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GERSTENHABER–SCHACK BIALGEBRAS

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Dedicated to the memory of Nodar Berikashvili

Abstract. A *Gerstenhaber–Schack (G-S) bialgebra* consists of a graded Hopf algebra H together with multilinear operations $\omega_m^n \in \{Hom^{-1}(H^{\otimes m}, H^{\otimes n}) : m+n=4\}$, whose sum is the degree -1 component of a 2-cocycle in the G-S complex of H. A *G-S extension* of a graded Hopf algebra H is a G-S bialgebra containing H. G-S extensions of H are classified up to isomorphism by the degree -1 component of the G-S cohomology group $H_{GS}^2(H;H)$. We exhibit a space X and a non-trivial topologically induced G-S bialgebra structure on $H^*(\Omega X;\mathbb{Z}_2)$.

1. Introduction

A *Gerstenhaber–Schack (G-S) bialgebra* consists of a graded Hopf algebra (gHa) H together with multilinear operations $\omega_m^n \in \{Hom^{-1}(H^{\otimes m}, H^{\otimes n}) : m+n=4\}$, whose sum is the degree -1 component of a 2-cocycle in the G-S complex of H (antipodes are not assumed). A *G-S extension* of a gHa H is a G-S bialgebra containing H. G-S extensions of H are classified up to isomorphism by the degree -1 component of the G-S cohomology group $H_{GS}^2(H;H)$.

Let X be a $\mathbb{Z}_2$-formal space. The bar construction $BA := BH^*(X;\mathbb{Z}_2)$ with standard differential and cofree coproduct Δ_{BA} is a differential graded (dg) coalgebra model for the singular cochains $S^*(\Omega X;\mathbb{Z}_2)$. A homotopy Gerstenhaber algebra (hGa) structure on $H^*(X;\mathbb{Z}_2)$ lifts to BA and the induced product is Hopf compatible with Δ_{BA}. Furthermore, under the right conditions, the dgHa structure on BA lifts to $H := H^*(BA)$ so that H is a gHa model for $H^*(\Omega X;\mathbb{Z}_2)$.

When H is free, there is a cocycle-selecting homomorphism $g : H \to BA$ and an A_∞-bialgebra structure ω on H induced by transferring the dgHa structure on BA to H along g. Since H has zero differential, ω specializes to a G-S bialgebra by forgetting all operations $\{\omega_m^n : m+n>4\}$ and all A_∞-bialgebra structure relations encoded by the biassociahedra $\{KK_m^n : m+n>5\}$ (see Definitions 1 and 3).

The article is organized as follows: Section 2 reviews the definition of an A_∞-bialgebra and defines A_k-bialgebras for $3 \le k < \infty$. Section 3 reviews the definition of an A_∞-bialgebra morphism and defines morphisms of A_k-bialgebras for $3 \le k < \infty$. Section 4 reviews the G-S complex of a dgHa and presents our main result:

Theorem 1. *Given a gHa* (H,μ,Δ) *and multilinear operations* $\omega := \{\omega_3^1, \omega_2^2, \omega_1^3\} \subset Hom^{-1}(H^{\otimes m}, H^{\otimes n})$, *let* $z := \omega_3^1 + \omega_2^2 + \omega_1^3$. *Then*

1. (H,μ,Δ,ω) *is a G-S extension if and only if* z *is the degree* -1 *component of a* 2*-cocycle in the G-S complex of* H.
2. *G-S extensions* ω *and* ω' *are equivalent if and only if* $cls(z-z')=0$.

Section 5 reviews the Transfer Theorem and the relevant special case of its proof (the Transfer Algorithm), reviews the definition of a hGa, and exhibits a space X with a non-trivial topologically induced G-S bialgebra structure on $H^*(BA) \approx H^*(\Omega X;\mathbb{Z}_2)$.

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2. Biassociahedra and A_k-bialgebras

In his 1963 seminal papers "Homotopy associativity of H-spaces I, II" [9], Jim Stasheff constructed the associahedra $K := \{K_n\}_{n\geq 2}$ and used them to define A_n-algebras for $2 \leq n \leq \infty$. In [7] and [8], S. Saneblidze and the current author constructed the biassociahedra $KK := \{KK_m^n\}_{m+n\geq 3}$ and used them to define A_∞-bialgebras; A_k-bialgebras for $3 \leq k < \infty$ are defined in Definition 1 below.

The *biassociahedron* KK_m^n is a contractible $(m+n-3)$-dimensional polytope, and $KK_1^n \cong KK_n^1$ is Stasheff's associahedron K_n. The 2-cell and edges of KK_3^2 pictured in Figure 1 are labeled by upward-directed graphs, each representing some composition of ω-operations. In dimensions ≤ 3, the biassociahedra KK constructed in [8] agree with the polytopes under the same name and symbol constructed by M. Markl in [5].

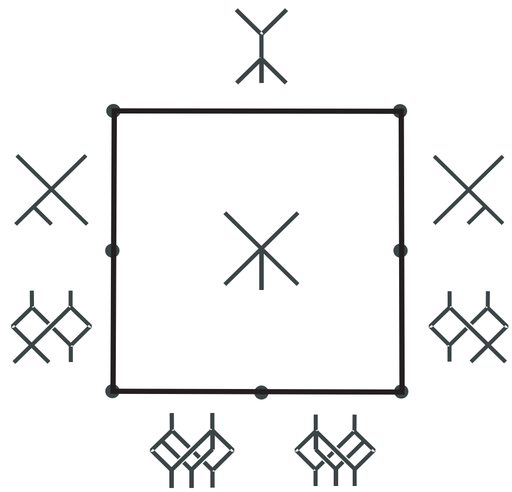

Figure 1. The biassociahedron KK_3^2.

Let R be a commutative ring with unity, let (A,d) be a dg R-module (dgm) with $|d| = +1$ and denote the tensor module of A by TA. The differential ∇ on $Hom^*(TA,TA)$ induced by d is defined for $f \in Hom^p(A^{\otimes m}, A^{\otimes n})$ by

$$\nabla f := d_{(n)} f - (-1)^p f\, d_{(m)},$$

where $d_{(k)} := \sum_{s=0}^{k-1} \mathbf{1}^{\otimes s} \otimes d \otimes \mathbf{1}^{\otimes k-s-1}$ is the linear extension of d to $A^{\otimes k}$. Denote the chain complex of cellular chains on a polytope P by $(CC_*(P), \partial)$ and the top-dimensional cell of KK_m^n by θ_m^n.

Definition 1. Let $3 \leq k \leq \infty$. An A_k**-bialgebra** consists of a **dgm** (A,d) together with **multilinear operations**

$$\omega = \{\omega_m^n \in Hom^{3-m-n}\left(A^{\otimes m}, A^{\otimes n}\right) : m+n \geq 3\},$$

where $m+n \leq k$ when $k < \infty$, and **structure maps**

$$\alpha = \{\alpha_m^n : (CC_*(KK_{m,n}), \partial) \to \left(Hom^{3-m-n}\left(A^{\otimes m}, A^{\otimes n}\right), \nabla\right)\},$$

where α_m^n is a chain map of matrads such that $\alpha_m^n(\theta_m^n) = \omega_m^n$. The KK_m^n **structure relation** is

$$\nabla \omega_m^n = (\nabla \circ \alpha_m^n)\theta_m^n = (\alpha_m^n \circ \partial)\theta_m^n.$$

An A_k-bialgebra A is **strict** if $\nabla \omega_m^n = 0$ for all m and n.

Stasheff's A_n-algebras are A_{n+1}-bialgebras with $\omega_i^j = 0$ for all $j > 1$. Just as the operadic structure of K encodes the structure relations in A_n-algebras, the matradic structure of KK encodes the structure relations in A_k-bialgebras.

For notational simplicity denote $\mu := \omega_2^1$ and $\Delta := \omega_1^2$. Let $\sigma_{m,n}$ denote the canonical permutation of tensor factors $(A_1 \otimes \cdots \otimes A_m)^{\otimes n} \mapsto A_1^{\otimes n} \otimes \cdots \otimes A_m^{\otimes n}$. The KK_m^n structure relations with $m+n \leq 4$ are

$$\begin{array}{rcl}\nabla\mu = 0 & \Leftrightarrow & d \text{ is a derivation} \\ \nabla\Delta = 0 & \Leftrightarrow & d \text{ is a coderivation} \\ \nabla\omega_3^1 = \mu\,(\mu \otimes \mathbf{1} - \mathbf{1} \otimes \mu) & \Leftrightarrow & \mu \text{ is homotopy associative} \\ \nabla\omega_2^2 = (\mu \otimes \mu)\,\sigma_{2,2}\,(\Delta \otimes \Delta) - \Delta\mu & \Leftrightarrow & \mu \text{ and } \Delta \text{ are homotopy compatible} \\ \nabla\omega_1^3 = (\mathbf{1} \otimes \Delta - \Delta \otimes \mathbf{1})\,\Delta & \Leftrightarrow & \Delta \text{ is homotopy coassociative.}\end{array} \tag{2.1}$$

The KK_m^n structure relations with $m + n = 5$ are displayed in (4.1).

While strict A_4-bialgebras are gHa's by the relations in (2.1), the operations ω_m^n with $m+n = 4$, are unconstrained. A "Gerstenhaber–Schack bialgebra" is an A_4-bialgebra with zero differential together with appropriately constrained operations ω_m^n with $m + n = 4$ (see Definition 3).

3. Bimultiplihedra and Morphisms of A_k-Bialgebras

In [9], J. Stasheff also introduced the multiplihedra $J := \{J_n\}_{n\geq1}$ and used them to define morphisms of A_n-algebras for $2 \leq n \leq \infty$. In [8], S. Saneblidze and the current author introduced the bimultiplihedra $JJ := \{JJ_m^n\}_{m+n\geq2}$ and used them to define morphisms of A_∞-bialgebras; morphisms of A_k-bialgebras are defined in Definition 2 below. The *bimultiplihedron* JJ_m^n is a contractible $(m + n - 2)$-dimensional polytope, and $JJ_1^n \cong JJ_n^1$ is Stasheff's multiplihedron J_n.

Given dgm's (A, d_A) and (B, d_B), let ∇ denote the induced differential on $Hom(TA, TB)$, and denote the top-dimensional cell of JJ_m^n by $\mathfrak{f}_m^n$.

Definition 2. Let (A, d_A, ω_A) and (B, d_B, ω_B) be A_k-bialgebras. A **morphism from** A **to** B consists of multilinear maps

$$G = \{g_m^n \in Hom^{2-m-n}(A^{\otimes m}, B^{\otimes n}) : m + n \geq 2\},$$

where $m + n \leq k$ when $k < \infty$, and **structure maps**

$$\beta = \{\beta_m^n : (CC_*(JJ_m^n), \partial) \to \big(Hom^{2-m-n}(A^{\otimes m}, B^{\otimes n}), \nabla\big)\},$$

where β_m^n is a chain map of relative matrads such that $\beta_m^n(\mathfrak{f}_m^n) = g_m^n$. The JJ_m^n-**structure relation** is

$$\nabla g_m^n = (\nabla \circ \beta)\mathfrak{f}_m^n = (\beta \circ \partial)\mathfrak{f}_m^n.$$

Denote a morphism G from A to B by $G : A \Rightarrow B$. A morphism $\Phi = \{\phi_m^n\} : A \Rightarrow B$ is an **isomorphism** if $\phi_1^1 : A \to B$ is an isomorphism of dgm's.

Stasheff's morphisms of A_n-algebras are morphisms of A_{n+1}-bialgebras with $g_i^j = 0$ for all $j > 1$. Just as the relative operadic structure of J encodes the structure relations in a morphism of A_n-algebras, the relative matradic structure of JJ encodes the structure relations in a morphism of A_k-bialgebras.

Remark 1. If $\Phi = \{\phi_m^n\} : A \Rightarrow A$ is an isomorphism, let $g = (\phi_1^1)^{-1}$ and define $\psi_m^n := g^{\otimes n}\phi_m^n$; then $\Psi = \{\psi_m^n\} : A \Rightarrow A$ is an isomorphism with $\psi_1^1 = \mathbf{1}_A$. Thus, when $\Phi : A \Rightarrow A$ is an isomorphism, we always assume that $\phi_1^1 = \mathbf{1}_A$.

To accommodate subscripts let $\omega^{n,m} := \omega_m^n$, and for notational simplicity let $\mu_X := \omega_X^{1,2}$ and $\Delta_Y := \omega_Y^{2,1}$. The JJ_m^n structure relations with $2 \leq m + n \leq 4$ are

$$\begin{array}{lcl}\nabla g_1^1 = 0 & \Leftrightarrow & g := g_1^1 \text{ is a chain map} \\ \nabla g_2^1 = g\mu_A - \mu_B\,(g \otimes g) & \Leftrightarrow & g \text{ is homotopy multiplicative} \\ \nabla g_1^2 = \Delta_B g - (g \otimes g)\,\Delta_A & \Leftrightarrow & g \text{ is homotopy comultiplicative}\end{array}$$

$$\nabla g_3^1 = g\omega_A^{1,3} - \mu_B\left(g \otimes g_2^1 - g_2^1 \otimes g\right) + g_2^1\,(\mu_A \otimes \mathbf{1} - \mathbf{1} \otimes \mu_A) - \omega_B^{1,3}g^{\otimes3}$$

$$\begin{aligned}\nabla g_2^2 = {} & (g \otimes g)\omega_A^{2,2} - (\mu_B \otimes \mu_B)\sigma_{2,2}(\Delta_B g \otimes g_1^2 + g_1^2 \otimes (g \otimes g)\Delta_A) + g_1^2\mu_A \\ & - \left(\mu_B(g \otimes g) \otimes g_2^1 + g_2^1 \otimes g\mu_A\right)\sigma_{2,2}(\Delta_A \otimes \Delta_A) + \Delta_B g_2^1 - \omega_B^{2,2}(g \otimes g)\end{aligned}$$

$$\nabla g_1^3 = g^{\otimes 3}\omega_A^{3,1} + \left(g\otimes g_1^2 - g_1^2\otimes g\right)\Delta_A + \left(\mathbf{1}\otimes\Delta_B - \Delta_B\otimes\mathbf{1}\right)g_1^2 - \omega_B^{3,1}g.$$

4. The G-S Complex of a DG Hopf Algebra

Let (H,d,μ,Δ) be a dgHa with $|d|=+1$ (when $|d|=-1$ the construction is completely dual). For $m\geq 1$, define left and right H-comodule actions $\lambda_m,\rho_m : H^{\otimes m}\to H^{\otimes m+1}$ by

$$\begin{aligned}
&\lambda_1=\rho_1:=\Delta\\
&\lambda_m:=\Big(\mu\left(\mu\otimes\mathbf{1}\right)\cdots\left(\mu\otimes\mathbf{1}^{\otimes m-2}\right)\otimes\mathbf{1}^{\otimes m}\Big)\sigma_{2,m}\Delta^{\otimes m}\\
&\rho_m:=\Big(\mathbf{1}^{\otimes m}\otimes\mu\left(\mathbf{1}\otimes\mu\right)\cdots\left(\mathbf{1}^{\otimes m-2}\otimes\mu\right)\Big)\sigma_{2,m}\Delta^{\otimes m}.
\end{aligned}$$

For $n\geq 1$, define left and right H-module actions $\lambda^n,\rho^n : H^{\otimes n+1}\to H^{\otimes n}$ by

$$\begin{aligned}
&\lambda^1=\rho^1:=\mu\\
&\lambda^n:=\mu^{\otimes n}\sigma_{n,2}\Big(\left(\Delta\otimes\mathbf{1}^{\otimes n-2}\right)\cdots\left(\Delta\otimes\mathbf{1}\right)\Delta\otimes\mathbf{1}^{\otimes n}\Big)\\
&\rho^n:=\mu^{\otimes n}\sigma_{n,2}\Big(\mathbf{1}^{\otimes n}\otimes\left(\mathbf{1}^{\otimes n-2}\otimes\Delta\right)\cdots\left(\mathbf{1}\otimes\Delta\right)\Delta\Big).
\end{aligned}$$

Then $H^{\underline{\otimes} m}:=(H^{\otimes m},\lambda_m,\rho_m)$ is an H-bicomodule, $H^{\overline{\otimes} n}:=\left(H^{\otimes n},\lambda^{n-1},\rho^{n-1}\right)$ is an H-bimodule (when $n=1$ the bimodule actions are undefined and $H^{\overline{\otimes}1}:=H$), and $\{Hom^p(H^{\underline{\otimes} m},H^{\overline{\otimes} n}) : p\in\mathbb{Z}$ and $m,n\geq 1\}$ is a trigraded H-bidimodule.

The linear extension $d_{(k)}:=\sum_{s=0}^{k-1}\mathbf{1}^{\otimes s}\otimes d\otimes\mathbf{1}^{\otimes k-s-1}$ and the (co)bar differentials (forgetting shift of dimensions)

$$\partial_{(m)}:=\sum_{s=0}^{m-1}(-1)^s\ \mathbf{1}^{\otimes s}\otimes\mu\otimes\mathbf{1}^{\otimes m-s-1}\text{ and }\delta_{(n)}:=\sum_{s=0}^{n-1}(-1)^s\ \mathbf{1}^{\otimes s}\otimes\Delta\otimes\mathbf{1}^{\otimes n-s-1}$$

induce strictly commuting differentials ∇, ∂, and δ on $\{Hom^p(H^{\underline{\otimes} m},H^{\overline{\otimes} n})\}$, which act on an element f of tridegree (p,m,n) by

$$\begin{aligned}
\nabla f&:=d_{(n)}f-(-1)^p\,fd_{(m)}\\
\partial f&:=\lambda^n\left(\mathbf{1}\otimes f\right)-f\partial_{(m)}-(-1)^m\,\rho^n\left(f\otimes\mathbf{1}\right)\\
\delta f&:=\left(\mathbf{1}\otimes f\right)\lambda_m-\delta_{(n)}f-(-1)^n\left(f\otimes\mathbf{1}\right)\rho_m.
\end{aligned}$$

Note that $\nabla:(p,m,n)\mapsto(p+1,m,n)$, $\partial:(p,m,n)\mapsto(p,m+1,n)$, and $\delta:(p,m,n)\mapsto(p,m,n+1)$.

The *G-S complex of H* is the triple complex $(Hom^*(H^{\underline{\otimes}*},H^{\overline{\otimes}*}),\nabla,\partial,\delta)$. The subspace of total r-cochains in degree p is

$$C_{GS}^{r,p}(H,H):=\bigoplus_{p+m+n=r+1}Hom^p\left(H^{\underline{\otimes} m},H^{\overline{\otimes} n}\right)$$

and the total differential D acts on a cochain f of tridegree (p,m,n) by

$$Df:=(-1)^{m+n}\nabla f+\partial f+(-1)^m\,\delta f,$$

where the signs are chosen so that $D^2=0$ and the restriction of D to the subspace $p=0$ agrees with the total differential on the G-S double complex of an ungraded Hopf algebra [1].

The subspace of total r-cocycles in degree p is denoted by $Z_{GS}^{r,p}(H;H)$. A general 2-cocycle has components φ_m^n of tridegree (p,m,n) with $p+m+n=3$, and is an infinitesimal in the deformation theory of dgHa's [10]. A 2-cocycle with $m+n\leq 4$ is pictured in Figure 2. The r^{th} G-S cohomology group in degree p with coefficients in H is $H_{GS}^{r,p}(H;H):=H^*\left(C_{GS}^{r,p}(H,H),D\right)$.

It is truly remarkable that the KK_m^n structure relations with $m+n=5$ and the JJ_m^n structure relations with $m+n=4$ can be expressed in terms of G-S differentials.

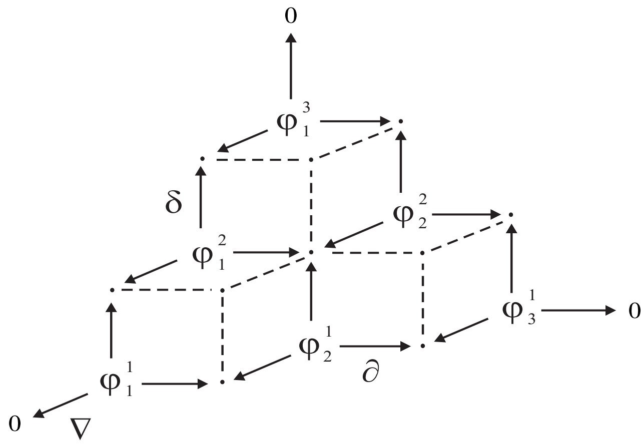

FIGURE 2. A 2-cocycle $\varphi_1^1 + \varphi_2^1 + \varphi_1^2 + \varphi_3^1 + \varphi_2^2 + \varphi_1^3$ with components of tridegree $(3 - m - n, m, n)$ and $m + n \leq 4$.

Example 1. To express the KK_3^2 structure relation in terms of G-S differentials, recall that $\alpha_m^n(\theta_m^n) = \omega_m^n$. Reading the graphical labels in Figure 1 from top-down and left-to-right, express each as a composition of ω-operations. Then up to sign

$$\nabla\omega_3^2 = \Delta\omega_3^1 + \omega_2^2(\mu \otimes \mathbf{1} + \mathbf{1} \otimes \mu) + (\mu \otimes \mu)\sigma_{2,2}(\omega_2^2 \otimes \Delta + \Delta \otimes \omega_2^2)$$
$$+\Big(\mu(\mu \otimes \mathbf{1}) \otimes \omega_3^1 + \omega_3^1 \otimes \mu(\mathbf{1} \otimes \mu)\Big)\sigma_{2,3}\Delta^{\otimes 3}.$$

By definition,

$$\partial\omega_2^2 = \omega_2^2(\mu \otimes \mathbf{1} + \mathbf{1} \otimes \mu) + (\mu \otimes \mu)\sigma_{2,2}(\omega_2^2 \otimes \Delta + \Delta \otimes \omega_2^2) \text{ and}$$
$$\delta\omega_3^1 = \Delta\omega_3^1 + \Big(\mu(\mu \otimes \mathbf{1}) \otimes \omega_3^1 + \omega_3^1 \otimes \mu(\mathbf{1} \otimes \mu)\Big)\sigma_{2,3}\Delta^{\otimes 3}$$

so that

$$\nabla\omega_3^2 = \partial\omega_2^2 + \delta\omega_3^1.$$

The KK_m^n structure relations with $m + n = 5$ are

$$\begin{array}{llcc}
KK_4^1: & \nabla\omega_4^1 = \partial\omega_3^1 & \overset{\nabla=0}{\Rightarrow} & \partial\omega_3^1 = 0 \\
KK_3^2: & \nabla\omega_3^2 = \partial\omega_2^2 - \delta\omega_3^1 & \Rightarrow & \partial\omega_2^2 - \delta\omega_3^1 = 0 \\
KK_2^3: & \nabla\omega_2^3 = \partial\omega_1^3 + \delta\omega_2^2 & \Rightarrow & \partial\omega_1^3 + \delta\omega_2^2 = 0 \\
KK_1^4: & \nabla\omega_1^4 = -\delta\omega_1^3 & \Rightarrow & \delta\omega_1^3 = 0.
\end{array} \tag{4.1}$$

The strict relations in (4.1) provide the linkage we need to form the degree -1 component $\omega_3^1 + \omega_2^2 + \omega_1^3$ of a strict G-S 2-cocycle (see Figure 3).

$$\begin{array}{ccccccc}
\delta\omega_1^3 = 0 & & & & & & \\
\uparrow & & & & & & \\
\omega_1^3 & \longrightarrow & \partial\omega_1^3 + \delta\omega_2^2 = 0 & & & & \\
 & & \uparrow & & & & \\
 & & \omega_2^2 & \longrightarrow & \partial\omega_2^2 - \delta\omega_3^1 = 0 & & \\
 & & & & \uparrow & & \\
 & & & & \omega_3^1 & \longrightarrow & \partial\omega_3^1 = 0
\end{array}$$

FIGURE 3. The degree -1 component of a strict G-S 2-cocycle.

Similarly, the JJ_m^n structure relations with $m+n=4$ for an isomorphism $\Phi:(H,d,\mu,\Delta,\omega_A)\Rightarrow(H,d,\mu,\Delta,\omega_B)$ of A_4-bialgebras are

$$\begin{array}{rr} JJ_3^1: & \nabla\phi_3^1=\omega_A^{1,3}-\partial\phi_2^1-\omega_B^{1,3} \\ JJ_2^2: & \nabla\phi_2^2=\omega_A^{2,2}-\partial\phi_1^2-\delta\phi_2^1-\omega_B^{2,2} \\ JJ_1^3: & \nabla\phi_1^3=\omega_A^{3,1}+\delta\phi_1^2-\omega_B^{3,1}. \end{array} \tag{4.2}$$

Indeed, the algebraic representations of the 2-dimensional biassociahedra and bimultiplihedra displayed in (4.1) and (4.2) appear quite naturally and were hiding in the G-S complex more than a decade before the corresponding polytopes appeared in [7].

Remark 2. The G-S differentials ∇, ∂, and δ capture the interactions of a higher order operation with the underlying dgHa structure but completely miss its interactions with the higher order structure. Consequently, the KK_m^n structure relations cannot be expressed in terms of G-S differentials when $m+n\geq 6$.

Now by definition, an A_4-bialgebra $(H,\mu,\Delta,\omega_3^1,\omega_2^2,\omega_1^3)$ (with zero differential) is a gHa with three higher order operations of degree -1. By homogeneity, $D(\omega_3^1+\omega_2^2+\omega_1^3)=0$ if and only if $\delta\omega_1^3=\partial\omega_2^2-\delta\omega_3^1=\partial\omega_1^3+\delta\omega_2^2=\delta\omega_1^3=0$.

Definition 3. An A_4-bialgebra $(H,\mu,\Delta,\omega_3^1,\omega_2^2,\omega_1^3)$ is a **Gerstenhaber–Schack bialgebra** if

$$D(\omega_3^1+\omega_2^2+\omega_1^3)=0. \tag{4.3}$$

A **G-S extension** of a gHa (H,μ,Δ) is a G-S bialgebra of the form $(H,\mu,\Delta,\omega:=\{\omega_3^1,\omega_2^2,\omega_1^3\})$; we sometimes refer to ω as a G-S extension when the context is clear. G-S extensions ω and ω' are **equivalent** if there exists an isomorphism $\Phi:(H,\mu,\Delta,\omega)\Rightarrow(H,\mu,\Delta,\omega')$ of A_4-bialgebras. A G-S extension ω is **trivial** if $(H,\mu,\Delta,\omega)\cong(H,\mu,\Delta)$.

Theorem 2. *Given a gHa (H,μ,Δ) and multilinear operations $\omega:=\{\omega_3^1,\omega_2^2,\omega_1^3\}\subset Hom^{-1}(H^{\otimes m},H^{\otimes n})$, let $z:=\omega_3^1+\omega_2^2+\omega_1^3$. Then*

1. *ω is a G-S extension if and only if $D(z)=0$.*
2. *G-S extensions ω and ω' are equivalent if and only if $cls(z-z')=0$.*

Proof. The proof of Part 1 is trivial.
Proof of part 2: $\omega\sim\omega'$ if and only if there exists an isomorphism $\Phi=\{\mathbf{1}_H,\phi_m^n:n+m=3,4\}:(H,\mu,\Delta,\omega)\Rightarrow(H,\mu,\Delta,\omega)$ of A_4-bialgebras if and only if Φ satisfies the JJ_m^n structure relations for $m+n=3,4$, which hold trivially when $m+n=3$. Since $\nabla=0$, the JJ_m^n structure relations in (4.2) reduce to

$$\begin{aligned} \partial\phi_2^1&=\omega_3^1-(\omega')_3^1 \\ \partial\phi_1^2+\delta\phi_2^1&=\omega_2^2-(\omega')_2^2 \\ -\delta\phi_1^2&=\omega_1^3-(\omega')_1^3. \end{aligned} \tag{4.4}$$

Therefore $\omega\sim\omega'$ if and only if there exists a $(1,-1)$-cochain $\phi_2^1+\phi_1^2$ such that the structure relations in (4.4) hold if and only if

$$D\left(\phi_2^1+\phi_1^2\right)=\partial\phi_2^1+\left(\partial\phi_1^2+\delta\phi_2^1\right)-\delta\phi_1^2=\omega-\omega'. \qquad \square$$

Corollary 1. *A G-S extension ω is trivial if and only if $cls(z)=0$.*

Proof. Set $\omega'=0$ and apply Theorem 1, Part 2. $\square$

Corollary 2. *G-S extensions of a gHa H are parametrized by $Z_{GS}^{2,-1}(H;H)$ and classified up to isomorphism by $H_{GS}^{2,-1}(H;H)$.*

Example 2. Consider the $\mathbb{Z}_2$-dg algebra (dga)

$$A=\langle 1,a_2,a_3,b_3,a_2a_3=a_3a_2\rangle,$$

where $|x_i|=i$, and the bar construction BA with standard differential d_{BA}, shuffle product sh, and cofree coproduct Δ_{BA}. Denote a homogeneous element $\downarrow x_1\otimes\cdots\otimes\downarrow x_n\in BA$ by $[x_1|\cdots|x_n]$. Then

BA is a dgHa such that $d_{BA}\left([a_2|a_3]+[a_3|a_2]\right)=0$, and $H_0:=H^*\left(BA\right)$ is a gHa with induced product μ and coproduct Δ. Let $\alpha_i:=cls[a_{i+1}]$, $\beta:=cls[b_3]$, and $\gamma:=\mu\left(\alpha_1\otimes\alpha_2\right)=cls([a_2|a_3]+[a_3|a_2])$. Then μ acts as the shuffle product except

$$\mu\left(\alpha_i\otimes\gamma\right)=\mu\left(\gamma\otimes\alpha_i\right)=0$$

(by associativity) and Δ acts as the free coproduct except

$$\Delta\gamma=1\otimes\gamma+\alpha_1\otimes\alpha_2+\alpha_2\otimes\alpha_1+\gamma\otimes 1$$

(by Hopf compatibility). Define ϕ_2^1, ω_3^1, and ω_2^2 to be zero except

$$\phi_2^1\left(\beta\otimes\beta\right):=\gamma,\quad \omega_2^2\left(\beta\otimes\beta\right):=\alpha_1\otimes\alpha_2+\alpha_2\otimes\alpha_1,\quad\text{and}$$
$$\omega_3^1\left(\beta\otimes\beta\otimes\beta\right):=\mu\left(\beta\otimes\gamma\right).$$

By direct calculation,

$$\left(\partial\phi_2^1\right)\left(\beta\otimes\beta\otimes\beta\right)=\mu\left(\beta\otimes\gamma\right)=\omega_3^1\left(\beta\otimes\beta\otimes\beta\right)\quad\text{and}$$

$$\begin{aligned}\left(\delta\phi_2^1\right)\left(\beta\otimes\beta\right)&=\left(\left(\mu\otimes\psi_2^1+\psi_2^1\otimes\mu\right)\sigma_{2,2}\left(\Delta\otimes\Delta\right)+\Delta\psi_2^1\right)\left(\beta\otimes\beta\right)\\&=\alpha_1\otimes\alpha_2+\alpha_2\otimes\alpha_1=\omega_2^2\left(\beta\otimes\beta\right).\end{aligned}$$

Therefore

$$D\phi_2^1=\partial\phi_2^1+\delta\phi_2^1=\omega_3^1+\omega_2^2.$$

Since $cls\left(\omega_3^1+\omega_2^2\right)=0$, the G-S extension $\tilde{H}:=\left(H,\mu,\Delta,\omega_3^1,\omega_2^2\right)$ is trivial by Theorem 2, Part 2, and indeed, $\Phi=\left\{\mathbf{1}_A,\psi_2^1\right\}:\tilde{H}\Rightarrow H_0$ is an isomorphism of A_4-bialgebras.

The remainder of this article considers an induced A_∞-bialgebra structure ω on a particular loop cohomology H and applies Theorem 2 to obtain a non-trivial G-S extension of the underlying gHa structure.

5. A Topological Application

5.1. **The Transfer Theorem and Algorithm.** Let X be a space. Under mild conditions, the Transfer Algorithm induces a canonical A_∞-bialgebra structure on $A:=H^*(BA)\approx H^*\left(\Omega X;\mathbb{Z}_2\right)$. We state the Transfer Theorem when A is free; the Transfer Algorithm appears in the proof. For the general case and a proof of uniqueness see [8].

Theorem 3 (The Transfer Theorem)**.** *Let (A,d_A) be a free dgm, let $(B,\,d_B,\omega_B)$ be an A_∞-bialgebra, and let $g:A\to B$ be a chain map/homology isomorphism. Then g induces an A_∞-bialgebra structure $\omega_A=\{\omega_A^{n,m}\}$ on A and extends to a map $G=\{g_m^n:g_1^1=g\}:A\Rightarrow B$ of A_∞-bialgebras. Furthermore, ω_A and G are unique up to isomorphism.*

Proof. (The Transfer Algorithm). For $f\in Hom(A^{\otimes m},A^{\otimes n})$ define $\tilde{g}(f):=g^{\otimes n}f$ and note that $\tilde{g}$ is a homology isomorphism since A is free. We obtain an induced A_∞-bialgebra structure by simultaneously constructing a chain map $\alpha_A:CC_*\left(KK\right)\to Hom(TA,TA)$ of matrads and a chain map $\beta:CC_*\left(JJ\right)\to Hom(TA,TB)$ of relative matrads.

Thinking of JJ_m^n as a subdivision of the cylinder $KK_m^n\times I$, denote the top dimensional cells of KK_m^n and JJ_m^n by θ_m^n and $\mathfrak{f}_m^n$, and identify the faces $KK_m^n\times 0$ and $KK_m^n\times 1$ of JJ_m^n with $\theta_m^n\left(\mathfrak{f}_1^1\right)^{\otimes m}$ and $\left(\mathfrak{f}_1^1\right)^{\otimes n}\theta_m^n$, respectively. By hypothesis, there is a map of matrads $\alpha_B:CC_*(KK)\to(U_B,\nabla)$ such that $\alpha_B(\theta_m^n)=\omega_B^{n,m}$.

To initialize the induction, define $\beta:CC_*\left(JJ_1^1\right)\to Hom^0\left(A,B\right)$ by $\beta\left(\mathfrak{f}_1^1\right)=g_1^1=g$, and extend β to $CC_*\left(JJ_2^1\right)\to Hom^{-1}\left(A\otimes A,B\right)$ and $CC_*\left(JJ_1^2\right)\to Hom^{-1}(A,\,B\otimes B)$ in the following way: On the vertices $\theta_2^1\left(\mathfrak{f}_1^1\otimes\mathfrak{f}_1^1\right)\in JJ_2^1$ and $\theta_1^2\mathfrak{f}_1^1\in JJ_1^2$, define $\beta\left(\theta_2^1\left(\mathfrak{f}_1^1\otimes\mathfrak{f}_1^1\right)\right)=\mu_B\left(g\otimes g\right)$ and $\beta\left(\theta_1^2\mathfrak{f}_1^1\right)=\Delta_B g$. Since $\mu_B\left(g\otimes g\right)$ and $\Delta_B g$ are ∇-cocycles, and $\tilde{g}_*$ is an isomorphism, there exist cocycles $\mu_A\in Hom^0(A\otimes A,A)$ and $\Delta_A\in Hom^0(A,A\otimes A)$ such that $\tilde{g}_*[\mu_A]=[\mu_B\left(g\otimes g\right)]$ and $\tilde{g}_*[\Delta_A]=[\Delta_B g]$. Thus $[g\mu_A-\mu_B\left(g\otimes g\right)]=[\Delta_B g-\left(g\otimes g\right)\Delta_A]=0$, and there exist cochains $g_2^1\in Hom^{-1}(A,B\otimes B)$ and $g_1^2\in Hom^{-1}(A\otimes A,B)$ such that $\nabla g_2^1=g\mu_A-\mu_B\left(g\otimes g\right)$ and $\nabla g_1^2=\Delta_B g-\left(g\otimes g\right)\Delta_A$.

For $m+n=3$, define $\alpha_A : CC_*(KK_m^n) \to Hom^0(A^{\otimes m}, A^{\otimes n})$ by $\alpha_A(\theta_m^n) := \omega_A^{n,m}$ and $\beta : CC_*(JJ_m^n) \to Hom^*(A^{\otimes m}, B^{\otimes n})$ by

$$\begin{aligned} \beta(\mathfrak{f}_m^n) &:= g_m^n \in Hom^{-1}(A^{\otimes m}, A^{\otimes n}) \\ \beta(\mathfrak{f}_1^1\,\theta_2^1) &:= g\,\mu_A \in Hom^0(A\otimes A, A) \\ \beta\big((\mathfrak{f}_1^1\otimes\mathfrak{f}_1^1)\,\theta_1^2\big) &:= (g\otimes g)\,\Delta_A \in Hom^0(A, A\otimes A)\,. \end{aligned}$$

Inductively, given $m+n\geq 4$, assume that for $i+j<m+n$ there exists a map of matrads $\alpha_A : CC_*(KK_i^j) \to Hom^{3-i-j}\left(A^{\otimes i}, A^{\otimes j}\right)$ and a map of relative matrads $\beta : CC_*(JJ_i^j) \to Hom^{2-i-j}(A^{\otimes i}, B^{\otimes j})$ such that $\alpha_A(\theta_i^j) = \omega_A^{j,i}$ and $\beta(\mathfrak{f}_i^j) = g_i^j$. Thus we are given chain maps $\alpha_A : CC_*(\partial KK_m^n) \to Hom^{4-m-n}(A^{\otimes m}, A^{\otimes n})$ and $\beta : CC_*(\partial JJ_m^n \smallsetminus \operatorname{int} KK_m^n \times 1) \to Hom^{3-m-n}(A^{\otimes m}, B^{\otimes n})$. We wish to extend α_A to the top cell θ_m^n of KK_m^n, and β to the codimension 1 cell $\left(\mathfrak{f}_1^1\right)^{\otimes n}\theta_m^n$ and the top cell $\mathfrak{f}_m^n$ of JJ_m^n. Since α_A is a map of matrads, the components of the cocycle

$$z = \alpha_A\big(CC_*(\partial KK_m^n)\big) \in Hom^{4-m-n}\left(A^{\otimes m}, A^{\otimes n}\right)$$

are expressed in terms of $\omega_A^{j,i}$ with $i+j<m+n$; similarly, since β is a map of relative matrads, the components of the cochain

$$\varphi = \beta\big(CC_*(\partial JJ_m^n \smallsetminus \operatorname{int} KK_m^n \times 1)\big) \in Hom^{3-m-n}\left(A^{\otimes m}, B^{\otimes n}\right)$$

are expressed in terms of ω_B, $\omega_A^{j,i}$ and g_i^j with $i+j<m+n$. Clearly $\tilde{g}(z) = \nabla\varphi$; and $[z]=[0]$ since $\tilde{g}$ is a homology isomorphism. Now choose a cochain $b\in Hom^{3-m-n}(A^{\otimes m}, A^{\otimes n})$ such that $\nabla b = z$; then $\nabla(\tilde{g}(b)-\varphi) = \nabla\tilde{g}(b) - \tilde{g}(z) = 0$. Choose a class representative $u\in\tilde{g}_*^{-1}[\tilde{g}(b)-\varphi]$, set $\omega_A^{n,m} = b-u$, and define $\alpha_A(\theta_m^n) := \omega_A^{n,m}$. Then $[\tilde{g}(\omega_A^{n,m})-\varphi] = [\tilde{g}(b-u)-\varphi] = [\tilde{g}(b)-\varphi] - [\tilde{g}(u)] = [0]$. Choose a cochain $g_m^n \in Hom^{2-m-n}(A^{\otimes m}, B^{\otimes n})$ such that $\nabla g_m^n = g^{\otimes n}\omega_A^{n,m} - \varphi$, and define $\beta(\mathfrak{f}_m^n) := g_m^n$. To extend β as a map of relative matrads, define $\beta\big((\mathfrak{f}_1^1)^{\otimes n}\,\theta_m^n\big) := g^{\otimes n}\,\omega_A^{n,m}$. Passing to the limit we obtain the desired maps α_A and β. □

5.2. **Homotopy Gerstenhaber Algebras.** When a 1-connected dga $(A, d, \cdot)$ over a field $\mathbf{k}$ admits a hGa structure, it lifts to the bar construction BA and induces a Hopf compatible product μ_{BA} so that BA is a dgHa. Furthermore, the dgHa structure on BA lifts to a gHa structure on $H^*(BA;\mathbf{k})$. Since such liftings are required in the application below, we include a brief review of hGa's for completeness. To avoid sign complications, we limit our discussion to $\mathbb{Z}_2$-dga's and follow the exposition given by Kadeishvili in [3]; for a general exposition see [2].

A (not necessarily 1-connected or commutative) $\mathbb{Z}_2$-dga $(A, d, \cdot)$ is a *homotopy Gerstenhaber algebra* (hGa) if there exist multilinear operations

$$E := \{E_{0,1} = E_{1,0} = \mathbf{1}_A\} \cup \{E_{1,q} : A\otimes A^{\otimes q} \to A\}_{q\geq 1}$$

such that $|E_{1,q}| = -q$, and satisfy the following relations:

$$\begin{aligned} dE_{1,q}(a;b_1,\dots,b_q) + E_{1,q}(da;b_1,\dots,b_q) &+ \textstyle\sum_i E_{1,q}(a;b_1,\dots,db_i,\dots,b_q) \\ = b_1\cdot E_{1,q-1}(a;b_2,\dots b_q) &+ E_{1,q-1}(a;b_1,\dots,b_{q-1})\cdot b_q \\ &+ \textstyle\sum_i E_{1,q-1}(a;b_1,\dots,b_i\cdot b_{i+1},\dots,b_q) \end{aligned} \tag{5.1}$$

$$\begin{aligned} E_{1,q}(a_1\cdot a_2;b_2,\dots b_q) &= a_1\cdot E_{1,q}(a_2;b_1,\dots b_q) + E_{1,q}(a_1;b_1,\dots b_q)\cdot a_2 \\ &+ \textstyle\sum_{p=1}^{q-1} E_{1,p}(a_1;b_1,\dots b_p)\cdot E_{1,q-p}(a_2;b_{p+1},\dots b_q) \end{aligned} \tag{5.2}$$

$$\begin{aligned} E_{1,n}(E_{1,m}(a;b_2,\dots b_m;c_1,\dots,c_n)) &= \textstyle\sum_{0\leq i_1\leq j_1\leq\cdots\leq i_m\leq j_m\leq n} \\ E_{1,m+n+(i_1+\cdots+i_m)-(j_1+\cdots+j_m)}&(a;c_1,\dots,c_{i_1},E_{1,j_1-i_1}(b_1;c_{i_1+1},\dots,c_{j_1}), \\ c_{j_1+1},\dots,c_{i_2},&E_{1,j_2-i_2}(b_2;c_{i_2+1},\dots,c_{j_2}),c_{j_2+1},\dots,c_{i_m}, \\ &E_{1,j_m-i_m}(b_m;c_{i_m+1},\dots,c_{j_m}),c_{j_m+1},\dots,c_n). \end{aligned} \tag{5.3}$$

Denote $E_{1,1}$ by $\smile_1$; setting $q=1$, relations (5.1) and (5.2) reduce to

$$d\,(a\smile_1 b)+da\smile_1 b+a\smile_1 db=a\cdot b+b\cdot a \quad\text{and}$$
$$(a\cdot b)\smile_1 c=a\cdot(b\smile_1 c)+(a\smile_1 c)\cdot b.$$

Thus $\smile_1$ measures the deviation of $\cdot$ from commutativity and is a right derivation of the product. Setting $q=2$, relation (5.1) reduces to

$$\begin{aligned}&dE_{1,2}\,(a;b,c)+E_{1,2}\,(da;b,c)+E_{1,2}\,(a;db,c)+E_{1,2}\,(a;b,dc)\\&=a\smile_1(b\cdot c)+(a\smile_1 b)\cdot c+b\cdot(a\smile_1 c)\,.\end{aligned}$$

Thus $\smile_1$ is a left derivation up to homotopy.

Let $(A,d,\cdot)$ be a 1-connected dga with an hGa structure E. Consider the tensor coalgebra $BA\otimes BA$ with coproduct $\psi:=\sigma_{2,2}\,(\Delta_{BA}\otimes\Delta_{BA})$. Define $\psi^{(0)}:=1$ and $\psi^{(k)}:=\left(\psi\otimes\mathbf{1}^{\otimes k-1}\right)\cdots(\psi\otimes\mathbf{1})\,\psi$, where $\mathbf{1}$ denotes the identity on $BA\otimes BA$. Comultiplicatively extend the hGa structure maps $E_{0,1}=E_{1,0}=\mathbf{1}_A$ as coalgebra maps $E_{0,1}:[\,]\otimes BA\to BA$ and $E_{1,0}:BA\otimes[\,]\to BA$. Then $E_{0,1}$ and $E_{1,0}$ have degree zero, are undefined except with respect to units, i.e., $E_{0,1}\,([\,]\otimes[x])=E_{1,0}\,([x]\otimes[\,])=[x]$, and generate the shuffle product

$$sh:=\sum_{k\geq 1}(E_{0,1}+E_{1,0})^{\otimes k}\,\psi^{(k-1)}:BA\otimes BA\to BA.$$

For example, $sh\,([a|b]\otimes[c])=(E_{0,1}+E_{1,0})^{\otimes 3}\,\psi^{(2)}\,([a|b]\otimes[c])=[a|b|c]+[a|c|b]+[c|a|b]$.

In general, the dgHa structure of $(BA,d_{BA},\Delta_{BA},sh)$ fails to induce a gHa structure on $H=H^*\,(BA)$. However, an induced gHa structure (H,Δ,μ) is obtained by comultiplicatively extending the hGa structure and perturbing the shuffle product, i.e.,

$$\mu_{BA}:=\sum_{k\geq 1}(E_{0,1}+E_{1,0}+E_{1,1}+E_{1,2}+\cdots)^{\otimes k}\,\psi^{(k-1)}:BA\otimes BA\to BA.$$

Then for example, $\mu_{BA}\,([a]\otimes[b])=[a|b]+[b|a]+[a\smile_1 b]$, and in particular, $\mu_{BA}\,([a]\otimes[a])=[a\smile_1 a]$.

5.3. **A Non-trivial G-S Extension of Loop Cohomology.** Let $Y:=\left(S^2\times S^3\right)\vee\Sigma\mathbb{C}P^2$ and consider the total space X of the 2-stage Postnikov system

$$\begin{array}{ccccc}K(\mathbb{Z}_2,4)&\longrightarrow&X&&\\&&\downarrow&&\\&&Y&\overset{f}{\longrightarrow}&K(\mathbb{Z}_2,5)\;.\\&&a_2a_3+Sq^2b&\underset{f^*}{\longleftarrow}&\iota_5\end{array}$$

Denote the generators of $A:=H^*\,(X;\mathbb{Z}_2)$ by $a_i\in H^i\left(S^i;\mathbb{Z}_2\right)$, $\left\{b,Sq^2b\right\}\in H^*(\Sigma\mathbb{C}P^2;\mathbb{Z}_2)$, and $\left\{Sq^1\iota_4,Sq^2\iota_4,\ldots\right\}\in H^*\,(\mathbb{Z}_2,4;\mathbb{Z}_2)$. The hGa structure of A is non-degenerate with $E_{1,1}:A\otimes A\to A$ given by

$$E_{1,1}\,(b\otimes b)=Sq^2b=a_2a_3.$$

The bar construction BA with standard differential d and cofree coproduct Δ_{BA} is a dg coalgebra. Note that $d([a_2|a_3]+[a_3|a_2])=0$. Lift $E_{1,0}$, $E_{0,1}$, and $E_{1,1}$ to BA and extend as coalgebra maps. Then μ_{BA} acts as the shuffle product except

$$\mu_{BA}\,([b]\otimes[b])=[a_2a_3]=d\,[a_2|a_3]\,,$$

$(BA,d,\Delta_{BA},\mu_{BA})$ is a dgHa, and $H:=H^*\,(BA;\mathbb{Z}_2)\approx H^*\,(\Omega X;\mathbb{Z}_2)$ as modules.

Let $\alpha_{i-1}:=cls[a_i]$, $\beta:=cls[b]$, and $\gamma:=cls([a_2|a_3]+[a_3|a_2])$; then the induced product and coproduct μ and Δ on H act as in Example 2 so that (H,μ,Δ) is a gHa. Represent γ by $\bar{\gamma}:=[a_2|a_3]+[a_3|a_2]$, a generator $x\neq\gamma$ by $\bar{x}:=[\uparrow x]$, and a general class $y_1|\cdots|y_n$ by $\bar{y}_1|\cdots|\bar{y}_n$. Define a cocycle-selecting homomorphism $g:H\to BA$ by $g\,(y_1|\cdots|y_n):=\bar{y}_1|\cdots|\bar{y}_n$; then the Transfer Algorithm transfers the dgHa structure on BA to an A_∞-bialgebra structure on H along g, which specializes to a strict A_k-bialgebra structure for each $k\geq 3$.

S. Saneblidze was the first to consider hGa's with non-trivial actions of the Steenrod algebra $\mathcal{A}_2$ in [6]. In general, the Steenrod $\smile_1$-cochain operation together with other higher cochain operations induce a non-trivial hGa structure on $S^*(X;\mathbb{Z}_2)$, but the failure of the differential to be a $\smile_1$-derivation prevents an immediate lifting of the hGa structure to cohomology (for some remarks on the history of lifting a $\smile_1$-operation on homology see [4] and [6]).

When no multiplicative map $A \to C$ of dga's exists, as is the case when $A = BH^*(X;\mathbb{Z}_2)$ and $C = S^*(\Omega X;\mathbb{Z}_2)$, there may exist a family of dga's $\{B_i\}$ and a zig-zag of multiplicative maps $A \leftarrow B_1 \cdots B_k \to C$. Indeed, in our application we have $BH^*(X;\mathbb{Z}_2) \leftarrow B(RH^*(X;\mathbb{Z}_2)) \leftarrow B(R_aH^*(X;\mathbb{Z}_2)) \to B(S^*(X;\mathbb{Z}_2) \to S^*(\Omega X;\mathbb{Z}_2)$, where the first is induced by the Hirsch resolution map $H^*(X;\mathbb{Z}_2) \leftarrow RH^*(X;\mathbb{Z}_2)$, the second is induced by the Hirsch resolution projection $RA \leftarrow R_aH^*(X;\mathbb{Z}_2)$, where $R_aH^*(X)$ denotes the Hirsch (absolute) resolution of $H^*(X)$, the third is induced by the Hirsch modeling map $R_aH^*(X;\mathbb{Z}_2) \to S^*(X;\mathbb{Z}_2)$, and the fourth is standard. Under this zig-zag, $H := H^*(BA;\mathbb{Z}_2)$ is a gHa model for $H^*(\Omega X;\mathbb{Z}_2)$.

Proposition 1. *The gHa model $H \approx H^*(\Omega(X);\mathbb{Z})$ admits a topologically invariant induced G-S bialgebra structure $\{\omega_3^1, \omega_2^2, \omega_1^3\}$ such that*

$$\omega_3^1 \neq 0,\ \omega_2^2 \neq 0,\ \text{and } \omega_1^3 \equiv 0.$$

Thus $\left(H, \mu, \Delta, \omega_2^2, \omega_3^1\right)$ is a G-S extension of H.

Proof. First, by the Transfer Algorithm Theorem, there is a cochain homotopy $g_1^2 : H \to BA \otimes BA$ satisfying the JJ_1^2 structure relation $\nabla g_1^2 = \Delta_{BA} g + (g \otimes g)\Delta$. Since $\nabla g_1^2 = 0$ by the comultiplicativity of g, we may choose $g_1^2 = 0$. Dually, note that

$$(g\mu + \mu_{BA}(g \otimes g))\,(x \otimes y) = \begin{cases} [a_2a_3]\,, & x \otimes y = \beta \otimes \beta \\ 0, & \text{otherwise.} \end{cases}$$

By the Transfer Algorithm, there is a cochain homotopy $g_2^1 : H \otimes H \to BA$ satisfying the JJ_2^1 structure relation $\nabla g_2^1 = g\mu + \mu_{BA}(g \otimes g)$ such that for some $i \in \{2,3\}$

$$g_2^1(x \otimes y) = \begin{cases} [a_i|a_{5-i}]\,, & x \otimes y = \beta \otimes \beta \\ 0, & \text{otherwise.} \end{cases}$$

Choose $i = 2$ so that $g_2^1(\beta \otimes \beta) = [a_2|a_3]$ (the choice $i = 3$ gives rise to an isomorphic structure); the analysis in [12] implies

$$\omega_2^2\,(\beta \otimes \beta) = \alpha_1 \otimes \alpha_2.$$

Second, by the Transfer Algorithm, there is a cochain homotopy $g_1^3 : H \to BA^{\otimes 3}$ satisfying the JJ_1^3 structure relation

$$\nabla g_1^3 = g^{\otimes 3}\omega_1^3 + \left(g \otimes g_1^2 + g_1^2 \otimes g\right)\Delta + \left(\Delta_{BA} \otimes \mathbf{1} + \mathbf{1} \otimes \Delta_{BA}\right) g_1^2 + \omega_{BA}^{3,1} g. \tag{5.4}$$

Since $\omega_{BA}^{3,1} = 0$ and $g_1^2 = 0$ by the choice above, (5.4) reduces to $\nabla g_1^3 = g^{\otimes 3}\omega_1^3 = \tilde{g}\left(\omega_1^3\right)$. Since H is free as a $\mathbb{Z}_2$-module, $\tilde{g} : Hom^*(H,\ H^{\otimes 3}) \to Hom^*\left(H, BA^{\otimes 3}\right)$ is a cohomology isomorphism, and $\tilde{g}\left(\omega_1^3\right)$ vanishes in cohomology, it follows that

$$\omega_1^3 \equiv 0.$$

Dually, there is a cochain homotopy $g_3^1 : H^{\otimes 3} \to BA$ satisfying the JJ_3^1 structure relation

$$\nabla g_3^1 = g\omega_3^1 + \mu_{BA}\left(g \otimes g_2^1 + g_2^1 \otimes g\right) + g_2^1\left(\mu \otimes \mathbf{1} + \mathbf{1} \otimes \mu\right) + \omega_{BA}^{1,3} g^{\otimes 3}. \tag{5.5}$$

For simplicity let $\phi_3^1 := \mu_{BA}\left(g \otimes g_2^1 + g_2^1 \otimes g\right) + g_2^1\left(\mu \otimes \mathbf{1} + \mathbf{1} \otimes \mu\right)$ and note that

$$\phi_3^1\,(\beta \otimes \beta \otimes \sigma) = \phi_3^1\,(\sigma \otimes \beta \otimes \beta) = \begin{cases} \mu_{BA}\left([a_2|a_3] \otimes \bar{\sigma}\right), & \sigma \neq 1, \beta \\ 0, & \text{otherwise.} \end{cases}$$

Since $\omega_{BA}^{1,3} = 0$, it follows that $\nabla g_3^1 = g\omega_3^1 + \phi_3^1$. Furthermore, since $g\omega_3^1$ and ϕ_3^1 are cohomologous in $Hom\left(H^{\otimes 3}, BA\right)$ and $\tilde{g}$ is a cohomology isomorphism, we have

$$\omega_3^1\,(\beta \otimes \beta \otimes \sigma) = \omega_3^1\,(\sigma \otimes \beta \otimes \beta) = \begin{cases} \mu\left(\alpha_1|\alpha_2 \otimes \sigma\right), & \sigma \neq 1, \beta \\ 0, & \text{otherwise.} \end{cases}$$

Finally, ω is invariant by uniqueness in the Transfer Theorem. □

Proposition 2. *The G-S extension in Proposition* 1 *is non-trivial.*

Proof. By Theorem 2, Part 1, the cochain $\omega_2^2 + \omega_3^1 \in Z_{GS}^{2,-1}(H;H)$. I claim $cls(\omega_2^2 + \omega_3^1) \neq 0$.

Suppose $f = f_1^2 \in Hom^{-1}(H, H^{\overline{\otimes}2})$ satisfies $\partial f = \omega_2^2$. Since $\omega_2^2(\beta\otimes\beta) = \alpha_1 \otimes \alpha_2$ by the proof of Proposition 1, evaluating at $\beta\otimes\beta$ gives

$$(\mu\otimes\mu)\sigma_{2,2}\big(\Delta(\beta)\otimes f(\beta) + f(\beta)\otimes\Delta(\beta)\big) = \alpha_1\otimes\alpha_2.$$

Since each component of the left-hand side has a factor involving β, this is impossible.

Suppose $f = f_2^1 \in Hom^{-1}(H^{\otimes 2}, H)$ satisfies $\delta f = \omega_2^2$; then

$$(\mu\otimes f + f\otimes\mu)\sigma_{2,2}(\Delta\otimes\Delta) + \Delta f = \omega_2^2.$$

Note that $\mu(\beta\otimes\beta) = 0$ since μ acts as the shuffle product, and $f(1\otimes 1) = 0$ for dimensional reasons. Evaluating the left-hand-side at $\beta\otimes\beta$ we obtain

$$1\otimes f(\beta\otimes\beta) + \beta\otimes f(1\otimes\beta) + \beta\otimes f(\beta\otimes 1).$$

Since the required (primitive) component $f(\beta\otimes\beta)\otimes 1$ is missing from

$$\Delta f(\beta\otimes\beta) = 1\otimes f(\beta\otimes\beta) + \beta\otimes f(1\otimes\beta) + \beta\otimes f(\beta\otimes 1) + \alpha_1\otimes\alpha_2,$$

and this too is impossible.

Therefore $D(f_1^2 + f_2^1) \neq \omega_2^2 + \omega_3^1$ for all $f_1^2 \in Hom^{-1}(H, H^{\overline{\otimes}2})$ and all $f_2^1 \in Hom^{-1}(H^{\otimes \underline{2}}, H)$, and it follows that $cls(\omega_2^2 + \omega_3^1) \neq 0$ as claimed. The conclusion follows by Theorem 2, Part 2. □

There is a family of spaces $\{X_{(k)}\}_{k\geq 3}$ such that $H^*(\Omega X_{(k)};\mathbb{Z}_2)$ admits an induced topologically invariant A_∞-bialgebra structure $\{\omega_{(k)}^{j,i}\}_{i+j>3}$ with $\omega_{(k)}^{k,2} \neq 0$ for each $k \geq 2$ (see Example 12.5 in [8]). Unfortunately, when $k \geq 4$ the required KK structure relations cannot be expressed in terms of G-S differentials (see Remark 2). One possible remedy might be to extend the G-S complex to a multicomplex with additional differentials defined in terms of the higher order operations. Given such a construction, the deformation complex for A_k-bialgebras would be in place.

Finally, the dgHa model $H \approx H^*(\Omega X;\mathbb{Z}_2)$ in our application admits an A_k-bialgebra structure for each $k \geq 3$. It would be nice to have a family of spaces $\{X_k\}_{k\geq 3}$ such that $H^*(X_k)$ admits an A_k but not an A_{k+1}-bialgebra structure. We leave this problem for the reader.

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