Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 3, 353–360

NEW ESTIMATES ON THE WEIGHTED THREE-POINT QUADRATURE RULE

JOSIPA BARIĆ

Abstract. In this article, the weighted three-point integral quadrature formula is estimated by the new bounds using the new method of calculating estimates for quadrature rules applying the weighted Hermite–Hadamard inequality for higher-order convex functions and weighted version of the integral identity expressed by w-harmonic sequences of functions. The importance of those results lies in providing new estimates of the definite integral values by using weighted three-point formula for numerical integration. The obtained results are employed in establishing new estimates for the Legendre–Gauss three-point quadrature formula with the use of specific form of the weight function w.

1. Introduction

Convexity plays an important role in many aspects of mathematical researches and it has wide applications in physics, mechanics, economics etc.

Let I be an interval in $\mathbb{R}$. Then $\phi : I \to \mathbb{R}$ is said to be convex if for all $x, y \in I$ and all $\alpha \in [0, 1]$,

$$\phi(\alpha x + (1-\alpha)y) \le \alpha\phi(x) + (1-\alpha)\phi(y) \tag{1.1}$$

holds. If inequality (1.1) is strict for all $x \ne y$ and $\alpha \in (0, 1)$, then ϕ is said to be strictly convex.

If the inequality in (1.1) is reversed, then ϕ is said to be concave. If it is strict for all $x \ne y$ and $\alpha \in (0, 1)$, then ϕ is said to be strictly concave.

One of the most important inequalities for convex functions, related to the integral mean of a convex function, is the following Hermite–Hadamard inequality.

If $f : [a, b] \to \mathbb{R}$ is a convex function, then

$$f\Big(\frac{a+b}{2}\Big) \le \frac{1}{b-a}\int_a^b f(x)\ dx \le \frac{1}{2}f(a) + \frac{1}{2}f(b). \tag{1.2}$$

If f is concave, then inequalities in (1.2) are reversed.

The weighted Hermite–Hadamard inequality for a convex function is given in the following theorem [8, 9].

Theorem 1.1. *Let $p : [a, b] \to \mathbb{R}$ be a nonnegative function. If f is a convex function given on an interval I, then we have*

$$f(\lambda) \le \frac{1}{P(b)}\int_a^b p(x)f(x)\ dx \le \frac{b-\lambda}{b-a}f(a) + \frac{\lambda-a}{b-a}f(b)$$

or

$$P(b)f(\lambda) \le \int_a^b p(x)f(x)\ dx \le P(b)\left[\frac{b-\lambda}{b-a}f(a) + \frac{\lambda-a}{b-a}f(b)\right], \tag{1.3}$$

where

$$P(t) = \int_a^t p(x)\ dx \quad \textit{and} \quad \lambda = \frac{1}{P(b)}\int_a^b p(x)x\ dx.$$

2020 *Mathematics Subject Classification.* 11B83, 26A51, 26D15, 65D30, 65D32.

Key words and phrases. Higher order convex function; Harmonic sequence; Three-point formula; Weighted function.

Recently, the refinements and applications for the Hermite–Hadamard inequality have attracted the attention of many researchers (see [2,5,7] and references cited therein).

Various weighted versions of the general integral identities that are used for the approximation of an integral $\int_a^b f(t)\,dt$, using the harmonic sequences of polynomials and w-harmonic sequences of functions, are obtained in [6]. To introduce one of those identities, let us consider a subdivision $\sigma = \{a = x_0 < x_1 < \cdots < x_m = b\}$ of the segment $[a,b]$, $m \in \mathbb{N}$. Let $w : [a,b] \to \mathbb{R}$ be an arbitrary integrable function. For each segment $[x_{k-1}, x_k]$, $k = 1, \ldots, m$, we define w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$ by:

$$\begin{cases} w'_{k1}(t) = w(t), & t \in [x_{k-1}, x_k], \\ w'_{kj}(t) = w_{k,j-1}(t), & t \in [x_{k-1}, x_k], \quad j = 2,3,\ldots,n. \end{cases} \tag{1.4}$$

Also, we define the function $W_{n,w}$ as follows:

$$W_{n,w}(t,\sigma) = \begin{cases} w_{1n}(t), & t \in [a, x_1], \\ w_{2n}(t), & t \in (x_1, x_2], \\ \cdots\cdots \\ \cdots\cdots \\ \cdots\cdots \\ w_{mn}(t), & t \in (x_{m-1}, b]. \end{cases} \tag{1.5}$$

Now, we state the integral identity proved in [6].

Lemma 1.1. *Let $g : [a,b] \to \mathbb{R}$ be a function such that $g^{(n)}$ is a piecewise continuous on $[a,b]$. Then the following identity:*

$$\begin{aligned} \int_a^b w(t)g(t)\,dt = {} & \sum_{j=1}^{n} (-1)^{j-1} \Bigg[w_{mj}(b) g^{(j-1)}(b) \\ & + \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Bigg] \\ & + (-1)^n \int_a^b W_{n,w}(t,\sigma) g^{(n)}(t)\,dt \end{aligned}$$

holds.

More recently obtained results on the weighted versions of the general integral identities and harmonic sequences of polynomials or w-harmonic sequences of functions can be found in [1,4,10] and in their references.

In [3] the authors derived the following Hermite–Hadamard type inequalities using a weighted version of the integral identity expressed by w-harmonic sequences of functions that are given in Lemma 1.1.

Theorem 1.2. *Suppose $w : [a,b] \to \mathbb{R}$ is an arbitrary integrable function and w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\ldots,n}$ are defined by (1.4). Let the function $W_{n,w}$, defined by (1.5), be nonnegative. Then:*

a) *if $g : [a,b] \to \mathbb{R}$ is an $(n+2)$-convex function, the following inequalities*

$$\begin{aligned} (-1)^n \cdot P(b) \cdot g^{(n)}(\lambda) \le {} & \int_a^b w(t)g(t)\,dt - \sum_{j=1}^{n} (-1)^{j-1} \Bigg[w_{mj}(b) g^{(j-1)}(b) \\ & + \sum_{k=1}^{m-1} \left[w_{kj}(x_k) - w_{k+1,j}(x_k) \right] g^{(j-1)}(x_k) - w_{1j}(a) g^{(j-1)}(a) \Bigg] \end{aligned}$$

$$\leq (-1)^n \cdot P(b) \cdot \left[\frac{b-\lambda}{b-a} g^{(n)}(a) + \frac{\lambda-a}{b-a} g^{(n)}(b)\right] \tag{1.6}$$

hold, where

$$P(b) = (-1)^n \left[\frac{1}{n!}\int_a^b w(t)\cdot t^n\ dt - \sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+1)!} \right.$$
$$\left. \times \left(w_{mj}(b) b^{n-j+1} + \sum_{k=1}^{m-1} \left(w_{kj}(x_k) - w_{k+1,j}(x_k)\right) x_k^{n-j+1} - w_{1j}(a) a^{n-j+1} \right)\right]$$

and

$$\lambda = (-1)^n \left[\frac{1}{(n+1)!P(b)}\int_a^b w(t)\cdot t^{n+1}\ dt - \frac{1}{P(b)}\sum_{j=1}^{n} \frac{(-1)^{j-1}}{(n-j+2)!} \right.$$
$$\left. \times \left(w_{mj}(b) b^{n-j+2} + \sum_{k=1}^{m-1} \left(w_{kj}(x_k) - w_{k+1,j}(x_k)\right) x_k^{n-j+2} - w_{1j}(a) a^{n-j+2} \right)\right];$$

b) *if g is an $(n+2)$-concave function, then* (1.6) *holds with the reversed sign of inequalities.*

In what follows, we will use the integral identity involving w-harmonic sequences of functions to obtain new bounds for weighted three-point integral quadrature formula.

2. Three-point Formula

In [1] the authors derived the weighted version of three-point quadrature formula given in the next

Theorem 2.1. *Let $w : [a,b] \to \mathbb{R}$ be an integrable function and $x \in \left[a, \frac{a+b}{2}\right)$. Let $n \in \mathbb{N}$ and $\{L_i\}_{i=0,1,\dots,n}$ be a sequence of harmonic polynomials such that deg $L_i \leq i-1$ and $L_0 = 0$. Assume that the segment $[a,b]$ is subdivided by x_0, x_1, x_2, x_3, x_4, where $x_0 = a$, $x_1 = x$, $x_2 = \frac{a+b}{2}$ and $x_4 = b$. Suppose $\{w_{kj}\}_{j=1,\dots,n}$ are w-harmonic sequences of functions on $[x_{k-1}, x_k]$, for $k = 1,2,3,4$, defined by:*

$$w_{1j}(t) = \frac{1}{(j-1)!}\int_a^t (t-s)^{j-1} w(s)\ ds, \quad t \in [a,x],$$

$$w_{2j}(t) = \frac{1}{(j-1)!}\int_x^t (t-s)^{j-1} w(s)\ ds + L_j(t), \quad t \in \left(x, \frac{a+b}{2}\right],$$

$$w_{3j}(t) = -\frac{1}{(j-1)!}\int_t^{a+b-x} (t-s)^{j-1} w(s)\ ds + (-1)^j L_j(a+b-t), \quad t \in \left(\frac{a+b}{2}, a+b-x\right],$$

$$w_{4j}(t) = -\frac{1}{(j-1)!}\int_t^b (t-s)^{j-1} w(s)\ ds, \quad t \in (a+b-x, b],$$

and $w_{k0}(t) = w(t)$ and

$$W_{n,w}(t,x) = \begin{cases} w_{1n}(t), & t \in [a,x] \\ w_{2n}(t), & t \in \left(x, \frac{a+b}{2}\right] \\ w_{3n}(t), & t \in \left(\frac{a+b}{2}, a+b-x\right] \\ w_{4n}(t), & t \in (a+b-x, b]. \end{cases} \tag{2.1}$$

If $g:[a,b]\to\mathbb{R}$ is such that $g^{(n)}$ is piecewise continuous, then we have

$$\int_a^b w(t)g(t)\,dt=\sum_{j=1}^{n}A_{j,w}(x)\left(g^{(j-1)}(x)+(-1)^{j-1}g^{j-1}\left(a+b-x\right)\right)$$
$$+\sum_{j=1}^{n}B_{j,w}(x)g^{(j-1)}\left(\frac{a+b}{2}\right)+(-1)^{n}\int_a^b W_{n,w}(t,x)g^{(n)}(t)\,dt, \tag{2.2}$$

where

$$A_{j,w}(x)=(-1)^{j-1}\left[\frac{1}{(j-1)!}\int_a^x (x-s)^{j-1}w(s)\,ds-L_j(x)\right],\quad j\geq 1,$$
$$B_{j,w}(x)=2\left[\frac{1}{(j-1)!}\int_x^{\frac{a+b}{2}}\left(\frac{a+b}{2}-s\right)^{j-1}w(s)\,ds+L_j\left(\frac{a+b}{2}\right)\right],\quad \textit{for odd } j\geq 1$$

and

$$B_{j,w}(x)=w_{2j}-w_{3j},\quad \textit{for even } j\geq 1.$$

Applying identity (2.2) to the results shown in the previous section, we now obtain new estimate for the weighted 3-point quadrature formula.

Theorem 2.2. *Suppose $w:[a,b]\to\mathbb{R}$ is an integrable function, $x\in\left[a,\frac{a+b}{2}\right)$ is fixed and $A_{j,w}$, $B_{j,w}$ are defined as in Theorem 2.1. If $g:[a,b]\to\mathbb{R}$ is $(n+2)$-convex, then*

$$(-1)^{n}\cdot P(b)\cdot g^{(n)}(\lambda)\leq\int_a^b w(t)g(t)\,dt-\sum_{j=1}^{n}A_{j,w}(x)\cdot\left(g^{(j-1)}(x)+(-1)^{j-1}g^{(j-1)}(a+b-x)\right)$$
$$-\sum_{j=1\ (odd\ j)}^{n}B_{j,w}(x)g^{(j-1)}\left(\frac{a+b}{2}\right)$$
$$\leq(-1)^{n}\cdot P(b)\cdot\left[\frac{b-\lambda}{b-a}g^{(n)}(a)+\frac{\lambda-a}{b-a}g^{(n)}(b)\right], \tag{2.3}$$

where

$$P(b)=(-1)^{n}\left(\frac{1}{n!}\int_a^b w(t)\cdot t^{n}\,dt\right.$$
$$-\sum_{j=1}^{n}\frac{x^{n-j+1}+(-1)^{j-1}(a+b-x)^{n-j+1}}{(n-j+1)!}\cdot A_{j,w}(x)$$
$$\left.-\sum_{j=1\ (odd\ j)}^{n}\frac{(a+b)^{n-j+1}}{2^{n-j+1}(n-j+1)!}\cdot B_{j,w}(x)\right)$$

and

$$\lambda=\frac{(-1)^{n}}{P(b)}\left(\frac{1}{(n+1)!}\int_a^b w(t)\cdot t^{n+1}\,dt\right.$$
$$-\sum_{j=1}^{n}\frac{x^{n-j+2}+(-1)^{n}(a+b-x)^{n-j+2}}{(n-j+2)!}\cdot A_{j,w}(x)$$
$$\left.-\sum_{j=1\ (odd\ j)}^{n}\frac{(a+b)^{n-j+2}}{2^{n-j+2}(n-j+2)!}\cdot B_{j,w}(x)\right).$$

Proof. To prove inequality (2.3), we use the weighted Hermite–Hadamard inequality for convex function from Theorem 1.1 and identity (2.2). In order to apply inequality (1.3), we need to assume that $W_{n,w}$, defined in (2.1), is a nonnegative function. Now, replacing in (1.3) function p by function $W_{n,w}$ and convex function f by convex function $g^{(n)}$ and then applying identity (2.2) to $(-1)^n \int_a^b W_{n,w}(t, x)g^{(n)}(t)\,dt$, we get inequality (2.3).

The value of $P(b)$ is derived by using the formula of $P(t)$ from Theorem 1.1 on the way that the function p is replaced by $W_{n,w}$ and then applying identity (2.2) to $g(t) = \frac{t^n}{n!}$. Then $g^{(n)}(t) = 1$, $g^{(j-1)}(t) = \frac{n(n-1)\cdot\dots\cdot(n-j+2)}{n!} \cdot t^{n-j+1} = \frac{1}{(n-j+1)!} \cdot t^{n-j+1}$.

In order to calculate λ, we again use the formula of λ from Theorem 1.1 replacing p by $W_{n,w}$ and then the function g in identity (2.2) takes the form $g(t) = \frac{t^{n+1}}{(n+1)!}$. So, $g^{(n)}(t) = t$, $g^{(j-1)}(t) = \frac{(n+1)n\cdot\dots\cdot(n-j+3)}{(n+1)!} \cdot t^{n-j+2} = \frac{1}{(n-j+2)!} \cdot t^{n-j+2}$. □

Let us recall that the integral mean value theorem states that if the functions $f, g : [a, b] \to \mathbb{R}$ are continuous and $g(x) \geq 0$, $x \in [a, b]$, then there exists $\nu \in (a, b)$ such that

$$\int_a^b f(t)g(t)\,dt = f(\nu)\int_a^b g(t)\,dt.$$

Since $W_{2n,w}(t, x) \geq 0$, for $x \in [a, b]$, we can apply the integral mean value theorem to $\int_a^b W_{2n,w}(t, x)g^{(2n)}(t)\,dt$ and get

$$\begin{aligned}
\int_a^b W_{2n,w}(t, x)g^{(2n)}(t)\,dt &= g^{(2n)}(\nu)\int_a^b W_{2n,w}(t, x)\,dt \\
&= g^{(2n)}(\nu)\Bigg(\int_a^x w_{1,2n}(t)\,dt + \int_x^{\frac{a+b}{2}} w_{2,2n}(t)\,dt \\
&\quad + \int_{\frac{a+b}{2}}^{a+b-x} w_{3,2n}(t)\,dt + \int_{a+b-x}^b w_{4,2n}(t)\,dt\Bigg) \\
&= g^{(2n)}(\nu)\left(w_{1,2n+1}(x) - w_{4,2n+1}(a+b-x)\right), \qquad (2.4)
\end{aligned}$$

since $w_{2,2n+1}\left(\frac{a+b}{2}\right) - w_{3,2n+1}\left(\frac{a+b}{2}\right) = B_{2n+1,w}\left(\frac{a+b}{2}\right) = 0$. Using the definitions of w-harmonic sequences of functions $\{w_{kj}\}_{j=1,\dots,n}$ given in Theorem 2.1, we get

$$\int_a^b W_{2n,w}(t, x)g^{(2n)}(t)\,dt = \frac{1}{(2n)!}\left[\int_a^x (x-s)^{2n}\,w(s)\,ds + \int_{a+b-x}^b (a+b-x-s)^{2n}\,w(s)\,ds\right]. \qquad (2.5)$$

According to this integral identity, we get the following special case of the result from Theorem 2.2.

Theorem 2.3. *Let $w : [a, b] \to \mathbb{R}$ be an integrable function, $x \in \left[a, \frac{a+b}{2}\right)$ and $A_{j,w}$, $B_{j,w}$ defined as in Theorem 2.1. If $g : [a, b] \to \mathbb{R}$ is $(2n+2)$-convex, then there exists $\nu \in (a, b)$ such that*

$$\begin{aligned}
P(b)\cdot g^{(2n)}(\lambda) &\leq \frac{g^{(2n)}(\nu)}{(2n)!}\left(\int_a^x (x-s)^{2n}\,w(s)\,ds + \int_{a+b-x}^b (a+b-x-s)^{2n}\,w(s)\,ds\right) \\
&\leq P(b)\cdot\left(\frac{b-\lambda}{b-a}g^{(2n)}(a) + \frac{\lambda-a}{b-a}g^{(2n)}(b)\right), \qquad (2.6)
\end{aligned}$$

where $P(b)$ and λ are the same as given in Theorem 2.2.

Proof. Inequality (2.6) follows from Theorem 2.2 since, regarding integral identity in (2.2), we can replace the middle term in (2.3) by $(-1)^n \int_a^b W_{n,w}(t,x) g^{(n)}(t)\, dt$ and then apply (2.5) with an additional condition that n in (2.2) and in Theorem 2.2 must be replaced by $(2n)$. □

In [11], the author derived the Hermite–Hadamard–Fejér type inequalities for the weighted three-point quadrature formula regarding the weight function that is symmetric arround $\frac{a+b}{2}$. Namely, taking $w(t) = w(a+b-t)$, $t \in [a,b]$, it follows that for $k = 1, \ldots, n$, we have $w_{1k}(t) = (-1)^k w_{4k}(a+b-t)$, $t \in [a,x]$ and

$$\begin{aligned} U_{n,w}(x) \cdot g^{(2n)}\Big(\frac{a+b}{2}\Big) &\le \int_a^b w(t) g(t)\, dt - \sum_{j=1}^{2n} A_{j,w}(x) \cdot \Big(g^{(j-1)}(x) + (-1)^{j-1} g^{(j-1)}(a+b-x)\Big) \\ &\quad - \sum_{j=1\ (odd\ j)}^{2n} B_{j,w}(x) g^{(j-1)}\Big(\frac{a+b}{2}\Big) \\ &\le U_{n,w}(x) \cdot \Big[\frac{1}{2} g^{(2n)}(a) + \frac{1}{2} g^{(2n)}(b)\Big], \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} U_{n,w}(x) = \frac{1}{(2n)!} \int_a^b w(t) \cdot t^{2n}\, dt &- \sum_{j=1}^{2n} \frac{x^{2n-j+1} + (-1)^{j-1} (a+b-x)^{2n-j+1}}{(2n-j+1)!} \cdot A_{j,w}(x) \\ - \sum_{j=1\ (odd\ j)}^{2n} \frac{(a+b)^{2n-j+1}}{2^{2n-j+1}(2n-j+1)!} \cdot B_{j,w}(x). \end{aligned}$$

Calculating $\int_a^b W_{2n,w}(t,x) g^{(2n)}(t)\, dt$ as in (2.4) and taking into account the symmetry conditions, we find that there exists $\nu \in (a,b)$ such that

$$\begin{aligned} \int_a^b W_{2n,w}(t,x) g^{(2n)}(t)\, dt &= g^{(2n)}(\nu) \int_a^b W_{2n,w}(t,x)\, dt \\ &= g^{(2n)}(\nu) \left[w_{1,2n+1}(x) - w_{4,2n+1}(a+b-x)\right] \\ &= g^{(2n)}(\nu) \left[w_{1,2n+1}(x) + w_{1,2n+1}(x)\right] \\ &= g^{(2n)}(\nu) 2 w_{1,2n+1}(x) \\ &= \frac{2 g^{(2n)}(\nu)}{(2n)!} \int_a^x (x-s)^{2n} w(s)\, ds. \end{aligned}$$

Since by identity (2.2) the integral $\int_a^b W_{2n,w}(t,x) g^{(2n)}(t)\, dt$ is equal to the middle term in (2.7), we obtain the following special case of (2.7) [11, Theorem 5]:

$$\begin{aligned} U_{n,w}(x) \cdot g^{(2n)}\Big(\frac{a+b}{2}\Big) &\le \frac{2 g^{(2n)}(\nu)}{(2n)!} \int_a^x (x-s)^{2n} w(s)\, ds \\ &\le U_{n,w}(x) \cdot \left(\frac{1}{2} g^{(2n)}(a) + \frac{1}{2} g^{(2n)}(b)\right), \end{aligned} \tag{2.8}$$

where $w : [a,b] \to \mathbb{R}$ is an integrable function with $w(t) = w(a+b-t)$, $t \in [a,b]$.

3. Legendre–Gauss Three-point Formula

Applying the results obtained in the previous section to specific form of the weight function w, we now derive new estimates for the Legendre–Gauss three-point quadrature formula.

Example 3.1. Applying Theorem 2.1 to the function $w(t)=1$, $t\in[a,b]$, for $x\in\left[a,\frac{a+b}{2}\right)$, we get

$$W_n(t,x)=\begin{cases}\frac{(t-a)^n}{n!}, & t\in[a,x]\\ \frac{(t-x)^n}{n!}+L_n(t), & t\in\left(x,\frac{a+b}{2}\right],\\ \frac{(t-a-b+x)^n}{n!}+(-1)^n\cdot L_n(a+b-t), & t\in\left(\frac{a+b}{2},a+b-x\right],\\ \frac{(t-b)^n}{n!}, & t\in(a+b-x,b]\end{cases}$$

and

$$A_{j,w}(x)=(-1)^{j-1}\left[\frac{(x-a)^j}{j!}-L_j(x)\right],\qquad \text{for } j=1,\dots,n,$$

$$B_{j,w}(x)=2\cdot\left[\frac{\left(\frac{a+b}{2}-x\right)^j}{j!}+L_j\left(\frac{a+b}{2}\right)\right],\quad \text{for odd } j=1,\dots,n,$$

$$B_{j,w}(x)=0,\quad \text{for even } j=1,\dots,n.$$

For establishing new estimates of the Legendre–Gauss three-point quadrature formula and for using Theorem 2.2 to the function $w(t)=1$, $t\in[a,b]$, we need to replace n in the definition of the W_n by $2n$ to provide the nonnegativity of W_n and then to assume that $g:[a,b]\to\mathbb{R}$ is $(2n+2)$-convex since then $g^{(2n)}$ is convex. Now, according to (2.3), it follows that

$$\begin{aligned}P(b)\cdot g^{(2n)}(\lambda)&\le\int_a^b g(t)\,dt-\sum_{j=1}^{2n}A_{j,w}(x)\cdot\left(g^{(j-1)}(x)+(-1)^{j-1}g^{(j-1)}(a+b-x)\right)\\&\quad-\sum_{j=1\ (odd\ j)}^{2n}B_{j,w}(x)g^{(j-1)}\left(\frac{a+b}{2}\right)\\&\le P(b)\cdot\left[\frac{b-\lambda}{b-a}g^{(2n)}(a)+\frac{\lambda-a}{b-a}g^{(2n)}(b)\right],\end{aligned}$$

where

$$\begin{aligned}P(b)=\frac{b^{2n+1}-a^{2n+1}}{(2n+1)!}&-\sum_{j=1}^{2n}\frac{x^{2n-j+1}+(-1)^{j-1}(a+b-x)^{2n-j+1}}{(2n-j+1)!}\cdot A_{j,w}(x)\\&-\sum_{j=1\ (odd\ j)}^{2n}\frac{(a+b)^{2n-j+1}}{2^{2n-j+1}(2n-j+1)!}\cdot B_{j,w}(x)\end{aligned}$$

and

$$\begin{aligned}\lambda=\frac{1}{P(b)}\Bigg[\frac{b^{2n+2}-a^{2n+2}}{(2n+2)!}&-\sum_{j=1}^{2n}\frac{x^{2n-j+2}+(a+b-x)^{2n-j+2}}{(2n-j+2)!}\cdot A_{j,w}(x)\\&-\sum_{j=1\ (odd\ j)}^{2n}\frac{(a+b)^{2n-j+2}}{2^{2n-j+2}(2n-j+2)!}\cdot B_{j,w}(x)\Bigg].\end{aligned}$$

Example 3.2. If the assumptions of Theorem 2.3 hold for $w(t)=1$, $t\in[a,b]$, from inequality (2.6), we get that there exists $\nu\in[a,b]$ such that

$$P(b)\cdot g^{(2n)}(\lambda)\le\frac{g^{(2n)}(\nu)}{(2n)!}\left(\int_a^x(x-s)^{2n}\ ds+\int_{a+b-x}^b(a+b-x-s)^{2n}\ ds\right)$$

$$\leq P(b) \cdot \left(\frac{b-\lambda}{b-a} g^{(2n)}(a) + \frac{\lambda - a}{b-a} g^{(2n)}(b) \right),$$

i.e.,

$$P(b) \cdot g^{(2n)}(\lambda) \leq \frac{2g^{(2n)}(\nu)}{(2n+1)!} \cdot (-a)^{2n+1}$$
$$\leq P(b) \cdot \left(\frac{b-\lambda}{b-a} g^{(2n)}(a) + \frac{\lambda - a}{b-a} g^{(2n)}(b) \right),$$

where $P(b)$ and λ have the same values as in the previous example.

Taking into account the symmetry conditions on the weighted function, from inequality (2.8), we get the following special case of (2.7) with $w(t) = 1$, $t \in fa, b]$.

$$U_{n,w}(x) \cdot g^{(2n)}\Big(\frac{a+b}{2}\Big) \leq \frac{2g^{(2n)}(\nu)}{(2n)!} \cdot \frac{(x-a)^{2n+1}}{2n+1}$$
$$\leq U_{n,w}(x) \cdot \left(\frac{1}{2} g^{(2n)}(a) + \frac{1}{2} g^{(2n)}(b) \right).$$

4. Conclusions

The results presented in this paper are an extension of the investigation started in [3], in which a new method was presented for calculating estimates for some quadrature rules, using the weighted Hermite–Hadamard inequality for higher-order convex functions. The obtained results were applied to a weighted three-point formula for numerical integration to derive new estimates of the definite integral values. The Hermite–Hadamard inequality is one of the most important inequalities, and several variants and improvements have been proposed in the literature. However, this paper offers new research directions that may be useful and may motivate applications to different types of convexity in future works and continuation of investigation on the four-point quadrature formula.

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(Received 10.08.2022)

University of Split, Faculty of Electrical Engineering, Mechanical Engineering and Naval Architecture, Ruđera Boškovića 32, 21 000 Split, Croatia

Email address: jbaric@fesb.hr