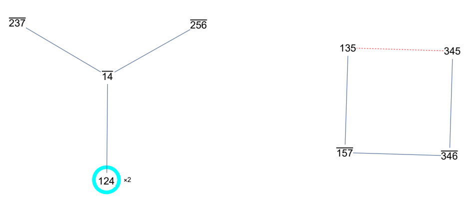
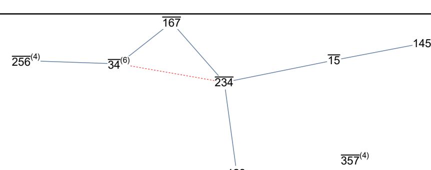
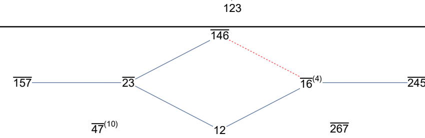
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
34	04004004	$E_6 + A_2$		4
35	40040004	$D_6 + 2A_1$		3
36	31131211	D_7		3
37	22202022	$D_8(a_2)$		3
38	22202024	$E_7(a_2) + A_1$		3
39	00400400	A_8		4
40	02022022	$D_6 + 2A_1$		4
41	13111110	$E_6(a_1) + A_1$		3, 1

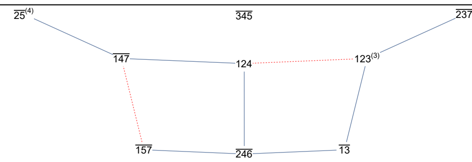
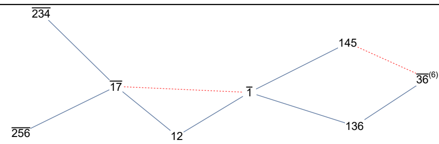
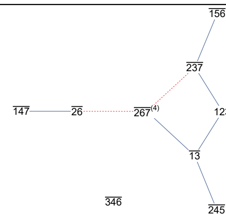
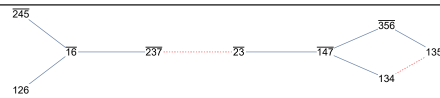
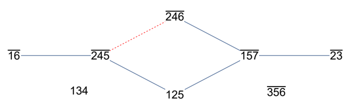
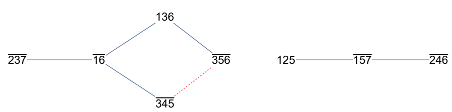
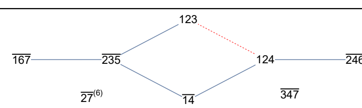
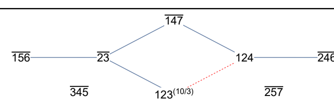
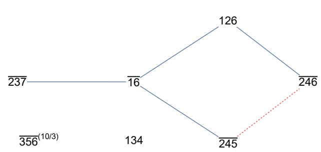
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
42	04004004	$E_6 + A_2$		4
43	22202022	$D_8(a_2)$		3, 1
44	13111110	$E_6(a_1) + A_1$		3, 1
45	13111110	$E_6(a_1) + A_1$		3, 1
46	02022022	$D_6 + 2A_1$		5
		$D_7(a_1)$		
47	04004000	$E_6(a_1) + A_2$		3, 1, 1
48	40040004	$D_7(a_1)$		$\mathfrak{t}^1 \ltimes ((-1)(0)(1))$
		$D_6 + 2A_1$		

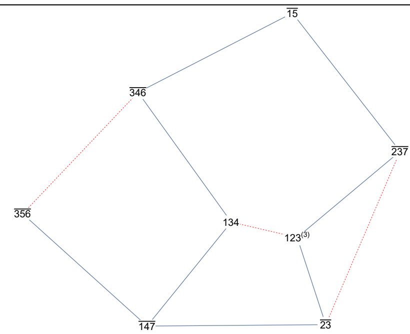
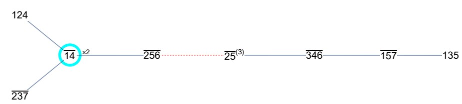
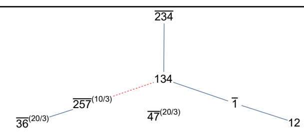
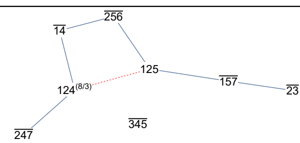
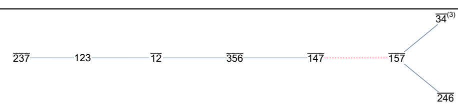
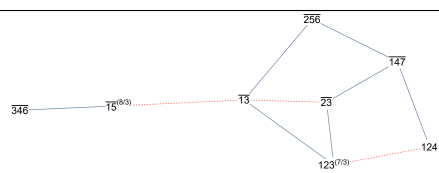
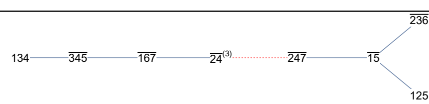
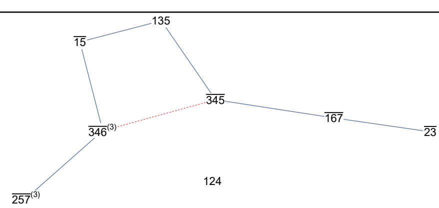
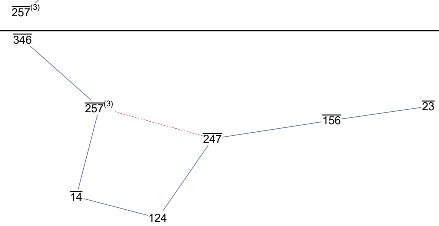
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
49	02022022	$D_7(a_1)$		5
		D_6+2A_1		
50	20220202	D_6+2A_1		5
51	00400004	D_5+A_2		4, 2
52	02002022	D_5+A_2		7
53	13111110	$E_6(a_1)+A_1$		3, 1
54	40040000	A_6		4

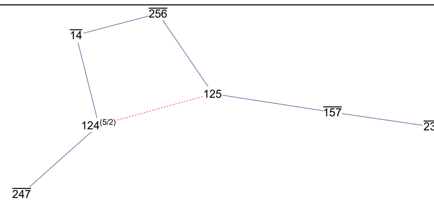
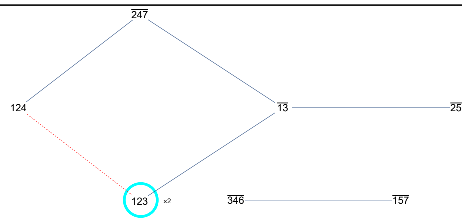
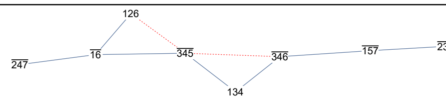
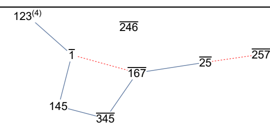
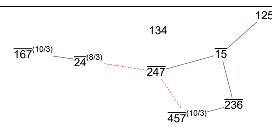
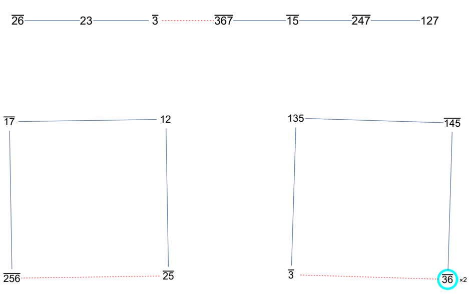
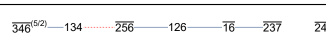
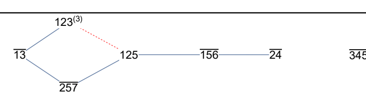
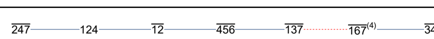
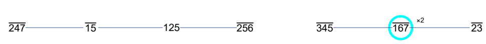
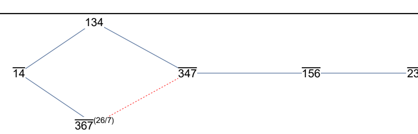
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
55	13111110	$E_6(a_1)+A_1$		4, 1
56	20202022	$D_8(a_3)$		5
57	12111111	A_7'		5
58	31010211	$D_7(a_2)$		6
59	11111111	D_5+A_3		5, 1
60	00400004	$D_6(a_1)+2A_1$		5, 2
		D_5+A_2		
61	11111111	D_5+A_3		6
62	40040000	A_6		$\mathfrak{t}^1 \ltimes \big((-1)^2(0)(1)^2\big)$
63	31010211	$D_7(a_2)$		6

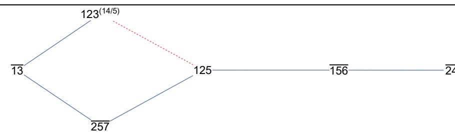
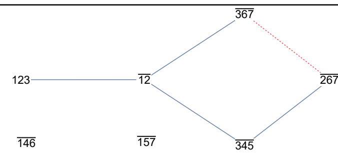
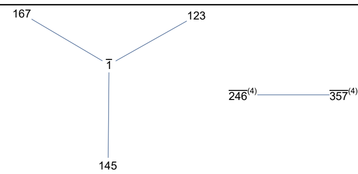
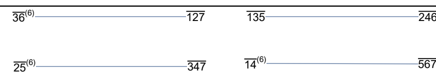
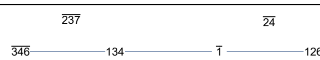
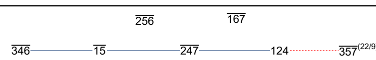
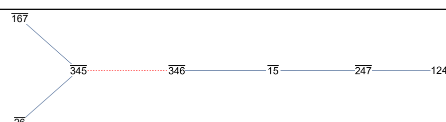
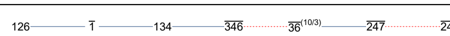
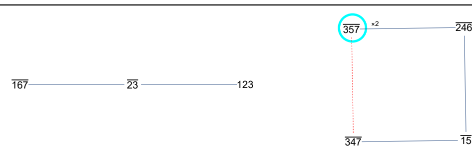
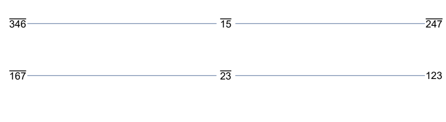
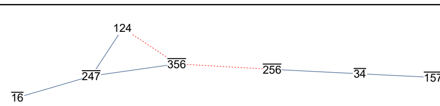
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
64	04020200	$E_6(a_1)$		4, 1, 1
		A_7''		
65	02002022	$D_5 + A_2$		7
		$D_6(a_1) + 2A_1$		
66	20200202	$D_6(a_1) + 2A_1$		4, 2, 1
67	00400004	$D_5 + A_2$		4, 3, 2
		$D_6(a_1) + 2A_1$		
68	11111110	$A_6 + A_1$		6, 1
69	02020200	A_6		7, 1
70	02002002	$D_6(a_2) + 2A_1$		7, 1
71	20200200	$D_5(a_1) + A_3$		3, 2, 2, 1

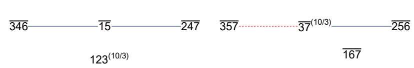
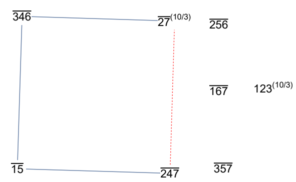
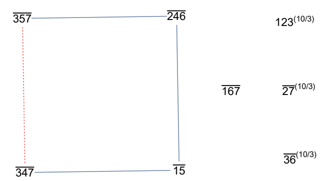
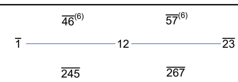
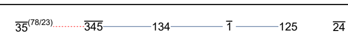
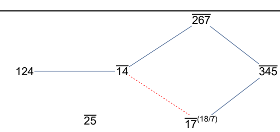
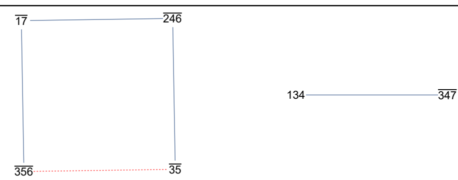
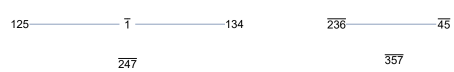
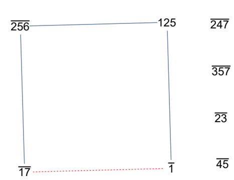
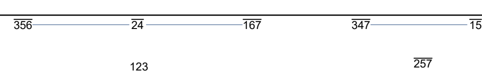
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{\mathrm{B}_7}$
72	11111110	$\mathrm{A}_6+\mathrm{A}_1$		$6,1,2$
		$\mathrm{A}_5+\mathrm{A}_2+\mathrm{A}_1$		
73	00004000	$2\mathrm{A}_4$		$5,6$
		$\mathrm{D}_5(a_1)+\mathrm{A}_3$		
		$\mathrm{D}_6(a_2)+2\mathrm{A}_1$		
		$\mathrm{D}_4(a_1)+\mathrm{D}_4$		
74	21011013	$\mathrm{D}_5+2\mathrm{A}_1$		$4,3,1$
75	02002020	$\mathrm{D}_6(a_2)+2\mathrm{A}_1$		$4,3,1$

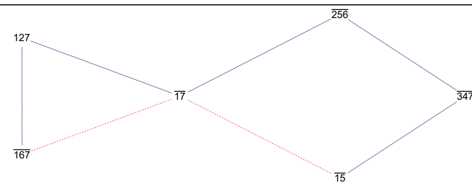
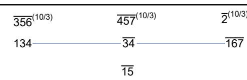
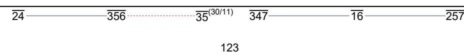
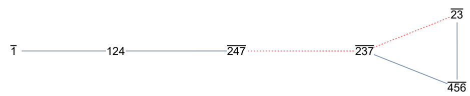
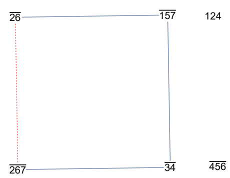
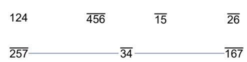
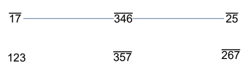
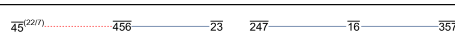
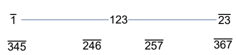
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
76	02002020	$D_6(a_2)+2A_1$		$5,3,1$
77	40000004	$D_5(a_1)+2A_1$		$\mathfrak{sl}_2\ltimes\left(\mathbf{1}^2\oplus\mathbf{2}^2\right)$
78	02002020	$A_5+A_2+A_1$		$4,3,1,1$
		$D_4+D_4(a_1)$		
79	20200200	$D_6(a_2)+2A_1$		$4,2,2,2$
		$D_5(a_1)+A_3$		
80	02002020	$D_6(a_2)+2A_1$		9
81	02002020	$D_6(a_2)+2A_1$		$5,4,1$
82	40000004	$D_5(a_1)+2A_1$		$4,2,2,2$

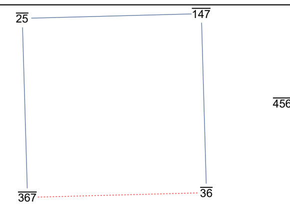
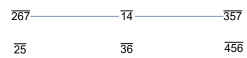
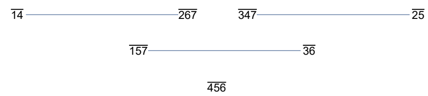
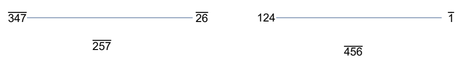
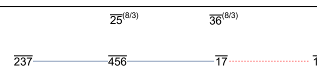
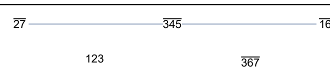
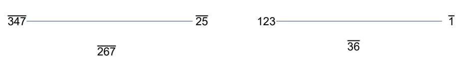
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
83	10102100	$D_6(a_2)$		5, 6
		$D_4 + A_3$		
84	21010100	$A_4 + 2A_1$		5, 4, 1
85	11110101	$A_5 + 2A_1$		4, 3, 3, 1
86	01011110	$A_5 + 2A_1$		5, 6
87	01011110	$A_5 + 2A_1$		5, 6
88	21011011	$D_4 + A_3$		4, 3, 2, 1, 1
89	11110101	$A_5 + 2A_1$		4, 3, 2, 2
90	10111011	$(A_5 + A_1)'$		7, 4

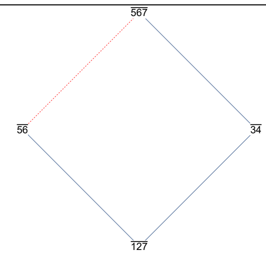
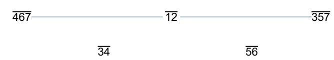
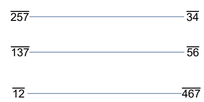
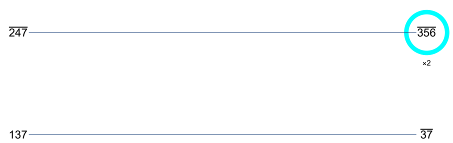
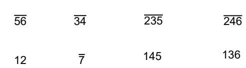
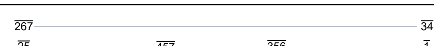
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
91	10111011	$(A_5+A_1)'$		$6, 5$
92	11010111	$D_5(a_1)+A_2$		$\mathfrak{t}^1 \ltimes ((-3)(-1)(0)(1)(2)(3), (-3)(-2)(-1)(0)(1))$
93	11101011	A_4+A_3		$4, 2, 2, 2, 2$
94	21010100	A_4+2A_1		$4, 2, 4, 2$
95	20002002	$D_5(a_1)+2A_1$		$5, 5, 3$
96	00400000	A_4+A_2, $2D_4(a_1)$		$11, 3$
97	21010100	A_4+2A_1		$4, 2, 4, 2$
98	02020002	$(A_5+A_1)''$		$3, 2, 3, 2, 2$
99	00020200	$(A_5+A_1)''$		$7, 5$
100	11101011	A_4+A_3		$\mathfrak{t}^1 \ltimes ((-5)(-3)(-1)(0)(1)(3), (-2)(-1)(0)(2)(3), (1))$
101	01020101	A_5		$\mathfrak{t}^1 \ltimes ((-2)(-1)^3(1)^3(2), (-1)(0)^2(1))$
102	21010100	A_4+2A_1		$4, 2, 3, 2, 2$

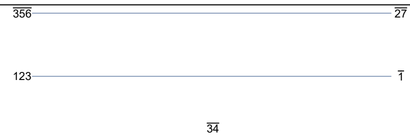
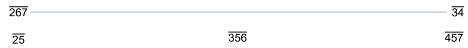
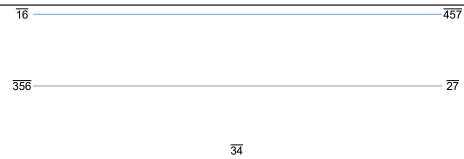
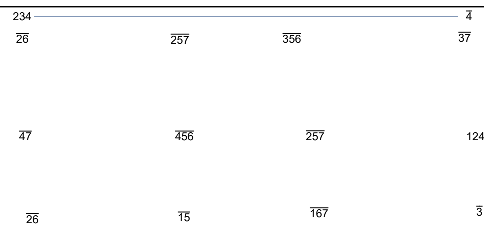
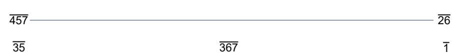
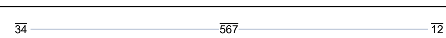
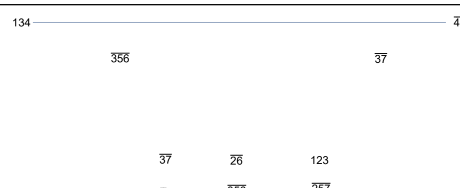
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
103	01020101	A_5		$\mathfrak{t}^1 \ltimes ((-1)^4(2),\ (-2)(1)^3,\ (0)^2,\ (-1))$
104	00200022	$D_5(a_1)+2A_1$		$\mathfrak{gl}_2 \ltimes (\mathbf{2}^{(-3)} \oplus \mathbf{1}^{(-2)} \oplus \mathbf{2}^{(-1)} \oplus \mathbf{2}^{(1)} \oplus \mathbf{1}^{(2)} \oplus \mathbf{2}^{(3)})$
105	40000004	D_4+A_2		$\mathfrak{sl}_3(\mathbf{3}^2)$
106	00000040	$4A_2$		$\mathfrak{t}^2 \ltimes \left(\binom{-2}{-1}\ \binom{-1}{-2}\ \binom{-1}{-1}\ \binom{-1}{0}\ \binom{0}{-1}\ \binom{-1}{1}\ \binom{0}{0}\ \binom{1}{-1}\ \binom{1}{0}\ \binom{0}{1}\ \binom{1}{1}\ \binom{2}{1}\ \binom{1}{2}\right)$
107	11010110	$A_4+A_2+A_1$		$5, 4, 3, 2$
108	00100100	A_4+2A_1		$\mathfrak{sl}_2 \ltimes (\mathbf{2}^3 \oplus \mathbf{1},\ \mathbf{2} \oplus \mathbf{1}^2)$
109	21010100	A_4+2A_1		$\mathfrak{t}^1 \ltimes ((-5)(-3)(-1)(0)(1)(3),\ (-5)(-1)(0)(2)(3),\ (-2)(1))$
110	02000200	A_4+A_2		$4, 4, 6, 1$
111	21010100	A_4+2A_1		$\mathfrak{t}^1 \ltimes ((-7)(-5)(-3)(-1)(1)(3)(5),\ (-2)(0)(2),\ (-3)(-1)(3),\ (0))$
112	01200100	A_4+2A_1		$10, 3, 2$
113	01200100	A_4+2A_1		$\mathfrak{t}^1 \ltimes ((-2)(-1)(1)^2(2),\ (-1)(0)^2(1),\ (-1)^2(0),\ (-1)(1),\ (0)(1))$
114	11110001	A_4+A_1		$6, 4, 4, 2$
115	10101011	$D_4(a_1)+A_3$ $2A_3+2A_1$	 	$\mathfrak{t}^1 \ltimes ((-1)(1),\ (0),\ (-1)(1),\ (-2)(0)(2),\ (-1)^2(1)^2,\ (0)^2,\ (-1)(1))$
116	11110001	A_4+A_1		$\mathfrak{t}^1 \ltimes ((-2)(-1)(0)(1)^2(2),\ (-1)^3(0)(1),\ (-1)(0)(1),\ (0)(1))$
117	10110100	$(2A_3)'$		$\mathfrak{t}^2 \ltimes \left(\binom{-2}{-1}\binom{-1}{-2}\binom{-1}{-1}\binom{-1}{0}\binom{0}{-1}\binom{1}{-1}\binom{0}{1}^2\binom{1}{1}^2,\ \binom{-1}{1}\binom{0}{0}\binom{-1}{2}\binom{2}{-1},\ \binom{1}{0}^2\right)$

#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
118	00200020	$A_3{+}A_2{+}2A_1$ $D_4(a_1)+4A_1$	  	$8, 7, 3$
119	20100011	A_3+4A_1		$(\mathfrak{sl}_2\oplus\mathfrak{sl}_2)\ltimes\big(\mathbf{2}_1\oplus\mathbf{2}_2\oplus\mathbf{1}^2,\ \mathbf{2}_1\oplus\mathbf{2}_2\oplus\mathbf{1},\ \mathbf{1}\ \big)$
120	11110001	A_4+A_1		$\mathfrak{sl}_2\ltimes\big(\mathbf{2}\oplus\mathbf{1}^4,\ \mathbf{2}\oplus\mathbf{1}^2,\ \mathbf{1}^3,\ \mathbf{1}^2\big)$
121	11110001	A_4+A_1		$\mathfrak{t}^1\ltimes\big((-2)(-1)^2(0)^2(1)^3(2),\ (-1)^2(1)^2,\ (-1)(0)(1),\ (0)\ \big)$
122	20002000	$D_4(a_1)+A_2$ $A_3{+}A_2{+}2A_1$ $D_4(a_1)+4A_1$	  	$\mathfrak{t}^1\ltimes\big((-1)^4(1)^4,\ (-2)(0)^5(2),\ (-1)^2(1)^2\big)$
123	00200020	$A_3{+}A_2{+}2A_1$		$3,2,2,2,2,1,1,1,2,1,1,1$

#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
124	02020000	A_4		$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \ltimes \big(\mathbf{2}_1 \oplus \mathbf{2}_2^2 \oplus \mathbf{1}^2,\ \mathbf{2}_1 \oplus \mathbf{1},\ \mathbf{2}_1 \oplus \mathbf{1}\ \big)$
125	00100030	$A_3+A_2+A_1$		$\mathfrak{t}^1 \ltimes ((-5)(-1)(6),\ (-6)(5),\ (-7)(0)(4),\ (-1)(3),\ (2),\ (1),\ (0),\ (-1),\ (-2)(5),\ (-3)(4),\ (3))$
126	01010100	A_3+4A_1		$\mathfrak{sl}_2 \ltimes \big(\mathbf{3} \oplus \mathbf{2}^3 \oplus \mathbf{1}^4,\ \mathbf{2}^2 \oplus \mathbf{1}\big)$
127	10101010	$A_3+A_2+A_1$		$4,4,5,3,3,2$
128	20100011	A_3+A_2		$\mathfrak{gl}_2 \ltimes \Big(\mathbf{1}^{(-2)^{\oplus 3}} \oplus \mathbf{2}^{(-1)} \oplus \mathbf{1}^{(2)^{\oplus 2}} \oplus \mathbf{2}^{(3)},\ \mathbf{1}^{(-4)} \oplus \mathbf{1}^{(0)^{\oplus 3}} \oplus \mathbf{2}^{(1)}, \mathbf{1}^{(-2)^{\oplus 2}} \oplus \mathbf{1}^{(2)}\Big)$
129	01010100	$D_4(a_1)+2A_1$ A_3+4A_1	 	$\mathfrak{t}^1 \ltimes \big((-2)(-1)(0)^2(1)(2),\ (-1)^2(0)(1)^2,\ (-2)(-1)(1)(2),\ (-1)(0)(1),\ (-1)(1),\ (0)\ \big)$
130	11001001	A_3+3A_1		$\mathfrak{t}^1 \ltimes ((-4)(-2)(0)(1)(2),\ (-3)(-1)(0)(3),\ (-2)(-1)(1)^2(2),\ (-3)(-1)(0)(3),\ (-1)(1)(2),\ (0))$
131	10001020	A_3+A_2		$\mathfrak{sl}_2 \ltimes \big(\mathbf{2}^2 \oplus \mathbf{1},\ \mathbf{2}^2 \oplus \mathbf{1}^2, \mathbf{3} \oplus \mathbf{1}^2,\ \mathbf{2}^2\big)$
132	20100011	A_3+4A_1		$\mathfrak{sl}_2 \oplus \mathfrak{t}^1 \oplus \mathfrak{sl}_2 \ltimes \Big(\mathbf{2}_1^{(-3)} \oplus \mathbf{1}^{(-2)} \oplus \mathbf{2}_2^{(-1)} \oplus \mathbf{2}_1^{(1)} \oplus \mathbf{1}^{(2)} \oplus \mathbf{2}_2^{(3)},\ \mathbf{2}_1^{(-1)} \oplus \mathbf{1}^{(0)} \oplus \mathbf{2}_2^{(1)},\ \mathbf{1}^{(-2)} \oplus \mathbf{1}^{(2)}\Big)$

#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
133	00100110	$D_4(a_1)+A_1$		$\mathfrak{t}^2 \ltimes \Big(\binom{-2}{-2}\binom{-1}{0}\binom{0}{-1}\binom{0}{2}\binom{1}{1}\binom{2}{0}, \binom{-1}{-1}^2\binom{0}{1}^2\binom{1}{0}^2, \binom{-2}{-1}\binom{-1}{-2}\binom{-1}{1}\binom{0}{0}\binom{1}{-1}\binom{1}{2}\binom{2}{1}, \binom{0}{-1}\binom{1}{1}\binom{-1}{0}\Big)$
		A_3+3A_1		
		$3A_2+A_1$		
134	10010011	$(A_3+2A_1)'$		$\mathfrak{t}^1 \ltimes ((-4)(-2)(-1)(0)(2), (-2)(-1)(1)(2), (-3)(0)(1)^2(2), (-3)(0)(1)(3), (-1)(0)(3), (-1)(2))$
135	10100100	$2A_2+2A_1$		$\mathfrak{t}^1 \ltimes ((-7)(-2)(1)(3)(6), (-9)(-4)(1)(4), (-6)(-1)^2(4), (-3)^2(2)(7), (-5)(0)^2(5), (-2)(3), (1))$
136	10010011	$(A_3+2A_1)'$		$\mathfrak{sl}_2 \ltimes \big(\mathbf{2}^4,\ \mathbf{3}\oplus\mathbf{1}^4,\ \mathbf{2}^3,\ \mathbf{1}\big)$
137	10010011	$(A_3+2A_1)'$		$\mathfrak{gl}_2 \ltimes \big(\mathbf{2}^{(-3)}\oplus\mathbf{2}^{(-1)}\oplus\mathbf{2}^{(0)}\oplus\mathbf{2}^{(1)}, \mathbf{1}^{(-3)}\oplus\mathbf{1}^{(0)}\oplus\mathbf{3}^{(1)}\oplus\mathbf{1}^{(1)}\oplus\mathbf{1}^{(2)}, \mathbf{2}^{(-2)}\oplus\mathbf{2}^{(1)}\oplus\mathbf{2}^{(2)},\ \mathbf{1}^{(-1)}\big)$
138	02000002	$(A_3+2A_1)''$		$\mathfrak{sl}_2\oplus\mathfrak{sl}_2 \ltimes \big(\mathbf{2}_1\otimes\mathbf{2}_2\oplus\mathbf{3}_1\oplus\mathbf{2}_1\oplus\mathbf{1}, \mathbf{2}_1\otimes\mathbf{2}_2\oplus\mathbf{3}_1\oplus\mathbf{2}_1,\ \mathbf{1}\big)$
139	10100100	$2A_2+2A_1$		$\mathfrak{t}^1 \ltimes \big((-9)(-5)(-1)^2(3)^2(7), (-6)(-2)^2(2)^2(6), (-7)(-3)^2(1)^2(5)^2, (-4)(0)^2(4), (-1)(3)\big)$
140	01000101	A_3+A_1		$\mathfrak{gl}_2 \ltimes \Big(\mathbf{1}^{(-2)^{\oplus 3}}\oplus\mathbf{2}^{(1)}\oplus\mathbf{3}^{(2)}, \mathbf{1}^{(-4)}\oplus\mathbf{2}^{(-1)}\oplus\mathbf{2}^{(3)}, \mathbf{2}^{(-3)}\oplus\mathbf{3}^{(0)}\oplus\mathbf{2}^{(1)},\ \mathbf{2}^{(-1)}\oplus\mathbf{1}^{(2)}\big)$
141	01010001	$2A_2+A_1$		$\mathfrak{t}^1 \ltimes \big((-4)(-1)^2(3)^2,\ (-1)^2(2), (-5)(-2)(2)^2, (-2)^2(1)(5), (-3)(1)^3, (-3)(0)(4), (0)^2, (3), (-1)\big)$

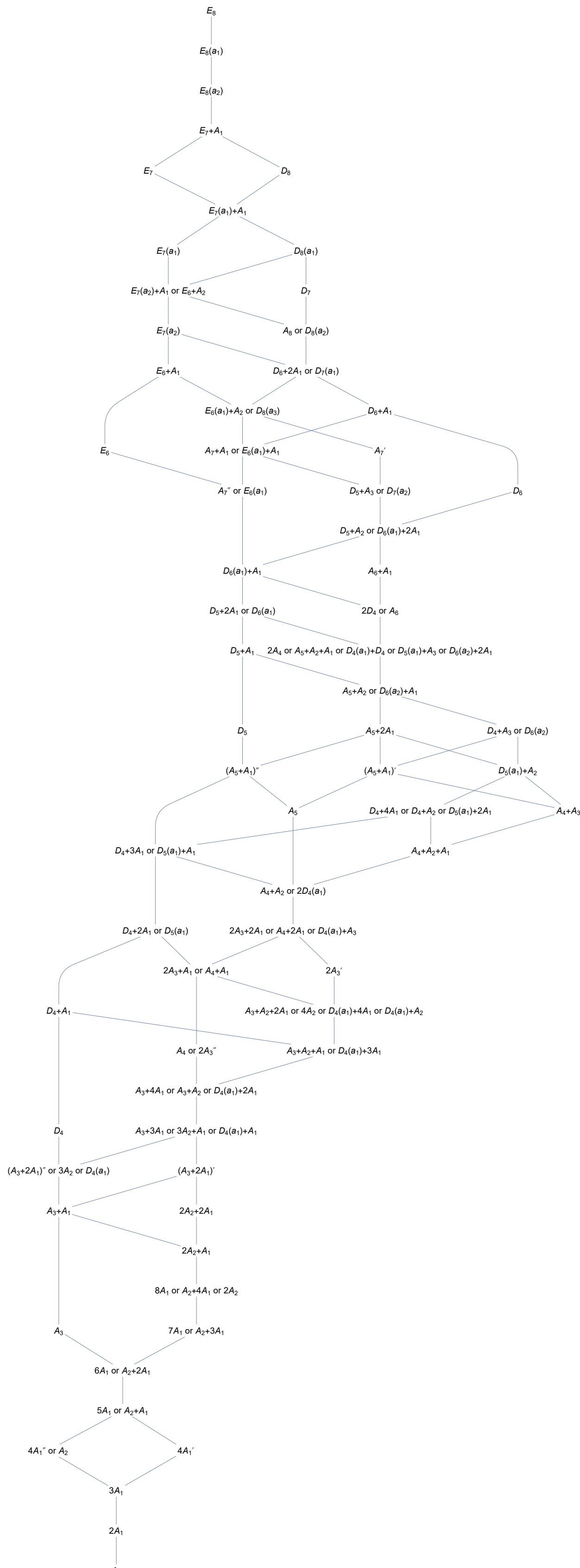
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
		$D_4(a_1)$		
142	00000200	$(A_3+2A_1)''$		$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \ltimes (\mathbf{2}_1 \oplus \mathbf{2}_2 \oplus \mathbf{2}_3 \oplus \mathbf{1}, \mathbf{2}_1{\otimes}\mathbf{2}_2 \oplus \mathbf{2}_1{\otimes}\mathbf{2}_3 \oplus \mathbf{2}_2{\otimes}\mathbf{2}_3)$
		$3A_2$		
		$2A_2$		
143	40000000	A_2+4A_1		$\mathfrak{sl}_3 \oplus \mathfrak{sl}_3 \ltimes \left(\mathbf{3}_1^{\oplus 2} \oplus \mathbf{3}_2^{\oplus 2} \oplus \mathbf{1}\right)$
		$8A_1$		
144	20000020	A_2+4A_1		$\mathfrak{sl}_3 \ltimes \left(\mathbf{8} \oplus \mathbf{3}^{\oplus 2} \oplus \mathbf{1}, \mathbf{3}^{\oplus 2}\right)$
		$8A_1$		
145	20000020	A_2+4A_1		$\mathfrak{sl}_2 \ltimes (\mathbf{2}\oplus\mathbf{1},\ \mathbf{2}\oplus\mathbf{1}, \mathbf{2}\oplus\mathbf{1}, \mathbf{3}\oplus\mathbf{2}, \mathbf{2}\oplus\mathbf{1}, \mathbf{2}\oplus\mathbf{1}, \mathbf{2}\oplus\mathbf{1}, \mathbf{2},\ \mathbf{1})$
146	01000101	A_3+A_1		$\mathfrak{sl}_2 \oplus \mathfrak{t}^1 \oplus \mathfrak{sl}_2 \ltimes \left(\mathbf{2}_1^{(-3)} \oplus \mathbf{1}^{(-2)} \oplus \mathbf{2}_2^{(1)} \oplus \mathbf{3}_2^{(2)}, (\mathbf{2}_1{\otimes}\mathbf{2}_2)^{(-2)} \oplus \mathbf{3}_2^{(0)} \oplus \mathbf{2}_2^{(3)}, (\mathbf{2}_1{\otimes}\mathbf{2}_2)^{(0)} \oplus \mathbf{1}^{(2)}\right)$

#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
147	01010001	$2A_2 + A_1$		$\mathfrak{gl}_2 \ltimes \big(\mathbf{1}^{(-8)} \oplus \mathbf{1}^{(-2)} \oplus \mathbf{2}^{(5)} \oplus \mathbf{1}^{(6)}, \mathbf{2}^{(-3)} \oplus \mathbf{1}^{(-2)}, \mathbf{1}^{(-10)} \oplus \mathbf{2}^{(3)} \oplus \mathbf{1}^{(4)}, \mathbf{2}^{(-5)} \oplus \mathbf{1}^{(-4)} \oplus \mathbf{1}^{(10)}, \mathbf{2}^{(1)} \oplus \mathbf{1}^{(2)^{\oplus 2}}, \mathbf{1}^{(-6)} \oplus \mathbf{1}^{(8)}, \mathbf{1}^{(0)^{\oplus 2}}, \mathbf{1}^{(6)}, \mathbf{1}^{(-2)}\big)$
148	30000010	$A_2 + 3A_1$		$\mathfrak{gl}_2 \ltimes \big(\mathbf{2}^{(-1)} \oplus \mathbf{1}^{(6)}, \mathbf{1}^{(-2)} \oplus \mathbf{2}^{(5)}, \mathbf{2}^{(-3)} \oplus \mathbf{1}^{(4)}, \mathbf{3}^{(-4)} \oplus \mathbf{2}^{(3)}, \mathbf{2}^{(-5)} \oplus \mathbf{1}^{(2)}, \mathbf{1}^{(-6)} \oplus \mathbf{2}^{(1)}, \mathbf{2}^{(-7)} \oplus \mathbf{1}^{(0)}, \mathbf{2}^{(-1)}, \mathbf{1}^{(-2)}\big)$
149	10010010	$A_2 + 3A_1$		$\mathfrak{gl}_2 \ltimes \big(\mathbf{2}^{(-5)} \oplus \mathbf{1}^{(-2)} \oplus \mathbf{3}^{(4)} \oplus \mathbf{1}^{(6)}, \mathbf{2}^{(-7)} \oplus \mathbf{2}^{(1)} \oplus \mathbf{1}^{(4)}, \mathbf{2}^{(-1)^{\oplus 2}} \oplus \mathbf{1}^{(2)}, \mathbf{1}^{(-6)} \oplus \mathbf{2}^{(-3)} \oplus \mathbf{2}^{(5)}, \mathbf{1}^{(0)} \oplus \mathbf{2}^{(3)}, \mathbf{1}^{(-2)}\big)$
150	01010001	$2A_2 + A_1$		$\mathfrak{gl}_2 \ltimes \big(\mathbf{1}^{(-6)} \oplus \mathbf{2}^{(-1)} \oplus \mathbf{1}^{(2)} \oplus \mathbf{3}^{(2)}, \mathbf{3}^{(-4)} \oplus \mathbf{2}^{(1)} \oplus \mathbf{3}^{(4)}, \mathbf{3}^{(-2)} \oplus \mathbf{1}^{(-2)} \oplus \mathbf{2}^{(3)}, \mathbf{1}^{(0)} \oplus \mathbf{3}^{(0)}, \mathbf{1}^{(2)}\big)$
151	00020000	$A_2 + 4A_1$ $8A_1$		$\mathfrak{sl}_2 \oplus \mathfrak{sl}_2 \ltimes (\mathbf{4}_1 \otimes \mathbf{2}_2 \oplus \mathbf{2}_1 \otimes \mathbf{2}_2 \oplus \mathbf{2}_1 \otimes \mathbf{4}_2, \mathbf{3}_1 \oplus \mathbf{3}_2)$
152	10010010	$A_2 + 3A_1$		$\mathfrak{t}^2 \ltimes \Big(\binom{-1}{-4}\binom{-1}{1}\binom{1}{1}, \binom{-2}{-3}\binom{0}{-3}\binom{0}{2}, \binom{-1}{-2}\binom{1}{-2}\binom{-1}{3}\binom{1}{3}, \binom{-2}{-1}\binom{0}{-1}^2\binom{2}{-1}, \binom{-1}{0}^2\binom{1}{0}^2, \binom{0}{1}^2\binom{2}{1}, \binom{-1}{-3}\binom{-1}{2}\binom{1}{2}^2, \binom{0}{-2}\binom{0}{3}, \binom{-1}{-1}\binom{1}{-1}, \binom{0}{0}, \binom{1}{1}\Big)$
153	20010000	$A_2 + 2A_1$		$\mathfrak{o}_5 \ltimes \big(\mathbf{4}^{(-3)} \oplus \mathbf{1}^{(-2)} \oplus \mathbf{1}^{(6)}, \mathbf{1}^{(-6)} \oplus \mathbf{4}^{(-5)} \oplus \mathbf{4}^{(3)} \oplus \mathbf{1}^{(4)}, \mathbf{1}^{(2)} \oplus \mathbf{4}^{(1)}, \mathbf{1}^{(0)}, \mathbf{1}^{(-2)}\big)$
154	00010002	A_3		$\mathfrak{sl}_3 \oplus \mathfrak{so}_5 \ltimes \mathbf{3} \otimes \mathbf{4} \oplus \mathbf{1} \otimes \mathbf{5}$
155	01000100	$A_2 + 2A_1$		$\mathfrak{sl}_2 \oplus \mathfrak{t}^1 \oplus \mathfrak{sl}_2 \ltimes \Big((\mathbf{2} \otimes \mathbf{2})^{(-3;1)} \oplus \mathbf{2}_2^{(0;4)} \oplus \mathbf{1}^{(2)}, \mathbf{3}_2^{(-2;2)} \oplus \mathbf{2}_1^{(1)} \oplus \mathbf{1}^{(-2)}, (\mathbf{2} \otimes \mathbf{2})^{(-1)} \oplus \mathbf{2}_2^{(2)}, \mathbf{1}^{(0)}\Big)$
156	20010000	$A_2 + 2A_1$ $6A_1$		$\mathfrak{sl}_2 \oplus \mathfrak{t}^1 \oplus \mathfrak{sl}_2 \ltimes \Big((\mathbf{2} \otimes \mathbf{4})^{(-1)} \oplus \mathbf{5}_2^{(-1)} \oplus \mathbf{3}_2^{(2)}, (\mathbf{2} \otimes \mathbf{2})^{(1)} \oplus \mathbf{3}_2^{(-2;1)}, \mathbf{3}_2^{(0)}\Big)$

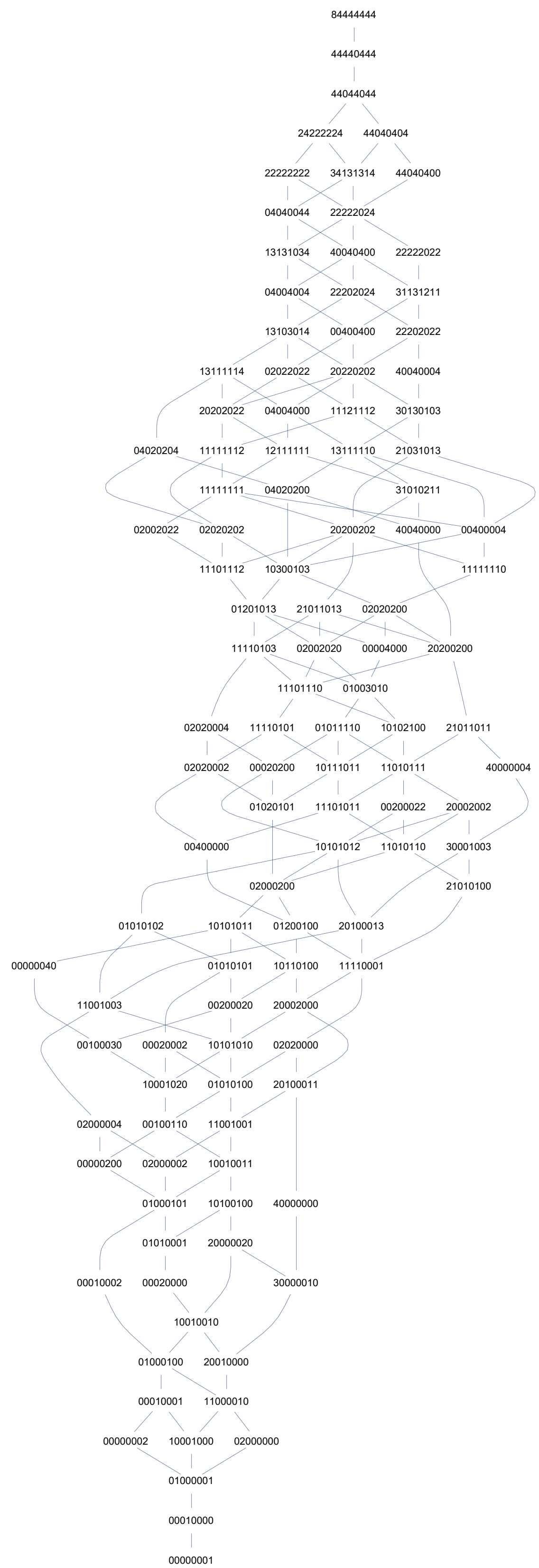
#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
157	01000100	$A_2 + 2A_1$ $6A_1$	$\overline{347}$ — $\overline{56}$, $\overline{17}$, $\overline{2}$ $\overline{2}$, $\overline{17}$, $\overline{357}$, $\overline{56}$, $\overline{467}$, $\overline{34}$	$\mathfrak{sl}_2 \oplus \mathfrak{t}^1 \oplus \mathfrak{sl}_2 \ltimes \big(\mathbf{2}_1^{(-5;-1;3)} \oplus \mathbf{2}_2^{(-3;1;5)}, (\mathbf{2}\otimes\mathbf{2})^{(0)} \oplus \mathbf{1}^{(-6;-2;-2;2;2;6)}, \mathbf{2}_1^{(-3;1)} \oplus \mathbf{2}_2^{(-1;3)}, \mathbf{1}^{(0)}\big)$
158	11000010	$5A_1$	$\overline{2}$, $\overline{34}$, $\overline{56}$, $\overline{467}$, $\overline{357}$	$\mathfrak{sl}_2 \oplus \mathfrak{t}^1 \oplus \mathfrak{sl}_2 \ltimes \big(\mathbf{2}_1^{(-1)^{\oplus 2}} \oplus \mathbf{2}_2^{(-1)^{\oplus 2}} \oplus \mathbf{1}^{(4)}, (\mathbf{2}\otimes\mathbf{2})^{(-2)} \oplus \mathbf{2}_1^{(3)} \oplus \mathbf{2}_2^{(3)} \oplus \mathbf{1}^{(-2)^{\oplus 3}}, \mathbf{2}_1^{(-3)} \oplus \mathbf{2}_2^{(-3)} \oplus \mathbf{1}^{(2)^{\oplus 3}}, \mathbf{2}_1^{(1)} \oplus \mathbf{2}_2^{(1)}, \mathbf{1}^{(0)}\big)$
159	00010001	$5A_1$	$\overline{567}$, $\overline{36}$, $\overline{47}$, $\overline{1}$, $\overline{25}$	$\mathfrak{sl}_3 \ltimes (\mathbf{6} \oplus \bar{\mathbf{3}}, \mathbf{3}^{\oplus 2}, \mathbf{8} \oplus \mathbf{1}, \bar{\mathbf{3}}^{\oplus 2}, \mathbf{3})$
160	11000010	$A_2 + A_1$	$\overline{34}$ — $\overline{567}$, $\overline{2}$	$\mathfrak{sl}_3 \oplus \mathfrak{t}^1 \oplus \mathfrak{sl}_2 \ltimes \big((\bar{\mathbf{3}}\otimes\mathbf{2})^{(-5)}\oplus(\mathbf{3}\otimes\mathbf{1})^{(-4)}\oplus (\mathbf{1}\otimes\mathbf{2})^{(-3)}\oplus\mathbf{1}^{(12)}, (\bar{\mathbf{3}}\otimes\mathbf{1})^{(-8)} \oplus \mathbf{1}^{(-6)} \oplus (\mathbf{3}\otimes\mathbf{1})^{(8)} \oplus (\mathbf{1}\otimes\mathbf{2})^{(9)}, (\mathbf{1}\otimes\mathbf{2})^{(-9)} \oplus (\bar{\mathbf{3}}\otimes\mathbf{1})^{(4)} \oplus \mathbf{1}^{(6)}, (\mathbf{1}\otimes\mathbf{2})^{(3)}, \mathbf{1}^{(0)}\big)$
161	10001000	$(4A_1)'$	$\overline{567}$, $\overline{25}$, $\overline{36}$, $\overline{47}$	$\mathfrak{gl}_3 \ltimes \big(\mathbf{3}^{(-2)} \oplus \mathbf{6}^{(-1)} \oplus \bar{\mathbf{3}}^{(5)}, \mathbf{8}^{(-3)} \oplus \mathbf{1}^{(3)} \oplus \mathbf{3}^{(4)}, \bar{\mathbf{3}}^{(-4;2)}, \mathbf{3}^{(1)}\big)$
162	10001000	$(4A_1)'$	$\overline{47}$, $\overline{267}$, $\overline{36}$, $\overline{5}$	$\mathfrak{o}_5 \ltimes \big(\mathbf{16}^{(-1)} \oplus \mathbf{4}^{(3)}, \mathbf{5}^{(-2;2)} \oplus \mathbf{1}^{(-2)}, \mathbf{4}^{(1)}\big)$
163	00000002	$(4A_1)''$	$\overline{1}$, $\overline{23}$, $\overline{45}$, $\overline{67}$	$\mathfrak{sp}_6 \ltimes (\mathbf{6}, \mathbf{14} \oplus \mathbf{1}, \mathbf{6})$
164	01000001	$3A_1$	$\overline{45}$, $\overline{67}$, $\overline{23}$	$\mathfrak{sp}_6 \oplus \mathfrak{t}^1 \ltimes \big(\mathbf{14}^{(-2)} \oplus \mathbf{1}^{(2)} \oplus \mathbf{6}^{(3)}, \mathbf{6}^{(1)}\big)$
165	02000000	A_2 $(4A_1)''$	127 — $\overline{7}$ $\overline{56}$, 125, $\overline{345}$, $\overline{7}$	$\mathfrak{sl}_5 \oplus \mathfrak{sl}_2 \ltimes (\mathbf{5}\otimes\mathbf{2} \oplus \bar{\mathbf{5}}\otimes\mathbf{2} \oplus \mathbf{1}\otimes\mathbf{2}, \mathbf{1})$
166	01000001	$3A_1$	$\overline{3}$, $\overline{67}$, $\overline{45}$	$\mathfrak{so}_5 \oplus \mathfrak{t}^1 \oplus \mathfrak{sl}_2 \ltimes \big((\mathbf{4}\otimes\mathbf{1})^{(-1)} \oplus (\mathbf{1}\otimes\mathbf{2})^{(3)}, (\mathbf{5}\otimes\mathbf{1})^{(-2)} \oplus \mathbf{1}^{(-2)} \oplus (\mathbf{4}\otimes\mathbf{2})^{(2)}, (\mathbf{4}\otimes\mathbf{1})^{(-3)} \oplus (\mathbf{1}\otimes\mathbf{2})^{(1)}, (\mathbf{4}\otimes\mathbf{2})^{(0)}, (\mathbf{1}\otimes\mathbf{2})^{(-1)}, \mathbf{1}^{(2)}\big)$
167	00010000	$2A_1$	$\overline{56}$, $\overline{47}$	$\mathfrak{so}_7 \oplus \mathfrak{sl}_3 \ltimes (\mathbf{8}\otimes\mathbf{3} \oplus \mathbf{1}\otimes\mathbf{3}, \mathbf{1}\otimes\bar{\mathbf{3}})$
168	00010000	$2A_1$	$\overline{67}$, $\overline{5}$	$\mathfrak{g}_2 \oplus \mathfrak{sl}_4 \ltimes (\mathbf{7}\otimes\mathbf{4}, \mathbf{1}\otimes\mathbf{6})$

#	D_8-orbit char.	E_8-orbit type	Dynkin scheme	$\mathfrak{z}_{B_7}$
169	00000001	A_1	$\overline{\varnothing}$	$\mathfrak{sl}_7 \ltimes (\mathbf{7},\ \mathbf{21})$

Closure diagram for E_8



Closure diagram for spin_{16}



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