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A RELIABLE NUMERICAL METHOD FOR THE SINGULARLY PERTURBED NONLINEAR DIFFERENTIAL EQUATION WITH AN INTEGRAL BOUNDARY CONDITION

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Abstract. This study purposes to present an efficient numerical method for the singularly perturbed nonlinear problems involving an integral boundary condition. Initially, some properties are given for the continuous problem. Then, using interpolating quadrature formulas [3], the finite difference scheme is established on the Bakhvalov–Shishkin mesh (B-S mesh). The error approximations of the suggested scheme are examined in the discrete maximum norm. Finally, some numerical examples are included to confirm the theory.

1. Introduction

Singularly perturbed problems are significant phenomena in many branches of science. Such problems and their applications emerge in computational neuroscience [15], optimal control theory [7, 16, 38], prey-predator systems [39], stochastic processes [35] and so on [17, 20, 26, 29, 30, 32].

These problems are classified by the highest-order derivative term multiplied by a small perturbation parameter ε. As the perturbation parameter tends to zero, the boundary layers occur in the solution. The solution behaves stable outside of the layer region, while it behaves irregularly within the layer region. Because of the layer behavior, traditional numerical methods do not yield accurate results. Therefore, uniform and stable numerical approaches are needed [17, 20, 26, 29, 30, 32]. To investigate the comprehensive theroetical analysis and numerical aspects of these problems, one may refer to [17, 20, 26, 29, 30, 32] and the references therein.

Recently, a large number of remarkable numerical methods have been proposed by many scholars. The authors in [27] have generated a weak Galerkin finite element technique on a polytopal mesh to solve convection-diffusion-reaction problems with layer behavior. In [13], using equidistributed monitor functions, a second-order finite difference scheme has been formulated on an adaptive mesh for singularly perturbed nonlinear problems including integral boundary condition. Babu and Bansal [4] have developed Mickens type discretization on a uniform mesh for singularly perturbed parabolic problems with time delay. In [18], using linear basis functions and interpolating quadrature rules, a second-order discretization has been obtained on Boglaev–Bakhvalov type mesh. In [15], using extended cubic B-splines, implicit Euler method has been suggested for singularly perturbed parabolic problems. Cui and Zhang [14] have used the quadratic Galerkin finite element approach on 2-D polygonal grids for singularly perturbed biharmonic equations. In [19], on the Shishkin-type mesh, singularly perturbed delay reaction-diffusion problems with integral boundary conditions have been analyzed and some stability results have been given. The authors in [1] have presented a higher-order Haar wavelet collocation approach for singularly perturbed nonlinear differential equations with integral boundary conditions. Cakir and Arslan [10] have established the first-order numerical scheme on the Shishkin mesh for singularly perturbed semilinear problems with two integral boundary conditions. In [37], a reproducing kernel method has been applied to the singularly perturbed nonlinear initial-boundary value problems. Subburayan and Ramanujam [36] have provided a first-order finite difference scheme for solving singularly perturbed problems with delay arguments by using Shishkin's

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decomposition procedure and linear interpolations. In [40], the virtual element method has been introduced to solve fourth-order singularly perturbed problems. In [12], exponential type finite difference scheme has been constructed on a uniform mesh for singularly perturbed three-point convection-diffusion problems. In [31], the Haar wavelet collocation method has been used for singularly perturbed convection-dominated problems involving delay parameters. Cakir and Amiraliyev [8] have proposed a second-order fitted difference scheme with exponential coefficient for singularly perturbed reaction-diffusion nonlocal boundary value problems. In [22, 23], singularly perturbed parameterized problems including integral boundary conditions have been discretized on layer-adapted meshes.

This paper concerns the following singularly perturbed problem of the nonlinear differential equation:

$$\varepsilon u' + f(t,u) = 0, \quad t \in I = (0,T], \quad T > 0, \tag{1.1}$$

with the integral boundary condition

$$u(0) = \mu u(T) + \int_0^T b(s)u(s)ds + d. \tag{1.2}$$

Here, ε is the perturbation parameter, $\bar{I} = [0,T]$, the functions $f(t,u)$ $\big((t,u) \in \bar{I} \times \mathbb{R}\big)$ and $b(t)$ $\big(t \in \bar{I}\big)$ are sufficiently smooth. Problem (1.1)–(1.2) has a boundary layer within the neighborhood of $t = 0$ (see [2, 9]). In papers [2, 9, 28], the problem (1.1)–(1.2) has been considered on layer-adapted meshes and some numerical results have been obtained. Motivated by the papers [2, 9, 12, 22, 23, 28], our aim is to design and analyze a stable finite difference scheme on B-S mesh for solving singularly perturbed nonlinear differential equations including an integral boundary condition. An extended overview of the differential equations with integral boundary conditions can be found in [6, 21, 34]. Furthermore, to analyze the layer adapted meshes in detail, please see [5, 20, 24–26, 29, 32]. Bakhvalov mesh have been introduced by N. S. Bakhvalov [5], Shishkin mesh have been mentioned in [20, 26, 29, 32] and T. Linß have used Bakhvalov–Shishkin mesh [24–26].

The plan of this paper is as follows: In Section 2, the analytical bounds for the problem (1.1)–(1.2) are presented. Then, using interpolating quadrature formulas [3], the finite difference scheme is constructed on the B-S mesh. Section 3 is devoted to the stability analysis and error estimates. In Section 4, some numerical examples are given to support the theoretical analysis. Finally, the paper ends with the concluding remarks.

2. The Mesh and Discrete Scheme

In this section, we give some analytical properties of the solution of problem (1.1)–(1.2). Moreover, the finite difference approximation is presented on B-S mesh.

Lemma 2.1 ([2])**.** *We assume that* $\frac{\partial f(t,u)}{\partial u}$ *is properly bounded and*

$$p(\varepsilon) = 1 - \mu A^+ - b^* B^+ \geq c_0 > 0.$$

Here,

$$A^+ = \begin{cases} 0, & \mu \leq 0, \\ \varepsilon e^{\frac{-\alpha T}{\varepsilon}}, & \mu > 0, \end{cases}$$

$$B^+ = \begin{cases} 0, & b^* \leq 0, \\ \alpha^{-1}\varepsilon\Big(1 - e^{\frac{-\alpha T}{\varepsilon}}\Big), & b^* > 0, \end{cases}$$

and $b^* = \max\limits_{\bar{I}} |b(t)|$. *Then, the following estimate*

$$\|u\|_\infty \leq C_0,$$

is satisfied, where

$$C_0 = c_0^{-1}\left(\|\mu\| + \|b\|_1\right)\alpha^{-1}\|F\|_\infty + c_0^{-1}\|d\|, \quad \|b\|_1 = \int_0^T |b(t)|\, dt.$$

Moreover, under the conditions $F(t) = f(t,0)$, $\left|\frac{\partial f}{\partial t}\right| \le C$ *and* $|u| \le C_0$, *the following relation*

$$|u'(t)| \le C\Big\{1 + \frac{1}{\varepsilon}e^{-\frac{\alpha t}{\varepsilon}}\Big\}, \quad t \in \overline{I} \tag{2.1}$$

holds.

Proof. For the proof of the lemma, please see [2,9,28]. □

Now, we give the mesh selection process and the finite difference scheme. Let ω_N be any non-uniform mesh on the interval I :

$$\omega_N = \{0 < t_1 < t_2 < \cdots < t_{N-1} < t_N = T\},$$
$$\overline{\omega}_N = \omega_N \cup \{t = 0\}.$$

For any mesh function $g_i = g(t_i)$, we use the difference rules in [33]:

$$g_{\bar{t},i} = \frac{g_i - g_{i-1}}{h_i}$$

and

$$\|g\|_\infty \equiv \|g\|_{\infty,\overline{\omega}_N} := \max_{0\le i\le N} |g_i|.$$

where $h_i = t_i - t_{i-1}$ is the mesh stepsize for $i \ge 1$. For an even mesh element N, we split the interval $[0,T]$ into two subintervals $[0,\sigma]$ and $[\sigma,T]$. The transition parameter σ is determined as $\sigma = \min\{\frac{T}{2}, \varepsilon\alpha^{-1}\ln N\}$ and a set of mesh points are described by (For Bakhvalov–Shishkin meshes, see [11,24–26])

$$\bar{\omega}_N = \begin{cases} t_i = -\alpha^{-1}\varepsilon\ln\left[1 - 2(1-N^{-1})\frac{i}{N}\right], & t_i \in [0,\sigma], \quad i = 0,1,\dots,\frac{N}{2}; \\ t_i = \sigma + \left(i - \frac{N}{2}\right)h, & h = \frac{2(T-\sigma)}{N}, \quad t_i \in [\sigma,T], \quad i = \frac{N}{2}+1,\dots,N. \end{cases}$$

Here, we assume that $\varepsilon \ll N^{-1}$ in the numerical experiments. Now, we use the interpolating quadrature rules [3] and numerical formulas in [2,9] to produce the numerical scheme. To construct the approximation for equation (1.1), we use the following integral identity:

$$\varepsilon u_{\bar{t},i} + h_i^{-1}\int_{t_{i-1}}^{t_i} f(t,u(t))\,dt = 0, \quad 1 \le i \le N,$$

whence we obtain

$$\varepsilon u_{\bar{t},i} + f(t_i,u_i) + R_i = 0, \quad 1 \le i \le N. \tag{2.2}$$

Here, the remainder term R_i is shown as

$$R_i = -h_i^{-1}\int_{t_{i-1}}^{t_i} (t - t_{i-1})\, f'(t,u(t))\,dt. \tag{2.3}$$

For the boundary condition (1.2), using the numerical integration rules [2,9] on $(0,T)$, it is found that

$$u_0 = \mu u_N + \sum_{i=1}^{N} h_i b_i u_i + d + r, \tag{2.4}$$

where the truncation error is expressed by

$$r = \sum_{i=1}^{N}\int_{t_{i-1}}^{t_i} (t - t_{i-1})\frac{d}{dt}(b(t)u(t))\,dt. \tag{2.5}$$

Neglecting the error terms R_i and r in (2.2) and (2.4), we present the following difference problem:

$$\ell y_i := \varepsilon y_{\bar{t},i} + f\left(t_i, y_i\right) = 0, \quad 1 \leq i \leq N, \tag{2.6}$$

$$y_0 = \mu y_N + \sum_{i=1}^{N} h_i b_i y_i + d. \tag{2.7}$$

3. Convergence Analysis

To examine the uniform convergence of the proposed method, let the error function $z_i = y_i - u_i$, $0 \leq i \leq N$ be the solution of the following discrete problem [2, 9]:

$$\varepsilon z_{\bar{t},i} + f\left(t_i, y_i\right) - f\left(t_i, u_i\right) = R_i, \quad 1 \leq i \leq N, \tag{3.1}$$

$$z_0 = \mu z_N + \sum_{i=1}^{N} h_i b_i z_i - r. \tag{3.2}$$

Here, R_i and r are denoted by (2.3) and (2.5), respectively.

Lemma 3.1. *Under the conditions of Lemma* 2.1*, for the error terms R_i and r, we have the estimate*

$$\|R\|_{\infty,\omega_N} \leq CN^{-1}, \tag{3.3}$$

$$|r| \leq CN^{-1}. \tag{3.4}$$

Proof. Here, we use the similar technique as in [2,9–11,22,23,28]. Initially, we show the proof of (3.3). From the relation (2.3), we can write

$$\begin{aligned}
|R_i| &\leq h_i^{-1} \int_{t_{i-1}}^{t_i} \left(t - t_{i-1}\right) \left| \frac{\partial f}{\partial t}\left(t, u(t)\right) + \frac{\partial f}{\partial u}\left(t, u(t)\right) u'(t) \right| dt \\
&\leq C h_i^{-1} \int_{t_{i-1}}^{t_i} \left(t - t_{i-1}\right)\left(1 + |u'(t)|\right) dt, \quad 1 \leq i \leq N.
\end{aligned}$$

Taking into account inequality (2.1), we find that

$$\begin{aligned}
|R_i| &\leq C\left\{ h_i + h_i^{-1}\varepsilon^{-1} \int_{t_{i-1}}^{t_i} \left(t - t_{i-1}\right) e^{-\frac{\alpha t}{\varepsilon}} dt \right\} \\
&\leq C\left\{ h_i + \varepsilon^{-1} \int_{t_{i-1}}^{t_i} e^{-\frac{\alpha t}{\varepsilon}} dt \right\}, \quad 1 \leq i \leq N.
\end{aligned}$$

Now, we estimate the remainder terms according to the mesh points of the Bakhvalov–Shishkin mesh. Firstly, we consider the truncation term R_i on the interval $[0, \sigma]$ for $\sigma \leq \frac{T}{2}$. Since

$$t_i = -\alpha^{-1}\varepsilon \ln\left[1 - 2(1 - N^{-1})\frac{i}{N}\right],$$

we get

$$h_i = -\alpha^{-1}\varepsilon \ln\left[1 - 2(1 - N^{-1})\frac{i}{N}\right] + \alpha^{-1}\varepsilon \ln\left[1 - 2(1 - N^{-1})\frac{i-1}{N}\right].$$

Next, using the mean value theorem according to $i_* \epsilon \, [i-1, i]$, we have

$$h_i \leq \frac{\varepsilon}{\alpha} \frac{2(1 - N^{-1})N^{-1}}{\left(1 - 2i_*\left(1 - N^{-1}\right)N^{-1}\right)} \leq CN^{-1}.$$

Secondly, for $\sigma \leq \frac{T}{2}$, on the interval $[\sigma, T]$, taking into account $t_i = \sigma + \left(i - \frac{N}{2}\right)h$, we write

$$h_i = \frac{2\left(T - \sigma\right)}{N} \leq CN^{-1}.$$

Thus, we find the estimate $|R_i| \leq CN^{-1}$, which proves the relation (3.3). Finally, we show the validity of the relation (3.4). From (2.5), we obtain

$$|r| \leq C \sum_{i=1}^{N} \int\limits_{t_{i-1}}^{t_i} (t_i - t_{i-1})\left(1 + |u'(t)|\right) dt, \quad 1 \leq i \leq N.$$

Considering (2.1), we obtain

$$|r| \leq C \sum_{i=1}^{N} h_i \int\limits_{t_{i-1}}^{t_i} \Big(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}}\Big) dt, \quad 1 \leq i \leq N.$$

Using the transition parameter of Bakhvalov–Shishkin mesh, we have

$$|r| \leq C \Bigg[\sum_{i=1}^{N/2} h_i \int\limits_{t_{i-1}}^{t_i} \Big(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}}\Big) dt + \sum_{i=\frac{N}{2}+1}^{N} h_i \int\limits_{t_{i-1}}^{t_i} \Big(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}}\Big) dt\Bigg]. \tag{3.5}$$

From the relation (3.5), we find that

$$|r| \leq C \Bigg[h_i \int\limits_{0}^{\sigma} \Big(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}}\Big) dt + h_i \int\limits_{\sigma}^{T} \Big(1 + \frac{1}{\varepsilon} e^{-\frac{\alpha t}{\varepsilon}}\Big) dt\Bigg].$$

Hence, we obtain

$$|r| \leq C\left(h_i + h_i\right) \leq CN^{-1},$$

which leads to the proof of the bound (3.4). So, the proof is completed. □

Lemma 3.2 ([2])**.** *We assume that*

$$1 - \mu A^* - b^* B^* \geq c_* > 0, \tag{3.6}$$

where

$$A^* = \begin{cases} 0, & \mu \leq 0, \\ \frac{1}{[(1+\alpha\rho_1)(1+\alpha\rho_2)]^{N/2}}, & \mu > 0, \end{cases}$$

and

$$B^* = \begin{cases} 0, & b^* \leq 0, \\ h_1 \sum\limits_{i=1}^{\frac{N}{2}} \left(\frac{1}{1+\alpha\rho_1}\right)^i + h_2 \left(\frac{1}{1+\alpha\rho_1}\right)^{\frac{N}{2}} \sum\limits_{i=\frac{N}{2}+1}^{N} \left(\frac{1}{1+\alpha\rho_2}\right)^{i-\frac{N}{2}}, & b^* > 0, \end{cases}$$

$$\rho_k = h^{(k)}/\varepsilon, \; k = 1, 2.$$

Then, the following estimate:

$$\|z\|_{\infty,\omega_N} \leq C\left(\|R\|_{\infty,\omega_N} + |r|\right)$$

holds.

Proof. See [2, 9, 28]. □

Remark 3.1 ([2])**.** To prove the relation (3.6), we can consider the following values:

$$A^* = \begin{cases} 0, & \mu \leq 0, \\ 1, & \mu > 0, \end{cases}$$

and

$$B^* = \begin{cases} 0, & b^* \leq 0, \\ \alpha^{-1}\varepsilon\left(1 + \frac{1}{1+\alpha\rho_1}\right) & b^* > 0. \end{cases}$$

Teorem 3.1. *Let $u(t)$ be the solution of problem* (1.1)–(1.2) *and y_i be the solution of problem* (2.6)–(2.7)*. Then we arrive at*

$$\|y-u\|_{\infty,\overline{\omega}_N} \leq CN^{-1},$$

which provides the main result of the paper.

4. Numerical Results

This section is devoted to the numerical calculations. Accordingly, three test problems are taken into account. For the nonlinear problem (2.6)–(2.7), the following iteration process is used [2]:

$$y_i^{(n)} = y_i^{(n-1)} - \frac{\left(y_i^{(n-1)} - y_{i-1}^{(n)}\right)\rho_i^{-1} + f\left(t_i, y_i^{(n-1)}\right)}{\frac{\partial f}{\partial u}\left(t_i, y_i^{(n-1)}\right) + \rho_i^{-1}}, \quad i = 1, \ldots, N,$$

$$y_0^{(n)} = \mu y_N^{(n-1)} + \sum_{i=1}^{N} h_i b_i y_i^{(n-1)} + d, \quad n = 1, 2, \ldots .$$

Here, $\rho_i = \frac{h_i}{\varepsilon}$ and $|\mu| + T\,\|b\|_\infty < 1$. Now, we test the numerical method on several examples.

Example 4.1 ([2, 9, 28])**.** Consider the first problem

$$\varepsilon u' + 2u - e^{-u} + t^2 = 0, \quad 0 < t \leq 1,$$

$$u(0) = \frac{1}{2}u(1) - \frac{1}{4}\int_0^1 e^{-s}u(s)ds + 1.$$

Table 1. Error approximations and the order of convergence on B-S-mesh.

ε		$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{-10}	e^N	0.01537017	0.00805098	0.00412783	0.00209043	0.00105207
	e^{2N}	0.00803328	0.00412783	0.00209043	0.00105201	0.00052776
	p^N	0.93607	0.96378	0.98157	0.99064	0.99528
2^{-12}	e^N	0.01556474	0.00818948	0.00420315	0.00212959	0.00107210
	e^{2N}	0.00817047	0.00419770	0.00212959	0.00107208	0.00053788
	p^N	0.92979	0.96417	0.98089	0.99016	0.99506
2^{-14}	e^N	0.01584503	0.00834673	0.00428768	0.00217371	0.00109461
	e^{2N}	0.00834673	0.00428768	0.00217355	0.00109461	0.00054926
	p^N	0.92474	0.96101	0.98014	0.98974	0.99484
2^{-16}	e^N	0.01617573	0.00851758	0.00438257	0.00222330	0.00111991
	e^{2N}	0.00851758	0.00437793	0.00222290	0.00111991	0.00056204
	p^N	0.92531	0.96019	0.97933	0.98931	0.99463
2^{-18}	e^N	0.01649406	0.00871911	0.00448919	0.00227908	0.00114838
	e^{2N}	0.00867967	0.00448919	0.00227838	0.00114838	0.00057643
	p^N	0.92623	0.95772	0.97844	0.98884	0.99439
2^{-20}	e^N	0.01661644	0.00880463	0.00453497	0.00230325	0.00116077
	e^{2N}	0.00875909	0.00453290	0.00230325	0.00116077	0.00058269
	p^N	0.92375	0.95782	0.97742	0.98859	0.99426

In iteration process, the initial estimate is taken as $y_0^{(n)} = 0,5$ and the stopping criterion is considered by $\max_i \left|y_i^{(n)} - y_i^{(n-1)}\right| \leq 10^{-5}$. The exact solution of this problem is unknown. Thus, we use the double-mesh principle [17, 20]. The error approximations are computed as

$$e_\varepsilon^N = \max_i \left|y_i^{\varepsilon,N} - \widetilde{y}_i^{\varepsilon,2N}\right|, \quad e^N = \max_\varepsilon e_\varepsilon^N$$

and the order of convergence is determined as

$$p^N = \ln\left(e^N/e^{2N}\right)/\ln 2.$$

The obtained results are summarized in Table 1.

This problem has been analyzed on the Shishkin-type mesh [2, 28] and Bakhvalov-type mesh in [9, 28]. Furthermore, the first-order convergence has been acquired. Here, we test the presented method on the Bakhvalov–Shishkin mesh. Accordingly, for the major values of N, the maximum pointwise errors are reduced, indicating that the proposed scheme is stable. The numerical solution profiles are given in Figures 1 and 2.

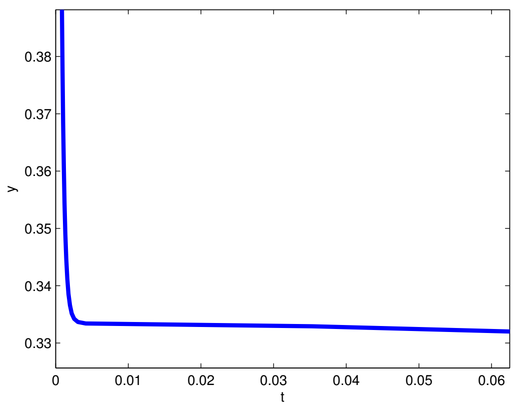

FIGURE 1. Approximate solution of Example 4.1 for $\varepsilon = 2^{-10}$ and $N = 64$.

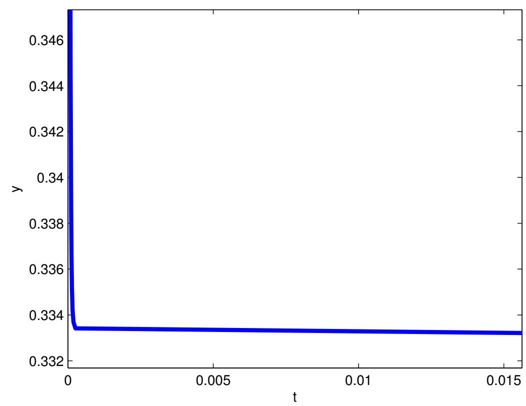

FIGURE 2. Approximate solution of Example 4.1 for $\varepsilon = 2^{-14}$ and $N = 32$.

Example 4.2. Take into account the second problem

$$\varepsilon u' + \sin(u) + 2u = 0, \quad 0 < t \le 1,$$

with

$$u(0) = \frac{1}{2}u(1) + \frac{1}{2}\int_0^1 \cos\left(\frac{\pi s}{4}\right)u(s)ds + 0,5.$$

The experimental results are given in Table 2.

The graphs of the numerical behavior are demonstrated in Figures 3 and 4.

Example 4.3. Consider the last problem

$$\varepsilon u' + \sinh(u) + 3u - e^{1-t} = 0, \quad 0 < t \le 1,$$

$$u(0) = \frac{1}{4}u(1) + \frac{1}{2}\int_0^1 e^{-s}u(s)ds + 1.$$

The numerical outcomes are presented in Table 3.

The numerical approximations are plotted in Figures 5 and 6.

In Tables 1–3, for the different values of ε and N, the error approximations and convergence rates are demonstrated. From these results, we observe that the order of convergence of the presented scheme is almost 1. Also, the layer behaviors of the numerical experiments are exhibited in Figures 1–6.

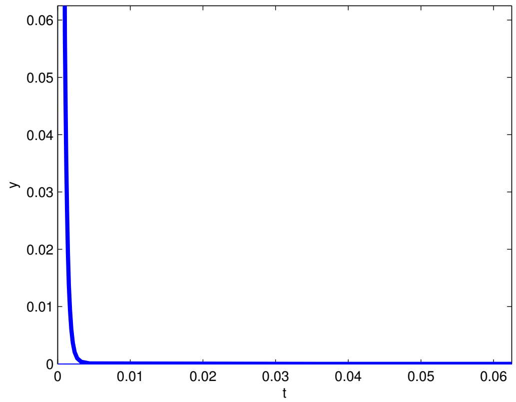

FIGURE 3. Numerical behavior of Example 4.2 for $\varepsilon = 2^{-10}$ and $N = 64$.

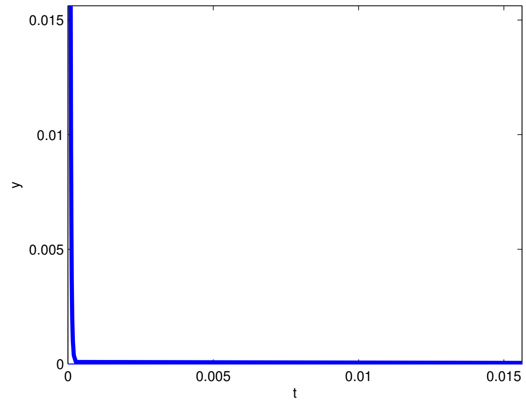

FIGURE 4. Numerical behavior of Example 4.2 for $\varepsilon = 2^{-14}$ and $N = 32$.

TABLE 2. Error approximations and the order of convergence on B-S-mesh.

ε		$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{-10}	e^N	0.02518971	0.01339619	0.00691383	0.00351289	0.00177071
	e^{2N}	0.01338138	0.00690868	0.00351140	0.00177071	0.00088900
	p^N	0.91260	0.95534	0.97743	0.98833	0.99406
2^{-12}	e^N	0.02524269	0.01342728	0.00693224	0.00352288	0.00177591
	e^{2N}	0.01342187	0.00692962	0.00352205	0.00177574	0.00089161
	p^N	0.91127	0.95431	0.97690	0.98833	0.99406
2^{-14}	e^N	0.02531859	0.01346236	0.00695061	0.00353286	0.00178110
	e^{2N}	0.01346236	0.00695054	0.00353268	0.00178103	0.00089423
	p^N	0.91126	0.95373	0.97637	0.98812	0.99404
2^{-16}	e^N	0.02539453	0.01350284	0.00697144	0.00354331	0.00178639
	e^{2N}	0.01350284	0.00697144	0.00354331	0.00178639	0.00089692
	p^N	0.91125	0.95373	0.97636	0.98804	0.99399
2^{-18}	e^N	0.02547050	0.01354329	0.00699233	0.00355392	0.00179174
	e^{2N}	0.01354329	0.00699233	0.00355392	0.00179174	0.00089961
	p^N	0.91124	0.95373	0.97636	0.98805	0.99399
2^{-20}	e^N	0.02550089	0.01355947	0.00700068	0.00355816	0.00179494
	e^{2N}	0.01355947	0.00700068	0.00355816	0.00179388	0.00090125
	p^N	0.91124	0.95373	0.97636	0.98805	0.99393

5. DISCUSSION AND CONCLUSION

In this paper, we have analyzed a difference scheme on Bakhvalov–Shishkin mesh for the singularly perturbed problem of the first-order nonlinear differential equation with an integral boundary condition. The stability of the presented scheme has been investigated and error bounds have been derived in the discrete maximum norm. It is proven that the scheme has $O(N^{-1})$ accuracy. The mentioned ideas in here can be applied to the different types of singularly perturbed nonlinear problems involving integro-differential equations, parameterized terms, etc.

TABLE 3. Error approximations and the order of convergence on B-S-mesh.

ε		$N=32$	$N=64$	$N=128$	$N=256$	$N=512$
2^{-10}	e^N	0.04740655	0.02554699	0.01328652	0.00677891	0.00342435
	e^{2N}	0.02554699	0.01328652	0.00677891	0.00342435	0.00172103
	p^N	0.89193	0.94318	0.97083	0.98522	0.99256
2^{-12}	e^N	0.04770634	0.02571046	0.01337202	0.00682265	0.00344647
	e^{2N}	0.02571046	0.01337202	0.00682265	0.00344647	0.00173215
	p^N	0.89182	0.94313	0.97081	0.98520	0.99255
2^{-14}	e^N	0.04770610	0.02571031	0.01337193	0.00682260	0.00344645
	e^{2N}	0.02571031	0.01337193	0.00682260	0.00344645	0.00173214
	p^N	0.89182	0.94313	0.97081	0.98520	0.99255
2^{-16}	e^N	0.04800778	0.02587470	0.01345787	0.00686655	0.00346908
	e^{2N}	0.02587470	0.01345787	0.00686655	0.00346868	0.00174365
	p^N	0.89172	0.94309	0.97079	0.98519	0.99244
2^{-18}	e^N	0.04812872	0.02594055	0.01349228	0.00688415	0.00347827
	e^{2N}	0.02594055	0.01349228	0.00688415	0.00347781	0.00174827
	p^N	0.89168	0.94307	0.97078	0.98509	0.99244
2^{-20}	e^N	0.04818924	0.02597349	0.01350949	0.00689294	0.00348287
	e^{2N}	0.02597349	0.01350949	0.00689294	0.00348255	0.00175058
	p^N	0.89167	0.94306	0.97078	0.98497	0.99244

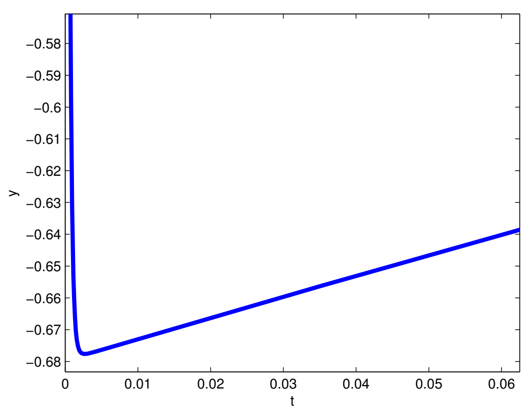

FIGURE 5. Numerical approximation of Example 4.3 for $\varepsilon=2^{-10}$ and N=64.

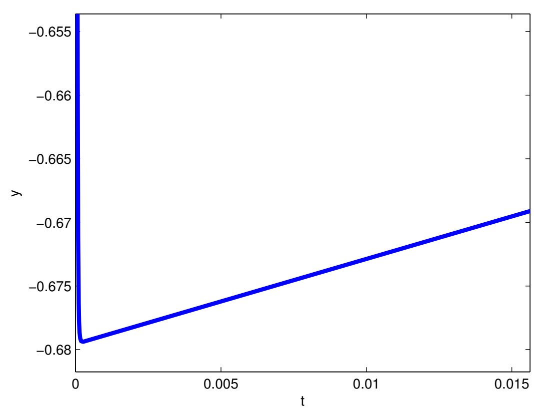

FIGURE 6. Numerical approximation of Example 4.3 for $\varepsilon=2^{-14}$ and N=32.

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