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# THE TAUTNESS PROPERTY FOR HOMOLOGY THEORIES

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*Dedicated to the memory of Academician Nodar Berikashvili*

**Abstract.** The tautness for a cohomology theory is formulated and studied by various authors. However, the analogous property is not considered for a homology theory. In this paper, we will define and study this very property for the Massey homology theory. Moreover, we will prove that the Kolmogoroff and Massey homologies are isomorphic on the category of locally compact, paracompact spaces and proper maps. Therefore, we will obtain the same result for the Kolmogoroff homology theory.

## 1. Introduction

Let $A$ be a closed subspace of a topological space $X$ and $\{U\}$ be a system of neighborhoods $U$ of $A$, directed by inclusion. Then for each cohomology theory $h^*$ there is a natural homomorphism

$$i^* : \varinjlim h^*(U) \longrightarrow h^*(A). \tag{$*$}$$

It is said that $A$ is tautly embedded in the space $X$ if the homomorphism $i^*$ is an isomorphism [15, §6.1]. The Alexander–Spanier cohomology on the category of paracompact Hausdorff spaces and continuous maps [15, Theorem 2 §6.6] and the Massey cohomology $H_c^*$ on the category of locally compact Hausdorff spaces and proper maps [10, Theorem 6.4, §6.4] are the examples of cohomologies for which any closed subspace $A$ is tautly embedded in $X$. It is natural to ask whether an analogous property holds for the exact homology theory, as well. Therefore, our aim is to investigate a natural homomorphism

$$i_* : h_*(A) \longrightarrow \varprojlim h_*(U) \tag{$**$}$$

for the homology theory. In this paper, it is proved that for the Massey homology $H_*^M$, there exists an infinite exact sequence on the category of locally compact Hausdorff spaces $X$, which includes the homomophism $i_*$. In particular, we have the following main

**Theorem 2.2.** *The system $\{N\}$ of closed neighborhoods $N$ of closed subspace $A$ of a locally compact Hausdorff space $X$, directed by an inclusion, induces the following exact sequence:*

$$\cdots \longrightarrow \varprojlim{}^{(2k+1)} H_{n+k+1}^M(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(3)} H_{n+2}^M(N) \longrightarrow \varprojlim{}^{(1)} H_{n+1}^M(N) \longrightarrow$$
$$\longrightarrow H_n^M(A,G) \xrightarrow{i_n} \varprojlim H_n^M(N) \longrightarrow \varprojlim{}^{(2)} H_{n+1}^M(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2k)} H_{n+k}^M(N) \longrightarrow \cdots,$$

*where $H_*^M(N) = H_*^M(N,G)$ is the Massey homology* [10, §4.6] *of closed neighborhood $N$ with a coefficient in an abelian group $G$.*

It is natural to study the same property for other exact homology theories [3,8,14,16]. Consequently, in the second part of the paper, it is proved that the Kolmogoroff [8, 11] and Massey [10] homologies are isomorphic on the category of locally compact, paracompact spaces and proper maps. Using the obtained result, we will show that for the Kolmogoroff [8], Milnor [14] and Steenrod [16] homology theories the following properties hold:

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**Corollary 3.5.** a) *If $X$ is a locally compact, paracompact Hausdorff space, then for the system $\{N\}$ of closed neighborhoods $N$ of a closed subspace $A$ of $X$, there is an infinite exact sequence*

$$\cdots \longrightarrow \varprojlim{}^{(2k+1)} H^K_{n+k+1}(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(3)} H^K_{n+2}(N) \longrightarrow \varprojlim{}^{(1)} H^K_{n+1}(N) \longrightarrow H^K_n(A,G) \longrightarrow$$
$$\xrightarrow{i_n} \varprojlim H^K_n(N) \longrightarrow \varprojlim{}^{(2)} H^K_{n+1}(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2k)} H^K_{n+k}(N) \longrightarrow \cdots,$$

*where $H^K_*(N) = H^K_*(N,G)$ is the Kolmogoroff homology.*

b) *If $X$ is a compact Hausdorff space, then for the system $\{N\}$ of closed neighborhoods $N$ of a closed subspace $A$ of $X$, there is an infinite exact sequence*

$$\cdots \longrightarrow \varprojlim{}^{(2k+1)} H^{Mi}_{n+k+1}(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(3)} H^M_{n+2}(N) \longrightarrow \varprojlim{}^{(1)} H^{Mi}_{n+1}(N) \longrightarrow H^{Mi}_n(A) \longrightarrow$$
$$\xrightarrow{i_n} \varprojlim H^{Mi}_n(N) \longrightarrow \varprojlim{}^{(2)} H^{Mi}_{n+1}(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2k)} H^{Mi}_{n+k}(N) \longrightarrow \cdots,$$

*where $H^{Mi}_*(N) = H^{Mi}_*(N,G)$ is the Milnor homology* [14].

**Corollary 3.6.** a) *If $X$ is a locally compact Hausdorff space with the second countable axiom, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} H^M_{n+1}(N_i) \longrightarrow H^M_n(A,G) \longrightarrow \varprojlim H^M_n(N_i) \longrightarrow 0,$$

*where $H^M_*$ is the Massey homology* [10].

b) *If $X$ is a locally compact, paracompact Hausdorff space with the second countable axiom, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} H^K_{n+1}(N_i) \longrightarrow H^K_n(A,G) \longrightarrow \varprojlim H^K_n(N_i) \longrightarrow 0,$$

*where $H^K_*$ is the Kolmogoroff homology* [8].

c) *If $X$ is a compact Hausdorff space with the second countable axiom, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} H^{Mi}_{n+1}(N_i) \longrightarrow H^{Mi}_n(A,G) \longrightarrow \varprojlim H^{Mi}_n(N_i) \longrightarrow 0,$$

*where $H^{Mi}_*$ is the Milnor homology* [14].

d) *If $X$ is a compact metric space, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$ there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} H^{st}_{n+1}(N_i) \longrightarrow H^{st}_n(A,G) \longrightarrow \varprojlim H^{st}_n(N_i) \longrightarrow 0,$$

*where $H^{st}_*$ is the Steenrod homology* [16].

## 2. Tautness

In the book [10, §1.1], W. Massey defined the cochain complex $C^*_c(X,G)$ for any locally compact Hausdorff spaces $X$ and any abelian group $G$. By Theorem 4.1 [10, §4.4], for each locally compact Hausdorff space $X$ and each integer $n$, the cochain group $C^n_c(X,\mathbb{Z})$ with integer coefficient is a free abelian group. The chain complex $C_*(X,G) = \mathrm{Hom}(C^*_c(X),G)$ is completely defined by the cochain complex $C^*_c(X)$ with the coefficient group $\mathbb{Z}$ of integers and therefore, by Theorem 4.1 [10, §4.4] and Theorem 4.1 (Universal Coefficients) [9, §III.4], there is an exact sequence [10, Corollary 4.18, §4.8]

$$0 \longrightarrow \mathrm{Ext}(H^{n+1}_c(X),G) \longrightarrow H^M_n(X,G) \longrightarrow \mathrm{Hom}(H^n_c(X),G) \longrightarrow 0, \tag{2.1}$$

where $H^M_n(X,G)$ is the Massey homology group and $H^{n+1}_c(X,G)$ is the Massey cohomology group, respectively [10, §4.6], i.e., $H^M_n(X,G) = H_n(\mathrm{Hom}(C^*_c(X),G))$ and $H^{n+1}_c(X) = H_n(C^*_c(X,\mathbb{Z})$. Moreover, this sequence is split. However, the splitting is natural only with respect to the coefficient homomorphisms.

Let $X$ be a locally compact space and $A$ be a closed subspace of $X$. In this case, for each closed neighborhood $N$ of $A$, there is a homomorphism $i_N : h^n(N) \to h^n(A)$. If $N_1 \subset N_2$, then there is a homomorphism $i_{N_1,N_2} : h^n(N_2) \to h^n(N_1)$. Therefore, there is the direct system $\{h^n(N)\}$ of abelian groups and homomorphisms $\{i_{N_1,N_2}\}$. Consequently, there exists a natural homomorphism

$$i^n : \varinjlim h^n(N) \longrightarrow h^n(A).$$

If $h^* = H_c^*$ is the Massey cohomology [10, §4.6], then (see Theorem 6.4 [10, §6.4]) there is an isomorphism

$$i^n : \varinjlim H_c^n(N,G) \xrightarrow{\sim} H_c^n(A,G). \tag{2.2}$$

In this case, a subspace $A$ is said to be taut with respect to the cohomology theory $H_c^*(-,G)$.

Let $h_*$ be a homology theory on the category of some topological spaces. Let $A$ be a closed subspace of $X$. In this case, for a neighborhood $N$ of $A$, there is a homomorphism $i_N : h_n(A) \to h_n(N)$. If $N_1 \subset N_2$, then there is a homomorphism $i_{N_1,N_2} : h_n(N_1) \to h_n(N_2)$. Therefore, there is the inverse system $\{h_n(N)\}$ of abelian groups and homomorphisms $\{i_{N_1,N_2}\}$. Consequently, there exists a natural homomorphism

$$i_n : h_n(A) \longrightarrow \varprojlim h_n(N).$$

**Definition 2.1.** A closed subspace $A$ of a space $X$ is said to be tautly embedded in $X$, if for some set $N$ of neighborhoods there exists a long exact sequence

$$\cdots \longrightarrow \varprojlim{}^{(2k+1)} h_{n+k+1}(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(3)} h_{n+2}(N) \longrightarrow \varprojlim{}^{(1)} h_{n+1}(N) \longrightarrow$$
$$\longrightarrow h_n(A,G) \xrightarrow{i_n} \varprojlim h_n(N) \longrightarrow \varprojlim{}^{(2)} h_{n+1}(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2k)} h_{n+k}(N) \longrightarrow \cdots ,$$

which contains the homomorphism $h_n(A) \xrightarrow{i_n} \varprojlim h_n(N)$.

Let $H_n^M(X,G) = H_n(\mathrm{Hom}(C_c^*(X),G))$ be the Massey homology group of locally compact Hausdorff spaces. Let $A$ be a closed subspace of $X$ and $N$ be the set of all closed neighborhoods of $A$. Then each homomorphism $i_{N_1,N_2} : N_1 \to N_2$ is a proper map (a map is proper if it is continuous and if an inverse image of any compact subspace is compact) and induces a homomorphism $i_{N_1,N_2} : H_n^M(N_1) \to H_n^M(N_2)$, which defines the homomorphism

$$i_* : H_n^M(A,G) \longrightarrow \varprojlim H_n^M(N,G).$$

Since the short exact sequence (2.1) is natural, there is a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & \mathrm{Ext}(H_c^{n+1}(A),G) & \longrightarrow & H_n^M(A,G) & \longrightarrow & \mathrm{Hom}(H_c^n(A),G) & \longrightarrow 0 \\
& \downarrow \rho' & & \downarrow i_n & & \downarrow \rho'' & \\
0 \longrightarrow & \varprojlim \mathrm{Ext}(H_c^{n+1}(N),G) & \longrightarrow & \varprojlim H_n^M(N,G) & \longrightarrow & \varprojlim \mathrm{Hom}(H_c^n(N),G) & \longrightarrow \\
\longrightarrow & \varprojlim{}^{(1)} \mathrm{Ext}(H_c^{n+1}(N),G) & \longrightarrow & \varprojlim{}^{(1)} H_n^M(N,G) & \longrightarrow & \varprojlim{}^{(1)} \mathrm{Hom}(H_c^n(N),G) & \longrightarrow \cdots
\end{array} \tag{2.3}$$

with exact arrows.

Using isomorphism (2.2) and the properties of functors $\mathrm{Hom}(-,G)$ and $\varinjlim$, there is an isomorphism

$$\mathrm{Hom}(H_c^n(A),G) \approx \mathrm{Hom}(\varinjlim H_c^n(N),G) \approx \varprojlim \mathrm{Hom}(H_c^n(N),G). \tag{2.4}$$

Therefore, a homomorphism $\rho''$ is an isomorphism.

Using the isomorphism (2.4) and the commutative diagram (2.3), we obtain the following commutative diagram:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Ker}\rho' & \xrightarrow{\sim} & \operatorname{Ker} i_n & & & & \\
 & & \downarrow & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Ext}(H_c^{n+1}(A),G) & \longrightarrow & H_n^M(A,G) & \longrightarrow & \operatorname{Hom}(H_c^n(A),G) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\rho'} & & \downarrow{\scriptstyle i_n} & & \downarrow{\scriptstyle\simeq} & & \\
0 & \longrightarrow & \varprojlim \operatorname{Ext}(H_c^{n+1}(N),G) & \longrightarrow & \varprojlim H_n^M(N,G) & \longrightarrow & \varprojlim \operatorname{Hom}(H_c^n(N),G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & & & \\
 & & \operatorname{Coker}\rho' & \xrightarrow{\sim} & \operatorname{Coker} i_n & & & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & 0 & & 0. & & & &
\end{array} \tag{2.5}$$

By Lemma 1 [13], if a complex $C_*$ is free, then there is an exact sequence

$$0 \longrightarrow \operatorname{Hom}(B_{n-1},G) \longrightarrow Z^n \longrightarrow \operatorname{Hom}(H_n,G) \longrightarrow 0,$$

where $B_{n-1} = \operatorname{Im}\,\partial_n$, $\partial : C_n \to C_{n-1}$ and $Z^n = \operatorname{Ker}\delta^{n+1}$, $\delta^{n+1} : C^n \to C^{n+1}$, where $C^* = \operatorname{Hom}(C_*,G)$. In our case, we have a dual version. In particular, the cochain complex $C_c^*(-)$ is free and hence there is an exact sequence

$$0 \longrightarrow \operatorname{Hom}(B_c^{n+1}(-),G) \longrightarrow Z_n \longrightarrow \operatorname{Hom}(H_c^n(-),G) \longrightarrow 0,$$

where $Z_n = \operatorname{Ker}\partial_n$, $\partial_n : C_n \to C_{n-1}$ and $B_c^{n+1} = \operatorname{Im}\delta^n$, $\delta^n : C_c^n \to C_c^{n+1}$, where $C_* = \operatorname{Hom}(C_c^*,G)$. Consequently, using [13, Lemma 2], for each $A \subset N$, there is a commutative diagram with exact arrows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}(B_c^{n+1}(N),G) & \longrightarrow & Z_n & \longrightarrow & \operatorname{Hom}(H_c^n(N),G) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \| & & \\
0 & \longrightarrow & \operatorname{Ext}(H_c^{n+1}(N),G) & \longrightarrow & H_n(N,G) & \longrightarrow & \operatorname{Hom}(H_c^n(N),G) & \longrightarrow & 0\,.
\end{array} \tag{2.6}$$

**Theorem 2.1.** *Let $\{C_c^*(N)\}$ be a direct system of free chain complexes $C_c^*(N)$ of closed neighborhoods $N$ of a closed subspace $A$ of locally compact Hausdorff spaces $X$ and let $G$ be an abelian group. In this case, for each $n \in \mathbb{Z}$ and $i \geq 1$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(i)} \operatorname{Ext}(H_c^{n+1}(N),G) \longrightarrow \varprojlim{}^{(i)} H_n^M(N,G) \longrightarrow \varprojlim{}^{(i)} \operatorname{Hom}(H_c^*(N),G) \longrightarrow 0, \tag{2.7}$$

*which splits for $i \geq 2$.*

*Proof.* Using the split sequence (2.1) and commutative diagram (2.6), we obtain the following commutative diagram with the exact arrows

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & \varprojlim{}^{(i)} \operatorname{Hom}(B_c^{n+1}(N),G) & \longrightarrow & \varprojlim{}^{(i)} Z_n & \longrightarrow & \varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N),G) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \| & & \\
\cdots & \longrightarrow & \varprojlim{}^{(i)} \operatorname{Ext}(H_c^{n+1}(N),G) & \longrightarrow & \varprojlim{}^{(i)} H_n^M(N,G) & \longrightarrow & \varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N),G) & \longrightarrow & \cdots\,.
\end{array} \tag{2.8}$$

In the paper [6], it is shown that for each direct system $\{A_\alpha\}$ of abelian groups $A_\alpha$, there exists a short exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(A_\alpha, G) \longrightarrow \operatorname{Ext}(\varinjlim A_\alpha, G) \longrightarrow \varprojlim \operatorname{Ext}(A_\alpha, G) \longrightarrow$$
$$\longrightarrow \varprojlim{}^{(2)} \operatorname{Hom}(A_\alpha, G) \longrightarrow 0, \tag{2.9}$$

and for each $i \geq 1$, there is an isomorphism

$$\varprojlim{}^{(i)} \operatorname{Ext}(A_\alpha, G) \approx \varprojlim{}^{(i+2)} \operatorname{Hom}(A_\alpha, G). \tag{2.10}$$

Consider a direct system $\{B_c^{n+1}(N)\}$ of free groups $B_c^{n+1}(N)$. In this case, by the exact sequence (2.9) and the isomorphism (2.10), we have

$$\varprojlim{}^{(i)} \operatorname{Hom}(B_c^{n+1}(N), G) = 0 \quad \text{for } \ i \geq 2. \tag{2.11}$$

By diagram (2.8) and equality (2.11), we have

a) an isomorphism $\varprojlim{}^{(i)} Z_n \approx \varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N), G)$ for each $i \geq 2$;

b) an epimorphism $\varprojlim{}^{(i)} H_n^M(N, G) \longrightarrow \varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N), G)$ for each $i \geq 1$;

c) a monomorphism $\varprojlim{}^{(i)} \operatorname{Ext}(H_c^{n+1}(N), G) \longrightarrow \varprojlim{}^{(i)} H_n^M(N, G)$ for each $i \geq 2$;

d) the trivial homomorphism $\varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N), G) \longrightarrow \varprojlim{}^{(i+1)} \operatorname{Ext}(H_c^{n+1}(N), G)$ for each $i \geq 1$.

By b) and c), for each $i \geq 2$, we have a short exact sequence

$$0 \longrightarrow \varprojlim{}^{(i)} \operatorname{Ext}(H_c^{n+1}(N), G) \longrightarrow \varprojlim{}^{(i)} H_n^M(N, G) \longrightarrow \varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N), G) \longrightarrow 0. \tag{2.12}$$

On the other hand, by a), for each $i \geq 2$, we can define a homomorphism

$$\varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N), G) \xrightarrow{\sim} \varprojlim{}^{(i)} Z_n \longrightarrow \varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N), G).$$

It is clear that the composition

$$\varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N), G) \xrightarrow{\sim} \varprojlim{}^{(i)} Z_n \longrightarrow \varprojlim{}^{(i)} H_n^M(N, G) \longrightarrow \varprojlim{}^{(i)} \operatorname{Hom}(H_c^n(N), G)$$

is the identity map. Therefore, for each $i \geq 2$, the sequence (2.12) splits. □

**Theorem 2.2.** *The system $\{N\}$ of closed neighborhoods $N$ of closed subspace $A$ of a locally compact Hausdorff space $X$, directed by an inclusion, induces the following exact sequence*

$$\cdots \longrightarrow \varprojlim{}^{(2k+1)} H_{n+k+1}^M(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(3)} H_{n+2}^M(N) \longrightarrow \varprojlim{}^{(1)} H_{n+1}^M(N) \longrightarrow$$
$$\longrightarrow H_n^M(A) \xrightarrow{i_n} \varprojlim H_n^M(N) \longrightarrow \varprojlim{}^{(2)} H_{n+1}^M(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2k)} H_{n+k}^M(N) \longrightarrow \cdots,$$

*where $H_*^M(-) = H_*^M(-, G)$ is the Massey homology with a coefficient abelian group $G$.*

*Proof.* By diagram (2.5) and the property d) from Theorem 2.1, we have the following exact sequence:

$$0 \longrightarrow \varprojlim \operatorname{Ext}(H_c^{n+1}(N), G) \longrightarrow \varprojlim H_n^M(N, G) \longrightarrow \varprojlim \operatorname{Hom}(H_c^n(N), G) \longrightarrow$$
$$\longrightarrow \varprojlim{}^{(1)} \operatorname{Ext}(H_c^{n+1}(N), G) \longrightarrow \varprojlim{}^{(1)} H_n^M(N, G) \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(H_c^n(N), G) \longrightarrow 0.$$

By isomorphism (2.4) and exact commutative diagram (2.3), we will obtain the following exact sequence:

$$0 \longrightarrow \varprojlim{}^{(1)} \operatorname{Ext}(H_c^{n+1}(N), G) \longrightarrow \varprojlim{}^{(1)} H_n^M(N) \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(H_c^n(N), G) \longrightarrow 0, \tag{2.13}$$

where by Theorem 2.1, the sequence (2.13) splits for each $i \geq 2$.

Note that by (5) and (9), there is an exact sequence

$$0 \longrightarrow \varprojlim{}^{(1)} \operatorname{Hom}(H_c^{n+1}(N), G) \longrightarrow H_n^M(A, G) \longrightarrow \varprojlim H_n^M(N, G) \longrightarrow$$
$$\longrightarrow \varprojlim{}^{(2)} \operatorname{Hom}(H_c^{n+1}(N), G) \longrightarrow 0. \tag{2.14}$$

By the exact sequences (2.7), (2.13) and (2.14), for $i \geq 1$, isomorphism (2.10) and Theorem 2.1 we have the exact sequence

$$
\begin{array}{ccccccccc}
 & & & & \vdots & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \varprojlim^{(3)} \operatorname{Ext}(H_c^{n+3}(N),G) & \longrightarrow & \varprojlim^{(3)} H^M_{n+2}(N) & \longrightarrow & \varprojlim^{(3)} \operatorname{Hom}(H_c^{n+2}(N),G) & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow \simeq & & \\
0 & \longleftarrow & \varprojlim^{(1)} \operatorname{Hom}(H_c^{n+1}(N),G) & \longleftarrow & \varprojlim^{(1)} H^M_{n+1}(N) & \longleftarrow & \varprojlim^{(1)} \operatorname{Ext}(H_c^{n+2}(N),G) & \longleftarrow & 0 \\
 & & \| & & \downarrow & & & & \\
0 & \longrightarrow & \varprojlim^{(1)} \operatorname{Hom}(H_c^{n+1}(N),G) & \longrightarrow & H^M_n(A,G) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \varprojlim H^M_n(N,G) & \longrightarrow & \varprojlim^{(2)} \operatorname{Hom}(H_c^{n+1}(N),G) & \longrightarrow & 0 \\
 & & & & \downarrow & & \| & & \\
0 & \longleftarrow & \varprojlim^{(2)} \operatorname{Ext}(H_c^{n+2}(N),G) & \longleftarrow & \varprojlim^{(2)} H^M_{n+1}(N) & \longleftarrow & \varprojlim^{(2)} \operatorname{Hom}(H_c^{n+1}(N),G) & \longleftarrow & 0 \\
 & & \downarrow \simeq & & \downarrow & & & & \\
0 & \longrightarrow & \varprojlim^{(4)} \operatorname{Hom}(H_c^{n+2}(N),G) & \longrightarrow & \varprojlim^{(4)} H^M_{n+2}(N) & \longrightarrow & \varprojlim^{(4)} \operatorname{Ext}(H_c^{n+3}(N),G) & \longrightarrow & 0 \\
 & & & & \downarrow & & & & \\
 & & & & \vdots & & & &
\end{array}
$$

The theorem is proved. □

## 3. The Kolmogoroff Homology

Our aim is to study the tautness property for other exact homology theories [3, 8, 14, 16]. Among them, one of the main places is taken by the Kolmogoroff homology, which was defined as early as in 1936 [8, 12]. A. N. Kolmogoroff defined homology on the category of locally compact Hausdorff spaces and proper maps with a compact coefficient group [8, 12]. Using the homology defined by all finite partitions, G. S. Chogoshvili in his paper [3] proved that the Kolmogoroff homology and Alexandroff-Čech homology groups are isomorphic on the category of compact Hausdorff spaces for a compact coefficient group [12]. Since the Steenrod and Alexandroff-Čech homologies are isomorphic on the category of compact metric spaces for a compact coefficient group [16], we have the isomorphisms

$$H^K_*(X,G) \overset{1}{\approx} H^{ch}_*(X,p,G) \overset{2}{\approx} H^{ch}_*(X,sp,G) \overset{3}{\approx} \check{H}_*(X,G) \overset{4}{\approx} H^{st}_*(X,G), \tag{3.1}$$

where $H^K_*(-,G)$ is the Kolmogoroff [8, 12], $H^{ch}_*(-,p,G)$ is the Chogoshvili projective [3, 12], $H^{ch}_*(-,sp,G)$ is the Chogoshvili spectral [3, 12], $\check{H}_*(-,G)$ is the Alexandroff-Čech [4] and $H^{st}_*(-,G)$ is the Steenrod [16] homology theory. Later, the Kolmogoroff and Chogoshvili homology theories were generalized and defined even for a discrete coefficient groups [12]. However, there are no isomorphisms 2 and 4 as in (3.1) [12]. Consequently, there was a natural interest in finding the connection between the Kolmogoroff and Steenrod homology groups for any discrete groups. Using the Uniqueness Theorem given by Milnor [14], it is proved in [11] that on the category of compact metric spaces the Kolmogoroff and the Steenrod homologies are isomorphic even for any discrete coefficient groups [12]. Therefore, to study the tautness properties for an exact homology theory, it is crucial to find a connection between the Kolmogoroff and Massey homology theories.

By Theorem 2.8 [10, §2.2], if $X$ is a locally compact Hausdorff oncompact space and $\dot{X}$ is its one-point Alexandroff compactification, then the inclusion $\mu : X \to \dot{X}$ induces an isomorphism

$$\mu^* : H_c^q(X, G) \xrightarrow{\sim} H_c^q(\dot{X}, *, G). \tag{3.2}$$

**Corollary 3.1.** *The inclusion $\rho : X \to (\dot{X}, *)$, where $\dot{X}$ is the one-point Alexandroff compactification of a locally compact Hausdorff space $X$, indices an isomorphism*

$$\rho_* : H_*^M(X, G) \xrightarrow{\sim} H_*^M(\dot{X}, *, G). \tag{3.3}$$

*Proof.* The inclusion $\rho : X \to (\dot{X}, *)$ induces a commutative diagram with the exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \operatorname{Ext}(H_c^{n+1}(X), G) & \longrightarrow & H_n^M(X, G) & \longrightarrow & \operatorname{Hom}(H_c^n(X), G) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \rho'} & & \downarrow{\scriptstyle \rho_n} & & \downarrow{\scriptstyle \rho''} & & \\ 0 & \longrightarrow & \operatorname{Ext}(H_c^{n+1}(\dot{X}, *), G) & \longrightarrow & H_n^M(\dot{X}, *, G) & \longrightarrow & \operatorname{Hom}(H_c^n(\dot{X}, *), G) & \longrightarrow & 0\,. \end{array}$$

By isomorphism (3.2), the homomorphisms $\rho'$ and $\rho''$ are the isomorphisms, as well. Therefore, by the Lemma of Five Homomorphisms, we obtain the required statement. □

Now, we will define the Kolmogoroff homology theory and, using the isomorphism (3.3), we will find its connection with the Massey homology theory.

Let $X$ be a locally compact Hausdorff space. A subset $A$ of space $X$ is called bounded if $\bar{A}$ is compact [4, Definition 6.1, §X.6].

**Definition 3.1.** Let $X$ be a locally compact space, $E_X$ be the set of all bounded subsets $E_i$ of $X$, and let $G$ be an abelian group. Denote by $E_X^{n+1} = E_X \times E_X \times \cdots \times E_X$ - a direct product of $E_X$. An $n$-dimensional Kolmogoroff chain of the space $X$ is called a function $f_n : E_X^{n+1} \to G$ satisfying the following conditions:

$K1)$ If $E_i = E_i' \cup E_i''$ and $E_i' \cap E_i'' = \varnothing$, then

$$f_n(E_0, \ldots, E_i, \ldots, E_n) = f_n(E_0, \ldots, E_i', \ldots, E_n) + f_n(E_0, \ldots, E_i'', \ldots, E_n);$$

$K2)$ $f_n$ will not change under even permutation and changes just the sign under odd permutation of argument; $f_n = 0$, if two arguments are the same;

$K3)$ If $\overline{E}_0 \cap \cdots \cap \overline{E}_n = \varnothing$, then $f_n(E_0, \ldots, E_n) = 0$.

The sum $f_n' + f_n''$ of two $f_n'$, $f_n''$ functions is defined by the following equation:

$$(f_n' + f_n'')(E_0, \ldots, E_n) = f_n'(E_0, \ldots, E_n) + f_n''(E_0, \ldots, E_n).$$

It is clear that the set of all $n$-dimensional functions $f_n$ is an abelian group, which is denoted by $K_n(X, G)$. The boundary operator $\Delta : K_n(X, G) \to K_{n-1}(X, G)$ is defined by the equation

$$\Delta f_n(E_0, \ldots, E_{n-1}) = f_n(U, E_0, \ldots, E_{n-1}),$$

where $U$ is an open bounded subset which includes $\bigcup_{i=1}^{n-1} \overline{E}_i$. Since the space $X$ is locally compact, such $U$ exists and the boundary operator $\Delta$ does not depend on the choice of $U$.

The homology of the chain complex $K_*(X, G) = \{K_n(X, G), \Delta\}$ is called the Kolmogoroff homology of a locally compact space $X$ and it is denoted by $H_*^K(X, G)$.

**Definition 3.2.** A locally finite system of bounded subspaces $e_i$ of space $X$, which are pairwise non-intersecting and their sum gives the whole space $X = \cup e_i$, is called a regular partition.

**Lemma 3.1.** *For each locally compact, paracompact space $X$ there exits a regular partition.*

*Proof.* Since $X$ is a locally compact space, for each point $x \in X$, there exists a bounded neighborhood $U_x$. Since the space $X$ is paracompact as well, an open covering $\{U_x\}_{x\in X}$ has a locally finite refinement $\{O_\lambda\}$, which is contained in bounded subspaces $O_\lambda$. If we write the elements of the covering $\{O_\lambda\}$ as a transfinite sequence $O_1, O_2, \dots, O_\lambda, \dots$, then we construct a regular covering in the following way: $O_1, O_2 \setminus O_1, \dots, O_\lambda \setminus \bigcup\limits_{i<\lambda} O_i$, where $O_i$ runs through all the ordinal numbers preceding $\lambda$. □

Denote by $S = \{S_\alpha\}$ the system of all regular partitions $S_\alpha$ of a space $X$.

**Lemma 3.2.** *Each compact subspace $F$ of a locally compact space has a nonempty intersection only with a finite number of closures $e_i^\alpha \in S_\alpha$.*

*Proof.* Since $S_\alpha$ is a locally finite system, for each point $x \in F$, there exists a neighborhood $U_x$, which has a nonempty intersection only with finitely many elements $e_i^\alpha \in S_\alpha$. From the collection $\{U_x\}_{x\in F}$ of the neighborhoods a finite subsystem can be chosen whose union covers the space $F$. Since for each open subspace $U$ and subspace $B$ there is an equivalence $U \cap B \neq \varnothing \Leftrightarrow U \cap \overline{B} \neq \varnothing$, we obtain the validity of the lemma. □

Denote by $N_\alpha$ the nerve of a regular partition $S_\alpha \in S$, which consists of simplexes $\sigma^n = (e_0^\alpha, \dots, e_n^\alpha)$, for which $\cap \overline{e}_i^\alpha \neq \varnothing$. By Lemma 3.2, the nerve $N_\alpha$ is locally finite [3, 12].

If $S_\alpha < S_\beta$, i.e., $S_\beta$ is a refinement of $S_\alpha$ and if for each vertex $e_j^\beta \in N_\beta$ we take the uniquely defined vertex $e_i^\alpha \in N_\alpha$, which contains $e_j^\beta$, then we obtain a simplicial map $\pi_{\beta\alpha} : N_\beta \to N_\alpha$. By Lemma 3.2, the map $\pi_{\beta\alpha}$ will be locally finite [4], i.e., an inverse image of each simplex contains only finitely many numbers of simplexes.

If we take for each $S_\alpha \in S$ the group of infinite chains $C_n^{inf}(N_\alpha, G)$ of the nerve $N_\alpha$ and homomorphisms $\pi_{\beta\alpha}^* : C_n^{inf}(N_\beta, G) \to C_n^{inf}(N_\alpha, G)$, induced by simplicial maps $\pi_{\beta\alpha}$, then we obtain an inverse system $\{C_n^{inf}(N_\alpha, G), \pi_{\beta\alpha}^*\}$, the inverse limit group of which is denoted by

$$C_n^{inf}(X, G) = \varprojlim \{C_n^{inf}(N_\alpha, G), \pi_{\beta\alpha}^*\}.$$

The boundary operator $\partial : C_n^{inf}(X, G) \to C_{n-1}^{inf}(X, G)$ is defined by the boundary operators $\partial_\alpha$ : $C_n^{inf}(N_\alpha, G) \to C_{n-1}^{inf}(N_\alpha, G)$, which commute with homomorphisms $\pi_{\beta\alpha}^*$. The homology group of the obtained complex $C_*^{inf}(X, G)$ is called the Chogoshvili projection homology group and denoted by $H_*^{ch}(X, p, G)$.

**Definition 3.3.** Let $A = \{A_i\}$ and $B = \{B_j\}$ be finite systems of sets such that $B = \{B_j\}$ consists of pairwise non-intersecting sets. We say that a system $B$ is a mosaic of the system $A$ if for each $B_j \in B$, there exists $A_i \in A$ such that $B_j \subset A_i$ and $A_i = \bigcup\limits_j B_{i_j}$, where $B_{i_j} \in B$.

**Lemma 3.3.** *For each finite system $A = \{A_i\}$, $i = 0, \dots, n$ of sets $A_i$, there exists a mosaic.*

*Proof.* The system consisting of the subspaces $\bigcap\limits_{i=0}^{n} A_i$, $\bigcap\limits_{t=1}^{n} A_t \setminus \bigcup\limits_{i_* \neq i_t} A_{i_*}$, where $i_1, \dots, i_p - p$ are different indices from the system $i = 0, \dots, n$, $1 \leq p \leq n$ and $i_*$ obtains all the value in the same system, except $i_1, \dots, i_p$, is a mosaic. □

**Lemma 3.4.** *If $\widetilde{f}_n$ is a function on the directed system $e_0, \dots, e_n$, mutually non-intersecting bounded subspaces $e_i$ of locally compact space $X$, which satisfies the conditions $K1)$–$K3)$, then it can be extended to the function $f_n \in K_n(X, G)$.*

*Proof.* By Lemma 3.3, for each directed system $E_0, \dots, E_n$ of bounded subspaces $E_i$, there exists a mosaic $\{e_{i_j}\}$ such that $E_i = \cup e_{i_j}$. Therefore, the function $f_n(E_0, \dots, E_n) = \sum\limits_{0_j, \dots, n_j} \widetilde{f}_n(e_{0_j}, \dots, e_{n_j})$ does not depend on the choice of mosaic. Indeed, let $\{e'_{i_j}\}$ be another mosaic of the system $E_0, \dots, E_n$ and $f'_n(E_0, \dots, E_n) = \sum\limits_{0_j, \dots, n_j} \widetilde{f}_n(e'_{0_j}, \dots, e_{n'_j})$. It is clear that the intersection $\{e_{i_j}\} \wedge \{e'_{i_j}\} = \{e''_{i_j}\}$ is a mosaic not only for $\{E_i\}$, but for each mosaic. Therefore, we have

$$f_n(E_0,\dots,E_n)=\sum \widetilde{f}_n(e_{0_j},\dots,e_{n_j})=\sum \widetilde{f}_n(e''_{0_j},\dots,e''_{n_j})$$
$$=\sum \widetilde{f}_n(e'_{0_j},\dots,e'_{n_j})=f'_n(e_0,\dots,e_n).$$

Thus, the defined function $f_n$, satisfies the conditions $K1)$–$K3)$ and so, $f_n\in K_n(X,G)$. □

**Theorem 3.1.** *Let $X$ be a locally compact, paracompact Hausdorff space. Then the Kolmogoroff homology $H^K_*(X,G)$ is isomorphic to the Chogoshvili projection homology $H^{ch}_*(X,p,G)$.*

*Proof.* We will prove much stronger statement. In particular, there is an isomorphism of chain complexes $K_*(X,G)$ and $C^{inf}_*(X,G)$,

$$K_*(X,G)\approx C^{inf}_*(X,G). \tag{3.4}$$

For each $S_\alpha\in S$, define a homomorphism $\xi_\alpha: K_*(X,G)\to C^{inf}_*(N_\alpha,G)$ by the formula $\xi_\alpha f_n(e^\alpha_0,\dots,e^\alpha_n)=f_n(e^\alpha_0,\dots,e^\alpha_n)$, where $f_n\in K_n(X,G)$, $(e^\alpha_0,\dots,e^\alpha_n)\in N_\alpha$. Therefore,

$$\xi_\alpha\Delta f_n(e^\alpha_0,\dots,e^\alpha_{n-1})=\Delta f_n(e^\alpha_0,\dots,e^\alpha_{n-1})=f_n(U,e^\alpha_0,\dots,e^\alpha_{n-1}).$$

By Lemma 3.2 and the property of the uniqueness of a function $f_n$ (property $K1)$), we have

$$f_n(U,e^\alpha_0,\dots,e^\alpha_{n-1})=f_n(\cup e^\alpha_{i_t},e^\alpha_0,\dots,e^\alpha_{n-1})=\sum_{i_t} f_n(e^\alpha_{i_t},e^\alpha_0,\dots,e^\alpha_{n-1}).$$

Therefore

$$\xi_\alpha\Delta f_n(e^\alpha_0,\dots,e^\alpha_{n-1})=\sum_{i_t} f_n(e^\alpha_{i_t},e^\alpha_0,\dots,e^\alpha_{n-1})=\partial_\alpha f_n(e^\alpha_0,\dots,e^\alpha_{n-1}),$$

i.e., $\xi_\alpha\Delta=\partial_\alpha\xi_\alpha$.

A homomorphism $\xi_\alpha$ induces an isomorphism

$$\xi: K_*(X,G)\longrightarrow C^{inf}_*(X,G).$$

Let $c_n=\{c_{\alpha,n}\}\in C^{inf}_n(X,G)$ and $E_0,\dots,E_n$ be a system of mutually non-intersecting bounded subspaces. If we add to this system the subspace $X\setminus\bigcup_i E_i$, then we obtain a finite partition $D$ of space $X$. Let $S_\alpha\in S$, then $D\wedge S_\alpha=S_{\alpha'}\in S$ for each $E_i=\cup e^{\alpha'}_{i_j}$, where $e^{\alpha'}_{i_j}\in S_{\alpha'}$.

Let $\widetilde{f}_n$ be a function of the system $E_0,\dots,E_n$ which is defined by

$$\widetilde{f}_n(E_0,\dots,E_n)=\sum_{0_i,\dots,n_j} c_{\alpha',n}(e^{\alpha'}_{0_j},\dots,e^{\alpha'}_{n_j}),$$

where $0_j,\dots,n_j$ get all values, where $(e^{\alpha'}_{0_j},\dots,e^{\alpha'}_{n_j})$ denotes a simplex in $N_{\alpha'}$. It is easy to show that such defined function $\widetilde{f}_n$ does not depend on the choice of $S_\alpha$ and it satisfies the properties $K1)$–$K3)$. By Lemma 3.4, a function $\widetilde{f}_n$ can be extended to a function $f_n\in K_n(X,G)$. If we define a homomorphism

$$\eta: C^{inf}_*(X,G)\longrightarrow K_*(X,G)$$

by $\eta(c_n)=f_n$, then it will be inverse of the homomorphism $\xi$. □

**Theorem 3.2.** *Let $\{G_\alpha,p_{\beta\alpha}\}_{\alpha\in\Lambda}$ be a direct system of free abelian groups $G_\alpha$, which satisfies the following conditions:*

1) *For each group $G_\alpha$, there exists a base $B=\{g^\alpha_1,g^\alpha_2,\dots,g^\alpha_\tau,\dots\}$;*
2) *For each pair $\alpha<\beta$, $\alpha,\beta\in\Lambda$, a set of indices $\{1,2,\dots,\tau(\beta),\dots\}$ of elements of a base $B_\beta$ can be decomposed with non-intersecting finite subspaces $I^{\alpha\beta}_1,I^{\alpha\beta}_2,\dots,I^{\alpha\beta}_{\tau(\alpha)}$ such that*

$$p_{\alpha\beta}(g^\alpha_i)=\begin{cases}\sum\limits_{j\in I^{\alpha\beta}_i} g^\beta_j, & \text{if}\quad I^{\alpha\beta}_i\neq\varnothing,\\ 0, & \text{if}\quad I^{\alpha\beta}_i=\varnothing,\end{cases}$$

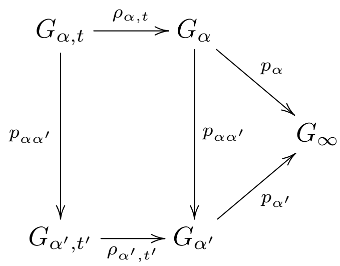
*for $i=1,2,\dots,\tau(\alpha),\dots$ .*

*Then the limit of the direct system $\{G_\alpha,p_{\alpha\beta}\}_{\alpha\in\Lambda}$ is a free group.*

*Proof.* Denote by $\overline{B}_\alpha$ the set of all finite subspaces $\alpha_t$ of a base $B_\alpha$ of a group $G_\alpha$ and by $G_{\alpha_t}$ a subgroup of group $G_\alpha$, generated by all elements $\alpha_t \in \overline{B}_\alpha$. It is possible to prove that such a group $G_\alpha$ is the direct limit group of the direct system of subgroups $G_{\alpha_t}$.

Let $\overline{\Lambda}$ be a set $\{(\alpha,t) \mid \alpha_t \in \overline{B}_\alpha\}$. It is considered that $(\alpha',t') > (\alpha,t)$, if $\alpha' > \alpha$ and $p_{\alpha\alpha'}G_{\alpha_t} \subset G_{\alpha'_{t'}}$. It is clear that $\overline{\Lambda}$ is a directed set and if we take $G_{\alpha,t} = G_{\alpha_t}$ for each pair $(\alpha,t) \in \overline{\Lambda}$, we obtain a direct system $\{G_{(\alpha,t)}, p_{\alpha\beta}\}$ which satisfies the condition of Theorem 3 [7]. Therefore, the direct limit of the given system is a free abelian group.

Let $G_\infty = \varinjlim G_\alpha$ and $G^*_\infty = \varinjlim G_{(\alpha,t)}$. Define a homomorphism $\varphi : G^*_\infty \to G_\infty$. Since $(\alpha',t') > (\alpha,t)$, we have the following commutative diagram:




A homomorphism $\varphi$ is induced by $\varphi_{\alpha,t} = p_\alpha \rho_{\alpha,t}$.

a) $\varphi$ is an epimorphism. Let $x \in G_\infty$ and $x_\alpha \in G_\alpha$ be their representatives. Since $G_\alpha = \varinjlim G_{\alpha,t}$, there is a representative $x_{\alpha,t} \in G_{\alpha,t}$ of an element $x_\alpha$. It is clear that a class $x^* \in G^*_\infty$, whose representative is $x_{\alpha,t}$, satisfies the properties $\varphi(x^*) = x$.

b) $\varphi$ is a monomorphism. Let $\varphi(x^*) = 0$. Since $p_\alpha \rho_{\alpha,t}(x_{\alpha,t}) = 0$, where $x_{\alpha,t}$ is a representative of an element $x^*$, there is $\beta > \alpha$ such that $p_{\alpha\beta}(\rho_{\alpha,t}(x_{\alpha,t})) = 0$. Let $G_{\beta,t'}$ be the subgroup of a group $G_\beta$ generated by all $g^\beta_j \in B_\beta$ such that $p_{\alpha\beta}(g^\alpha_i) = \sum g^\beta_j$, when $g^\alpha_i$ runs through the base $G_{\alpha,t}$. Since $\rho_{\beta',t'} p_{\alpha\beta}(x_{\alpha,t}) = p_{\alpha\beta}\rho_{\alpha,t}(x_{\alpha,t}) = 0$ and $\rho_{\beta,t'}$ are monomorphisms, $p_{\alpha\beta}(x_{\alpha,t}) = 0$ and so, $x^* = 0$. □

**Remark 3.1.** Theorem 3.2 is a generalization of Theorem 3 [7] proven in the case where the base $B_\alpha$ is finite.

**Theorem 3.3.** *Let $X$ be a locally compact, paracompact Hausdorff space, then there exists the universal coefficient formula for the Kolmogoroff homology group*

$$0 \longrightarrow \mathrm{Ext}(\check{H}^{n+1}(\dot{X},*),G) \longrightarrow H^K_n(X,G) \longrightarrow \mathrm{Hom}(\check{H}^n(\dot{X},*),G) \longrightarrow 0.$$

*Proof.* It is easy to see that the direct system $\{C^n_f(N_\alpha), \pi^{\alpha\beta}_*\}$ of the groups $C^n_f(N_\alpha)$ of cochains with an integer coefficient group of nerves $N_\alpha$, where $S_\alpha \in S$, satisfies the condition of Theorem 3.2. Therefore, the direct limit $C^n_f(X) = \varinjlim (C^n_f(N_\alpha), \pi^{\alpha\beta}_*)$ is a free group. By Theorem 4.1 [9, §III.4], for the homology group $H_n(\mathrm{Hom}(C^*_f(X),G))$, there exists the Universal Coefficient Formula

$$0 \longrightarrow \mathrm{Ext}(H^{n+1}_f(X),G) \longrightarrow H_n(\mathrm{Hom}(C^*_f(X),G)) \longrightarrow \mathrm{Hom}(H^n_f(X),G) \longrightarrow 0. \tag{3.5}$$

Since $\mathrm{Hom}(C^*_f(X),G) \approx C^{inf}_*(X,G)$, by isomorphism (3.4) and Theorem 3.1, there exists an isomorphism

$$H_*(\mathrm{Hom}(C^*_f(X),G)) \approx H^{ch}_*(X,G) \approx H^K_*(X,G). \tag{3.6}$$

On the other hand, by Theorem 2.1.1 [3] and Theorem 6.9 [4, §X.6], we obtain the isomorphisms

$$H^*_f(X,G) \approx H^*_\triangle(X,G) \approx \check{H}^*(\dot{X},*,G), \tag{3.7}$$

where $H^*_\triangle$ is the Alexandroff homology with proper subcomplexes. Using the exact sequence (3.5), by isomorphisms (3.6) and (3.7), we obtain the required statement. □

**Corollary 3.2.** *An inclusion $\rho : X \to (\dot{X},*)$, where $\dot{X}$ is the one-point Alexandroff compactification of locally compact, paracompact Hausdorff space $X$, induces an isomorphism*

$$H^K_*(X,G) \approx H^K_*(\dot{X},*,G).$$

**Corollary 3.3.** *Since the Kolmogoroff and the Massey homology theories satisfy the condition of uniqueness, in particular, the Universal Coefficient Formula* [2, Theorem 4.4], [1, Theorem 1.5], *they are isomorphisms on the category of compact spaces.*

**Corollary 3.4.** *By Corollaries* 3.1 *and* 3.2, *the Kolmogoroff and Massey homologies of locally compact, paracompact Hausdorff spaces are isomorphic to the Kolmogoroff and Massey homologies of a compact space, which is the one-point Alexadroff compactification of the given space. Therefore, by Corollary* 3.3, *there is an isomorphism*

$$H_*^K(X,G) \approx H_*^M(X,G)$$

*on the category of locally compact, paracompact Hausdorff spaces and proper maps.*

**Corollary 3.5.** a) *If $X$ is a locally compact, paracompact Hausdorff space, then for the system $\{N\}$ of closed neighborhoods $N$ of a closed subspace $A$ of $X$, there is an infinite exact sequence*

$$\cdots \longrightarrow \varprojlim{}^{(2k+1)} H_{n+k+1}^K(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(3)} H_{n+2}^K(N) \longrightarrow \varprojlim{}^{(1)} H_{n+1}^K(N) \longrightarrow H_n^K(A,G) \longrightarrow$$
$$\xrightarrow{i_n} \varprojlim H_n^K(N) \longrightarrow \varprojlim{}^{(2)} H_{n+1}^K(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2k)} H_{n+k}^K(N) \longrightarrow \cdots,$$

*where $H_*^K(N) = H_*^K(N,G)$ is the Kolmogoroff homology.*

b) *If $X$ is a compact Hausdorff space, then for the system $\{N\}$ of closed neighborhoods $N$ of a closed subspace $A$ of $X$, there is an infinite exact sequence*

$$\cdots \longrightarrow \varprojlim{}^{(2k+1)} H_{n+k+1}^{Mi}(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(3)} H_{n+2}^M(N) \longrightarrow \varprojlim{}^{(1)} H_{n+1}^{Mi}(N) \longrightarrow H_n^{Mi}(A) \longrightarrow$$
$$\xrightarrow{i_n} \varprojlim H_n^{Mi}(N) \longrightarrow \varprojlim{}^{(2)} H_{n+1}^{Mi}(N) \longrightarrow \cdots \longrightarrow \varprojlim{}^{(2k)} H_{n+k}^{Mi}(N) \longrightarrow \cdots,$$

*where $H_*^{Mi}(N) = H_*^{Mi}(N,G)$ is the Milnor homology* [14].

As it is known [6], for each countable inverse system $\{A_k\}$ of abelian groups $A_k$, there is $\varprojlim{}^{(i)}\{A_k\} = 0$ for $i \geq 2$. By virtue of this fact, Theorem 2.2 and Corollary 3.5, we have

**Corollary 3.6.** a) *If $X$ is a locally compact Hausdorff space with the second countable axiom, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} H_{n+1}^M(N_i) \longrightarrow H_n^M(A,G) \longrightarrow \varprojlim H_n^M(N_i) \longrightarrow 0,$$

*where $H_*^M$ is the Massey homology* [10].

b) *If $X$ is a locally compact, paracompact Hausdorff space with the second countable axiom, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} H_{n+1}^K(N_i) \longrightarrow H_n^K(A,G) \longrightarrow \varprojlim H_n^K(N_i) \longrightarrow 0,$$

*where $H_*^K$ is the Kolmogoroff homology* [8].

c) *If $X$ is a compact Hausdorff space with the second countable axiom, then for each countable system $\{N_i\}$ of closed neighborhoods of a closed subspace $A$ of $X$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} H_{n+1}^{Mi}(N_i) \longrightarrow H_n^{Mi}(A,G) \longrightarrow \varprojlim H_n^{Mi}(N_i) \longrightarrow 0,$$

*where $H_*^{Mi}$ is the Milnor homology* [14].

d) *If $X$ is a compact metric space, then for each countable system $\{N_i\}$ of a closed neighborhoods of closed subspace $A$ of $X$, there is a short exact sequence*

$$0 \longrightarrow \varprojlim{}^{(1)} H_{n+1}^{st}(N_i) \longrightarrow H_n^{st}(A,G) \longrightarrow \varprojlim H_n^{st}(N_i) \longrightarrow 0,$$

*where $H_*^{st}$ is the Steenrod homology* [16].

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