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ON RANGE OF AN ELEMENTARY OPERATOR

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Abstract. Let $\mathcal{L}(H)$ denote the algebra of all bounded linear operators on a complex infinite dimensional Hilbert space H into itself. Given $A, B, C, D \in \mathcal{L}(H)$, the elementary operator $\Delta_{(A,C),(B,D)} \in \mathcal{L}(\mathcal{L}(H))$ is defined by $\Delta_{(A,C),(B,D)}(X) = AXB - CXD$. In the present paper, we give necessary and sufficient conditions that (i) the range of the elementary operator $\Delta_{(A,C),(B,D)}$ is dense in the weak and the ultraweak operator topologies, (ii) the norm closure of the range of $\Delta_{(A,C),(B,D)}$ contains the ideal of compact operators. We initiate the study on the class of operators such that the norm closure of the range of $\Delta_{(A,C),(B,D)}$ is closed under taking adjoints. We establish some basic properties concerning these operators.

1. Introduction

Let $\mathcal{H}$ be a separable infinite-dimensional complex Hilbert space and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$ into itself.

The generalized derivation operator $\delta_{A,B}$ associated with (A, B), defined on $\mathcal{L}(\mathcal{H})$ by

$$\delta_{A,B}(X) = AX - XB,$$

was initially systematically studied by M. Rosenblum [23]. The properties of such operators have been well studied (see, for example, [3, 25, 26]).

If $A = B$, then $\delta_{A,A} = \delta_A : \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H})$ defined by

$$\delta_A(X) = AX - XA,$$

is called the inner derivation induced by A. The ranges and kernels of derivations have been studied intensively (see [4, 5, 12, 14, 19, 20]).

The elementary operator $\Delta_{A,B}$ associated with (A, B) is defined on $\mathcal{L}(\mathcal{H})$ by

$$\Delta_{A,B}(X) = AXB - X.$$

If $A = B$, we write simply Δ_A for $\Delta_{A,A}$. The properties of elementary operators, their spectrum [15,16], norm [21,22,24] and ranges [1,2,6,13,16–18] have been studied intensively, but many problems remain still open [16]. In particular, L. Fialkow [16] and Z. Genkai [18] studied the problem of characterizing operators $A, B \in \mathcal{L}(\mathcal{H})$ for which $R(\triangle_{A,B})$, the range of $\triangle_{A,B}$, is dense in $\mathcal{L}(\mathcal{H})$ in the norm topology. Given $A, B, C, D \in \mathcal{L}(\mathcal{H})$, we define the elementary operator $\triangle_{(A,C),(B,D)}$ as

$$\begin{aligned} \triangle_{(A,C),(B,D)} :\ & \mathcal{L}(\mathcal{H}) \longrightarrow \mathcal{L}(\mathcal{H}) \\ & X \longmapsto AXB - CXD. \end{aligned}$$

In [28] J. P. Williams obtained the necessary and sufficient conditions that the range $R(\delta_A)$ is dense in the weak and ultra-weak operator topologies, the norm closure of the range contains the ideal $\mathcal{K}$ of compact operators on $\mathcal{H}$. The extension of these results to $\Delta_{A,B}$ has been carried out by Bouali and Bouhafsi [7].

Our aim in this paper is a modest one. In Section 2, we provide a characterization of the case, where the range $R(\triangle_{(A,C),(B,D)})$ is weakly and ultra-weakly dense in $\mathcal{L}(\mathcal{H})$. Complementary results related to the range of the elementary operator $\triangle_{(A,C),(B,D)}$ are also given.

The D-symmetric operators (A is D-symmetric if $\overline{R(\delta_A)}$ is self-adjoint, where $\overline{R(\delta_A)}$ is the closure of the range $R(\delta_A)$ of δ_A in the norm topology) were studied by J. H. Anderson, J. W. Bunce,

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J. A. Deddens and J. P. Williams [1], S. Bouali and J. Charles [8,9], S. Bouali and M. Ech-chad [10,11] and J. G. Stampfli [26].

We consider the class of quadruplets (A, B, C, D) such that $\overline{R(\triangle_{(A,C),(B,D)})}$ is self-adjoint and call such quadruplet D-symmetric. In this paper we extend the results of the D-symmetric operators to D-symmetric quadruplets.

In Section 3, we give some properties and characterizations which concern the D-symmetric quadruplets. We prove that if (A, B, C, D) is D-symmetric, then $BTA = DTC$ implies $BT^*A = DT^*C$ for all $T \in C_1(\mathcal{H})$. In order to generalize these results, we initiate the study of a more general class of quadruplets (A, B, C, D) that have the following property: $BTA = DTC$ implies $BT^*A = DT^*C$ for all $T \in C_1(\mathcal{H})$. We give a characterization and some basic properties concerning this class of operators.

Notation and definitions

(1) Let $\mathcal{L}(\mathcal{H})$ be the algebra of all bounded linear operators acting on a complex separable Hilbert space $\mathcal{H}$, let $\mathcal{K}(\mathcal{H})$ denote the ideal of all compact operators on $\mathcal{H}$, and let $\mathcal{B}(\mathcal{H})$ be the class of all finite rank operators. Finally, let $\mathcal{C}(\mathcal{H}) = \mathcal{L}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ denote the Calkin algebra. For $A \in \mathcal{L}(\mathcal{H})$, let $[A]$ denote the coset of A in the Calkin algebra $\mathcal{C}(\mathcal{H})$.

(2) For T, a linear operator acting on the Banach space X, we denote by T^*, $\ker(T)$ and $R(T)$, respectively, the adjoint, the kernel and the range of T. Also, we denote by $\overline{R(T)}$, $\overline{R(T)}^{w}$ and $\overline{R(T)}^{w^*}$, respectively, the closure of the range of T with respect to the norm topology, the weak topology and the ultra-weak topology.

(3) Let $C_1(\mathcal{H})$ be the ideal of trace class operators. The ideal $C_1(\mathcal{H})$ admits a complex valued function $tr(T)$ which has the characteristic properties of the trace of matrices. The trace function is defined by

$$tr(T) = \sum_n < Te_n, e_n >,$$

where (e_n) is any complete orthonormal system in $\mathcal{H}$.

(4) As a Banach space, $C_1(\mathcal{H})$ may be identified with the conjugate space of the ideal $\mathcal{K}(\mathcal{H})$ of compact operators by means of the linear isometry $T \longmapsto f_T$, where $f_T(X) = tr(XT)$. Moreover, $\mathcal{H}$ is the dual of $C_1(\mathcal{H})$. The ultra-weak continuous linear functionals on $\mathcal{L}(\mathcal{H})$ are those of the form f_T for some $T \in C_1(\mathcal{H})$, and the weak continuous linear functionals on $\mathcal{L}(\mathcal{H})$ are those of the form f_T, where $T \in \mathcal{B}(\mathcal{H})$.

(5) If φ is a linear functional on $\mathcal{L}(\mathcal{H})$, then φ^*, the adjoint of φ, is defined by $\varphi^*(X) = \overline{\varphi(X^*)}$ for all $X \in \mathcal{L}(\mathcal{H})$.

(6) Recall that for $x, y \in \mathcal{H}$, the operator $x \otimes y \in \mathcal{L}(\mathcal{H})$ is defined by $(x \otimes y)z =< z, y > x$ for all $z \in \mathcal{H}$.

(7) For any subspace $\mathcal{S}$ of $\mathcal{L}(\mathcal{H})$, we denote the polar of $\mathcal{S}$ by

$$\mathcal{S}^\circ = \{f \in \mathcal{L}(\mathcal{H})' \mid f(X) = 0 \text{ for all } X \in \mathcal{S} \}.$$

2. The Range of the Elementary Operator $\triangle_{(A,B),(C,D)}$

Lemma 2.1. *Let S_1 and S_2 be two subspaces of $\mathcal{L}(\mathcal{H})$. Then $S_1^\circ \subset S_2^\circ$ if and only if $S_2 \subset \overline{S_1}$.*

Proof. This is an easy consequence of the bipolar theorem. □

Theorem 2.2. *Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. Then*

$$R(\triangle_{(A,C),(B,D)})^\circ \simeq R(\triangle_{(A,C),(B,D)})^\circ \cap \mathcal{K}(\mathcal{H})^\circ \oplus \ker(\triangle_{(B,D),(A,C)}) \cap C_1(\mathcal{H}).$$

Proof. Let $f = f_0 + f_T$ be the canonical decomposition of continuous linear functional $f \in \mathcal{L}(\mathcal{H})'$ into a trace form part and a functional vanishing on $\mathcal{K}(\mathcal{H})$ [28, p. 276]. Then we have $f \in R(\triangle_{(A,C),(B,D)})^\circ$

if and only if $f_0, f_T \in R(\triangle_{(A,C),(B,D)})^\circ$, and we have $f_T \in R(\triangle_{(A,C),(B,D)})^\circ$ if and only $T \in \ker(\triangle_{(B,D),(A,C)}) \cap C_1(\mathcal{H})$. Indeed, let $x, y \in \mathcal{H}$, then we have

$$\begin{aligned} f(A(x \otimes y)B) &= f_T(A(x \otimes y)B) \\ &= tr(TAx \otimes B^* y) \\ &= < TAx, B^* y > \end{aligned}$$

and

$$\begin{aligned} f(C(x \otimes y)D) &= f_T(C(x \otimes y)D) \\ &= tr(TCx \otimes D^* y) \\ &= < TCx, D^* y > \end{aligned}$$

It follows that

$$< TAx, B^* y > = < TCx, D^* y >,$$

for all $x, y \in \mathcal{H}$ and hence

$$f_T(AXB) = f_T(CXD),$$

for all finite rank operator X. Since the class of finite rank operators is dense in $\mathcal{L}(\mathcal{H})$ relative to the ultra-weak operator topology, it follows that $f_T \in R(\triangle_{(A,C),(B,D)})^\circ$. This implies that

$$f_0 = f - f_T \in R(\triangle_{(A,C),(B,D)})^\circ.$$

Conversely, the preceding computation shows that $BTA = DTC$ and $T \in C_1(\mathcal{H})$, then $f_T \in R(\triangle_{(A,C),(B,D)})^\circ$. The proof is complete. □

Corollary 2.3. *Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. Then the following statements are equivalent:*

(1) $\overline{R(\triangle_{(A,C),(B,D)})}^{w^*} = \mathcal{L}(\mathcal{H})$.
(2) $\mathcal{K}(\mathcal{H}) \subset \overline{R(\triangle_{(A,C),(B,D)})}$.
(3) $\ker(\triangle_{(B,D),(A,C)}) \cap C_1(\mathcal{H}) = \{0\}$.

Proof. The negation of (1) and (3) is equivalent to the fact that there exists a nonzero ultraweakly continuous linear form f_T such that $f_T \in R(\triangle_{(A,C),(B,D)})^\circ$. By Theorem 2.2, this occurs if and only if $R(\triangle_{(A,C),(B,D)})^\circ \not\subset \mathcal{K}(\mathcal{H})^\circ$. It follows from Lemma 2.1 that the last condition is equivalent to $\mathcal{K}(\mathcal{H}) \not\subset \overline{R(\triangle_{(A,C),(B,D)})}$. □

Corollary 2.4. *Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$, then*

$$\overline{R(\triangle_{(A,C),(B,D)})} \cap \mathcal{K}(\mathcal{H}) = \overline{R(\triangle_{(A,C),(B,D)})}^{w^*} \cap \mathcal{K}(\mathcal{H}).$$

Proof. Setting $S := R(\triangle_{(A,C),(B,D)})$, we have trivially $\overline{S} \cap \mathcal{K}(\mathcal{H}) \subset \overline{S}^{w^*} \cap \mathcal{K}(\mathcal{H})$, where

$$\overline{S} \cap \mathcal{K}(\mathcal{H}) = \cap\{\ker(f) \cap \mathcal{K}(\mathcal{H}) \mid f \in \mathcal{L}(\mathcal{H})', f(S) = 0\}$$

and

$$\overline{S}^{w^*} \cap \mathcal{K}(\mathcal{H}) = \cap\{\ker(f_T) \cap \mathcal{K}(\mathcal{H}) \mid T \in C_1(\mathcal{H}), f_T(S) = 0\}.$$

To establish a converse inclusion, we consider any $K \in \overline{S}^{w^*} \cap \mathcal{K}(\mathcal{H})$ and $f \in \mathcal{L}(\mathcal{H})'$ such that $f(S) = 0$ and prove that $f(K) = 0$. By Theorem 2.2, the canonical decomposition $f = f_0 + f_T$ satisfies $f_T(S) = f_0(S) = 0$. Since $K \in \mathcal{K}(\mathcal{H})$, we have $f_0(K) = 0$. On the other hand,

$$K \in \overline{S}^{w^*} \cap \mathcal{K}(\mathcal{H}) = \cap\{\ker(f_T) \cap \mathcal{K}(\mathcal{H}) : T \in C_1(\mathcal{H}), f_T(S) = 0\},$$

which entails $f_T(K) = 0$. Thus, indeed, $f(K) = f_0(K) + f_T(K) = 0$. □

Theorem 2.5. *Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. Then*

(1) *every finite rank operator in* $\overline{R(\triangle_{(A,C),(B,D)})}^{w} \cap \ker(\triangle_{(A^*,C^*),(B^*,D^*)})$ *vanishes;*
(2) *every trace class operator in* $\overline{R(\triangle_{(A,C),(B,D)})}^{w^*} \cap \ker(\triangle_{(A^*,C^*),(B^*,D^*)})$ *vanishes.*

Proof. (1) Let T be a finite rank operator in $\overline{R(\triangle_{(A,C),(B,D)})}^{w} \cap \ker(\triangle_{(A^*,C^*),(B^*,D^*)})$, then $T^* \in \ker(\triangle_{(B,D),(A,C)}) \cap \mathcal{B}(\mathcal{H})$. It follows that f_{T^*} vanishes on the range of $\triangle_{(A,C),(B,D)}$. In particular, $f_{T^*}(T) = tr(T^*T) = 0$, that is, $T^*T = 0$, thus $T = 0$.
(2) It suffices to replace $\mathcal{B}(\mathcal{H})$ with $C_1(\mathcal{H})$ in the above proof. □

Theorem 2.6. *Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. Then:*

(1) $\overline{R(\triangle_{(A,C),(B,D)})}^{w} = \mathcal{L}(\mathcal{H})$ *if and only if* $\ker(\triangle_{(B,D),(A,C)}) \cap \mathcal{B}(\mathcal{H}) = \{0\}$;
(2) $\overline{R(\triangle_{(A,C),(B,D)})}^{w^*} = \mathcal{L}(\mathcal{H})$ *if and only if* $\ker(\triangle_{(B,D),(A,C)}) \cap C_1(\mathcal{H}) = \{0\}$.

Proof. (1) Suppose that $\overline{R(\triangle_{(A,C),(B,D)})}^{w} = \mathcal{L}(\mathcal{H})$ and $T \in \ker(\triangle_{(B,D),(A,C)}) \cap \mathcal{B}(\mathcal{H})$. It follows that $T^* \in \overline{R(\triangle_{(A,C),(B,D)})}^{w} \cap \ker(\triangle_{(A^*,C^*),(B^*,D^*)})$, hence $T = 0$ by Theorem 2.5.

Conversely, assume that there exists $T \in \mathcal{L}(\mathcal{H}) \backslash \overline{R(\triangle_{(A,C),(B,D)})}^{w}$. It follows that there is an operator $S \in \mathcal{B}(\mathcal{H})$ such that $tr(ST) \neq 0$ and $tr(SX) = 0$ for each $X \in R(\triangle_{(A,C),(B,D)})$. Hence, we obtain that $S \in \ker(\triangle_{(B,D),(A,C)}) \cap \mathcal{B}(\mathcal{H})$ and $S \neq 0$.

(2) It suffices to replace $\mathcal{B}(\mathcal{H})$ by $C_1(\mathcal{H})$ in the preceding proof. □

Remark 2.7. If $A, B, C, D \in \mathcal{L}(\mathcal{H})$ such that C and D are invertible and

$$\| A \| \| B \| \| C^{-1} \| \| D^{-1} \| < 1,$$

then Corollary 2.3 and Theorem 2.6 show that

$$\overline{R(\triangle_{(A,C),(B,D)})}^{w} = \overline{R(\triangle_{(A,C),(B,D)})}^{w^*} = \mathcal{L}(\mathcal{H}).$$

Theorem 2.8. *Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. Then:*

(1) $\overline{R(\triangle_{(A_2,C_2),(B_2,D_2)})}^{w} \subset \overline{R(\triangle_{(A_1,C_1),(B_1,D_1)})}^{w}$ *if and only if* $\ker(\triangle_{(B_1,D_1),(A_1,C_1)}) \cap \mathcal{B}(\mathcal{H}) \subset \ker(\triangle_{(B_2,D_2),(A_2,C_2)}) \cap \mathcal{B}(\mathcal{H})$;
(2) $\overline{R(\triangle_{(A_2,C_2),(B_2,D_2)})}^{w^*} \subset \overline{R(\triangle_{(A_1,C_1),(B_1,D_1)})}^{w^*}$ *if and only if* $\ker(\triangle_{(B_1,D_1),(A_1,C_1)}) \cap C_1(\mathcal{H}) \subset \ker(\triangle_{(B_2,D_2),(A_2,C_2)}) \cap C_1(\mathcal{H})$.

Proof. (1) Assume that $\ker(\triangle_{(B_1,D_1),(A_1,C_1)}) \cap \mathcal{B}(\mathcal{H}) \subset \ker(\triangle_{(B_2,D_2),(A_2,C_2)}) \cap \mathcal{B}(\mathcal{H})$. Let f_T be a weakly continuous linear form that vanishes on $R(\triangle_{(A_1,C_1),(B_1,D_1)})$. Then it is easy to see that

$$\begin{aligned} f_T(A_1XB_1 - C_1XD_1) &= tr[T(A_1XB_1 - C_1XD_1)] \\ &= tr[(B_1TA_1 - D_1TC_1)X)] \\ &= 0 \end{aligned}$$

for all $X \in \mathcal{L}(\mathcal{H})$, hence $B_1TA_1 = D_1TC_1$ and

$$T \in \ker(\triangle_{(B_1,D_1),(A_1,C_1)}) \cap \mathcal{B}(\mathcal{H}) \subset \ker(\triangle_{(B_2,D_2),(A_2,C_2)}) \cap \mathcal{B}(\mathcal{H}).$$

Observe that

$$\begin{aligned} f_T(A_2XB_2 - C_2XD_2) &= tr[T(A_2XB_2 - C_2XD_2)] \\ &= 0, \end{aligned}$$

thus f_T annihilates $R(\triangle_{(A_2,C_2),(B_2,D_2)})$. It follows that $\overline{R(\triangle_{(A_2,C_2),(B_2,D_2)})}^{w} \subset \overline{R(\triangle_{(A_1,C_1),(B_1,D_1)})}^{w}$. For the converse implication we reverse the above argument.

(2) It suffices to replace $\mathcal{B}(\mathcal{H})$ by $C_1(\mathcal{H})$ in the preceding proof. □

3. D-symmetric Quadruplets

Definition 3.1. Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. We say that the quadruplet (A, B, C, D) is D-symmetric if

$$\overline{R(\triangle_{(A,C),(B,D)})} = \overline{R(\triangle_{(B^*,D^*),(A^*,C^*)})}.$$

Remark 3.2. (1) Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. The quadruplet (A, B, C, D) is D-symmetric if and only if $\overline{R(\triangle_{(A,C),(B,D)})}$ is a self-adjoint subspace of $\mathcal{L}(\mathcal{H})$. Equivalently, $R(\triangle_{(A,C),(B,D)})^\circ$, the annihilator of $R(\triangle_{(A,C),(B,D)})$, is a self-adjoint subspace of $\mathcal{L}(\mathcal{H})'$ in the sense that $f \in R(\triangle_{(A,C),(B,D)})^\circ$ implies $f^* \in R(\triangle_{(A,C),(B,D)})^\circ$.

(2) If (A, B, C, D) is D-symmetric, then (B^*, A^*, D^*, C^*) is D-symmetric.

(3) For all $A,\ B \in \mathcal{L}(\mathcal{H})$, (A, B, B^*, A^*) is D-symmetric.

(4) If U and V are self-adjoint operators, then (U, U, V, V) is D-symmetric.

Theorem 3.3. *For $A, B, C, D \in \mathcal{L}(\mathcal{H})$, the following statements are equivalent:*

(1) (A, B, C, D) *is D-symmetric.*

(2) **(i)** $([A], [B], [C], [D])$ *is D-symmetric, and*

(ii) $BTA = DTC$ *implies* $BT^*A = DT^*C$ *for all* $T \in C_1(\mathcal{H})$.

Proof. (1) $\Rightarrow$ (2). Suppose that (A, B, C, D) is D-symmetric.

(i) Let $\psi \in R(\triangle_{([A],[B]),([C],[D])})^\circ$. We define a bounded linear functional f on $\mathcal{L}(\mathcal{H})$ by

$$f(X) = \psi([X]).$$

It is clear that $f \in R(\triangle_{(A,C),(B,D)})^\circ$ if and only if $\psi \in R(\triangle_{([A],[B]),([C],[D])})^\circ$. Since (A, B, C, D) is D-symmetric, it follows from the above Remark that $f^* \in R(\triangle_{(A,C),(B,D)})^\circ$ and, consequently, $\psi^* \in R(\triangle_{([A],[B]),([C],[D])})^\circ$. Then $([A], [B], [C], [D])$ is D-symmetric.

(ii) If $BTA = DTC$ and $T \in C_1(\mathcal{H})$, then Theorem 2.2 implies that $f_T \in R(\triangle_{(A,C),(B,D)})^\circ$. Since (A, B, C, D) is D-symmetric, it follows that

$$(f_T)^* = f_{T^*} \in R(\triangle_{(A,C),(B,D)})^\circ,$$

whence we get $BT^*A = DT^*C$.

(2) $\Rightarrow$ (1). Let $f \in R(\triangle_{(A,C),(B,D)})^\circ$ We can write $f = f_0 + f_T$, where $f_0 \in R(\triangle_{(A,C),(B,D)})^\circ \cap \mathcal{K}(\mathcal{H})^\circ$ and $T \in \ker(\triangle_{(B,D),(A,C)}) \cap C_1(\mathcal{H})$. Using **(ii),** one obtains $BT^*A = DT^*C$, that is, $f_{T^*} \in R(\triangle_{(A,C),(B,D)})^\circ$. It remains to show that $f_0^* \in R(\triangle_{(A,C),(B,D)})^\circ$. Let φ be the linear functional on the Calkin algebra defined by

$$\varphi([X]) = f_0(X).$$

Since f_0 vanishes on $\mathcal{K}(\mathcal{H})$, it follows that φ is well defined. From **(i)**, $(([A], [B]), ([C], [D])$ is D-symmetric, hence $\varphi \in R(\triangle_{([A],[B]),([C],[D])})^\circ$ implies that $\varphi^* \in R(\triangle_{([A],[B]),([C],[D])})^\circ$, that is, $f_0^* \in R(\triangle_{(A,C),(B,D)})^\circ$. Thus we have shown that

$$f^* = f_0^* + f_{T^*} \in R(\triangle_{(A,C),(B,D)})^\circ,$$

consequently, (A, B, C, D) is D-symmetric. □

Definition 3.4. Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. We say that the (A, B, C, D) is P-symmetric if $BTA = DTC$ implies $BT^*A = DT^*C$ for all $T \in C_1(\mathcal{H})$.

Theorem 3.5. *Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. (A, B, C, D) is P-symmetric if and only if $\overline{R(\triangle_{(A,C),(B,D)})}^{w^*}$ is self-adjoint.*

Proof. Let $\mathcal{L}(\mathcal{H})'^{w^*}$ be the space of ultra-weakly continuous linear functionals on $\mathcal{L}(\mathcal{H})$. $\overline{R(\triangle_{(A,C),(B,D)})}^{w^*}$ is self-adjoint if and only if $R(\triangle_{(A,C),(B,D)})^\circ \cap \mathcal{L}(\mathcal{H})'^{w^*}$ is self-adjoint. It follows from Theorem 2.2 that

$$R(\triangle_{(A,C),(B,D)})^\circ \cap \mathcal{L}(\mathcal{H})'^{w^*} \cong \ker(\triangle_{(B,D),(A,C)}) \cap C_1(\mathcal{H}).$$

The ultra-weak topology is generated by all f_T with $T \in C_1(\mathcal{H})$ and so, $R(\triangle_{(A,C),(B,D)})^\circ$ is the intersection

$$\cap\{\ker f_T \mid f_T(AXB - CXD) = 0 \ \text{ for all } \ X \in \mathcal{L}(\mathcal{H})\}.$$

Since

$$f_T(AXB - CXD) = tr(T(AXB - CXD) = tr((BTA - DTC)X),$$

this intersection is

$$\ker(\triangle_{(B,D),(A,C)}) \cap C_1(\mathcal{H}).$$

If (A, B, C, D) is P-symmetric, then

$$\ker(\triangle_{(B,D),(A,C)}) \cap C_1(\mathcal{H}) = \ker(\triangle_{(A^*,C^*),(B^*,D^*)}) \cap C_1(\mathcal{H}),$$

and so, the ultra-weak closure of $R(\triangle_{(A,C),(B,D)})$ is self-adjoint.

Conversely, if $\overline{R(\triangle_{(A,C),(B,D)})}^{w^*}$ is self-adjoint, then the set of $T \in C_1(\mathcal{H})$ for which f_T vanishes on $R(\triangle_{(A,C),(B,D)})$ must be self-adjoint, $Y \in R(\triangle_{(A,C),(B,D)})$ implies

$$0 = f_T(Y^*) = tr(TY^*) = \overline{tr(TY^*)}.$$

Hence

$$\ker(\triangle_{(B,D),(A,C)}) \cap C_1(\mathcal{H}) = \ker(\triangle_{(A^*,C^*),(B^*,D^*)}) \cap C_1(\mathcal{H}).$$

Thus (A, B, C, D) is P-symmetric. $\square$

Remark 3.6.

(1) If (A, B, C, D) is P-symmetric, then (B^*, A^*, D^*, C^*) is P-symmetric.

(2) For all $A,\ B \in \mathcal{L}(\mathcal{H})$, (A, B, B^*, A^*) is P-symmetric.

(3) If $A, B, C, D \in \mathcal{L}(\mathcal{H})$ such that C and D are invertible and

$$\| A \| \| B \| \| C^{-1} \| \| D^{-1} \| < 1,$$

then (A, B, C, D) is P-symmetric.

Indeed, suppose that there exists $T \in C_1(\mathcal{H}) \setminus \{0\}$ such that $BTA = DTC$. Then

$$D^{-1}BTAC^{-1} = T.$$

It follows that $\| A \| \| B \| \| C^{-1} \| \| D^{-1} \| \geq 1$. This implies that (A, B, C, D) is P-symmetric if $\| A \| \| B \| \| C^{-1} \| \| D^{-1} \| < 1$.

Theorem 3.7. *Let $\{M, N\}$ be a commuting pair of normal operators. Then (M, M, N, N) is P-symmetric.*

Proof. Using Corollary 1 [27], we can show that

$$\|MXM - NXN\|_{C_2} = \|M^*XM^* - N^*XN^*\|_{C_2}$$

for all $X \in \mathcal{L}(\mathcal{H})$ and $\{M, N\}$, a commuting pair of normal operators, with C_2, is the Hilbert-Schmidt class. $\square$

Theorem 3.8. *Let $A, B, C, D \in \mathcal{L}(\mathcal{H})$. If there exist $\alpha, \beta, \gamma, \lambda \in \mathbb{C}$ with $\gamma \neq \lambda$ and nonzero vectors $f, g \in \mathcal{H}$ such that:*

(1) $Bf = \alpha f,\ Df = \alpha f,\ A^*f = C^*f \neq 0$ *and*

(2) $A^*g = \beta g,\ C^*g = \beta g,\ B^*g = \gamma g,\ D^*g = \lambda g,$

Then (A, B, C, D) is not P-symmetric.

Proof. (A, B, C, D) is P-symmetric if and only if $\overline{R(\triangle_{(A,C),(B,D)})}^{w^*}$ is self-adjoint. Under the preceding hypothesis, we must show that $\overline{R(\triangle_{(A,C),(B,D)})}^{w^*} \neq \overline{R(\triangle_{(B^*,D^*),(A^*,C^*)})}^{w^*}$. We consider the operator $T = g \otimes A^*f$. It is easily seen that

$$< (AXB - CXD)f, g >= 0$$

for all $X \in \mathcal{L}(\mathcal{H})$. On the other hand, one obtains that

$$< (B^*TA^* - D^*TC^*)f, g >= (\gamma - \lambda) \| A^*f \|^2 \| g \|^2 .$$

If $B^*TA^* - D^*TC^* \in \overline{R(\triangle_{(A,C),(B,D)})}^{w^*}$, then there exists a net $(X_\alpha)_\alpha$ in $\mathcal{L}(\mathcal{H})$ such that

$$AX_\alpha B - CX_\alpha D \longrightarrow B^*TA^* - D^*TC^*.$$

This implies that

$$0 =< (AX_\alpha B - CX_\alpha D)f, g >\longrightarrow< (B^*TA^* - D^*TC^*)f, g >= (\gamma - \lambda) \| A^*f \|^2 \| g \|^2 .$$

It follows that $(\gamma - \lambda) \| A^*f \|^2 \| g \|^2 = 0$ wich is absurd. $\square$

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