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TRIPLE WEAK SOLUTIONS FOR A KIRCHHOFF-TYPE PROBLEM

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Abstract. The aim of this work is to investigate the existence and multiplicity of solutions for a second order Kirchhoff-type problem. Under appropriate conditions, we prove the existence of an open interval of positive parameters under which the problem admits at least three distinct weak solutions. Three weak solutions follow from a recent G. Bonanno variational principle.

1. Introduction

The aim of the present paper is to investigate the existence of at least three positive weak solutions for the following second-order Kirchhoff-type problem

$$\begin{cases} -K\Big(\int\limits_a^b |u'(x)|^2 dx\Big)u''(x) = \lambda\ \alpha(x)f(x,u(x)) + h(x,u(x)), & x \in (a,b), \\ u(a) = 0, & u(b) = \beta u(\gamma), \end{cases} \tag{1.1}$$

where λ is a positive parameter, $K : [0,+\infty[\to \mathbb{R}$ is a continuous function such that there exist positive numbers m_0 and m_1 with $m_0 \le K(x) \le m_1$ for all $x \ge 0$, $a,b \in \mathbb{R}$ with $a < b$, $f : [a,b]\times\mathbb{R} \to \mathbb{R}$ is a non-negative L^1-Carathéodory function, $\alpha \in L^\infty([a,b])$, $\alpha(x) \ge 0$, for a.e. $x \in [a,b]$, $\alpha \not\equiv 0$, $h : [a,b]\times\mathbb{R} \to \mathbb{R}$ is a non-negative Carathéodory function , there exists $L > 0$ such that $h(x,t) \le L|t|$ for each $x \in [a,b]$ and $t \in \mathbb{R}$, $\beta \in \mathbb{R}$ and $\gamma \in (a,b)$.

In recent years, due to the widespread application of boundary value problems in engineering, the study of three-point boundary value problems has become the subject of research by many authors.

For example, Gupta [8], under natural conditions on f, using degree-theoretic arguments, obtained the existence and uniqueness theorems to the three-point nonlinear second-order boundary value problem

$$\begin{cases} u'' = f(x,u(x),u'(x)) - e(x), & 0 < x < 1, \\ u(0) = 0, & u(\eta) = u(1), \end{cases}$$

where $f : [0,1]\times\mathbb{R}^2 \to \mathbb{R}$ is a function satisfying Carathéodory's conditions, $e : [0,1] \to \mathbb{R}$ is a function in $L^1(0,1)$ and $\eta \in (0,1)$.

He and Ge in [9], based upon the Leggett–Williams fixed-point theorem, provided the conditions for the existence of three positive solutions to the nonlinear boundary value problem

$$\begin{cases} u'' + f(t,u) = 0, & t \in (0,1), \\ u(0) = 0, & au(\eta) = u(1), \end{cases} \tag{1.2}$$

where $0 < \eta < 1$, $0 < a$ and $a\eta < 1$. Also, $f : [0,1] \times [0,\infty) \to [0,\infty)$ is a continuous function and $f(t,.)$ does not vanish identically on any subset of $[0,1]$ with positive measure. Recently, Lin in [15] by using variational methods and a three-critical-point theorem, considered the existence of at least three solutions for problem (1.2). Different types of Kirchhoff equation are expressed in [14]. It was proposed as an extension of the classical D'Alembert's wave equation for free vibrations of elastic strings. The Kirchhoff's model takes into account the length changes of the string produced by transverse vibrations. Some interesting results can be found in [4, 7, 20]. Many researchers have studied the problems of Kirchhoff-type (we refer the reader to the papers [3, 10, 11, 18, 19] and the

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references therein). Also, a for three-point boundary value problem of Kirchhoff-type we refer the reader to papers [1,2,12]. For example, in [2], the authors, based on variational methods and a critical point theorem, established the existence of at least one weak solution for the problem

$$\begin{cases} -K\Big(\int\limits_a^b |u'(t)|^2 dt\Big)u''(t) = f(t,u(t)) + h(u(t)), & t \in (a,b), \\ u(a) = 0, & u(b) = \alpha u(\eta), \end{cases}$$

where $K : [0,+\infty[\to \mathbb{R}$ is a continuous function such that there exist positive numbers m and M with $m \le K(x) \le M$ for all $x \ge 0$, $a, b \in \mathbb{R}$ with $a < b$, $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $h : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, $\alpha \in \mathbb{R}$ and $\eta \in (a,b)$.

In this paper, using a three critical points theorem obtained in [5], we establish the existence of at least three weak solutions for problem (1.1).

2. Preliminaries

In this section, we recall some basic facts and introduce the necessary notations.

Definition 2.1. A function $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is said to be a Carathéodory function if:

(C_1) the function $x \to f(x,t)$ is measurable for every $t \in \mathbb{R}$;

(C_2) the function $t \to f(x,t)$ is continuous for a.e. $x \in [a,b]$. And $f : [a,b] \times \mathbb{R} \to \mathbb{R}$ is said to be an L^1-Carathéodory function if, in addition to conditions (C_1) and (C_2), the following condition is also satisfied:

(C_3) for every $\rho > 0$, there is the function $l_\rho \in L^1([a,b])$ such that $\sup_{|t|\le\rho} |f(x,t)| \le l_\rho(x)$ for almost every $x \in [a,b]$.

Denote

$$X := \big\{u \in W^{1,2}(a,b) | u(a) = 0, \quad u(b) = \beta u(\gamma)\big\},$$

endowed with the norm

$$\|u\| := \Big(\int\limits_a^b |u'(x)|^2 dx\Big)^{1/2}.$$

Theorem 2.1 ([15, Theorem 3.2]). *X is a separable and reflexive real Banach space.*

Theorem 2.2 ([6, Theorem 8.8]). *If I is a bounded subset of $\mathbb{R}$, then the injection $W^{1,p}(I) \subset C(\bar{I})$ is compact for all $1 < p \le \infty$.*

Remark 2.1. From Theorem 2.2, we see that the embedding $X \to C([a,b])$ is compact.

The following lemma is required in the proof of the main theorem of this paper.

Lemma 2.1 ([2, Lemma 2.4]). *For all $u \in X$, we have*

$$\max_{x\in[a,b]} |u(x)| \le \frac{(1+|\beta|)\sqrt{b-a}}{2} \|u\|. \tag{2.1}$$

Remark 2.2. If $\{u_n\}_{n\in\mathbb{N}}$ is a bounded sequence in X, then from the compact embedding $X \hookrightarrow C([a,b])$ it has a subsequence that pointwise converges to some $u \in X$ (it comes from the definition of compact embedding) . Also, since X is a reflexive space, then there exists a subsequence that weakly converges in X (see [6, Theorem 3.18]) and so, according to continuous embedding, $X \to L^\infty([a,b])$ weakly converges in $L^\infty([a,b])$.

In accordance with the functions f, K and h, we introduce the functions $F : [a,b] \times \mathbb{R} \to \mathbb{R}, \widetilde{K} : [0,\infty) \to \mathbb{R}$ and $H : [a,b] \times \mathbb{R} \to \mathbb{R}$, respectively, as follows:

$$F(x,t) = \int_0^t f(x,\xi)d\xi \quad \text{for all } \ (x,t) \in [a,b] \times \mathbb{R},$$

$$\widetilde{K}(t) = \int_0^t K(\xi)d\xi \quad \text{for all } \ t \geq 0$$

and

$$H(x,t) = \int_0^t h(x,\xi)d\xi \quad \text{for all } \ (x,t) \in [a,b] \times \mathbb{R}.$$

Moreover, set

$$F^\theta = \int_a^b \sup_{|\xi| \leq \theta} F(x,\xi)dx \quad \text{for all } \ \theta > 0.$$

In this paper, we assume that the following condition

$$4m_0 > L(1+|\beta|)^2(b-a)^2$$

holds. Let Φ, $\Psi : X \to \mathbb{R}$ be defined by

$$\Phi(u) = \frac{1}{2}\widetilde{K}(\|u\|^2) - \int_a^b H(x,u(x))dx$$

and

$$\Psi(u) = \int_a^b \alpha(x)F(x,u(x))dx,$$

for every $u \in X$.
Now, according to (2.1), we observe that

$$\frac{(4m_0 - L(1+|\beta|)^2(b-a)^2)}{8}\|u\|^2 \leq \Phi(u) \leq \frac{(4m_1 + L(1+|\beta|)^2(b-a)^2)}{8}\|u\|^2, \tag{2.2}$$

for every $u \in X$. Ψ is a differentiable functional whose differential at the point $u \in X$ is

$$\Psi'(u)(v) = \int_a^b \alpha(x)f(x,u(x))v(x)dx,$$

and Φ is a continuously Gâteaux differentiable functional whose differential at the point $u \in X$ is

$$\Phi'(u)(v) = K\left(\int_a^b |u'(x)|^2\right)\left(\int_a^b u'(x)v'(x)dx\right) - \int_a^b h(x,u(x))v(x)dx,$$

for every $v \in X$.

Definition 2.2. Let Φ and Ψ be defined as above and put $I_\lambda = \Phi - \lambda\Psi$. We say that $u \in X$ is a critical point of I_λ when $I'_\lambda(u) = 0_{\{X^*\}}$, that is, $I'_\lambda(u)(v) = \Phi'(u)(v) - \lambda\Psi'(u)(v) = 0$ for all $v \in X$.

Definition 2.3. A function $u \in X$ is a weak solution to problem (1.1) if

$$K\left(\int_a^b |u'(x)|^2\right)\left(\int_a^b u'(x)v'(x)dx\right) - \int_a^b h(x,u(x))v(x)dx - \lambda\int_a^b \alpha(x)f(x,u(x))v(x)dx = 0,$$

for every $v \in X$.

Remark 2.3. We observe that the weak solutions of problem (1.1) are exactly the solutions of the equation $I'_\lambda(u)(v) = 0$.

Lemma 2.2. *If $u_0 \not\equiv 0$ is a weak solution for problem* (1.1)*, then u_0 is non-negative.*

Proof. From Remark 2.3, one has $I'_\lambda(u_0)(v) = 0$, for all $v \in X$. Choosing $v(x) = \max\{-u_0(x), 0\}$, putting $E = \{x \in [a, b] : u_0(x) < 0\}$ and arguing by contradiction, we assume that E is a nonempty set. Then we have

$$K\left(\int_{E\cup E^c} |u_0'(x)|^2 dx\right)\left(\int_{E\cup E^c} u_0'(x)v'(x)dx\right)$$

$$= \lambda \int_a^b \alpha(x)f(x, u_0(x))v(x)dx + \int_a^b h(x, u_0(x))v(x)dx \geq 0,$$

that is,

$$-m_1 \int_E v'(x)v'(x)dx \geq 0$$

which means that $-m_1\|v\|^2 \geq 0$ and one has $\|v\| = 0$, therefore $v = 0$. But this is absurd and so, E is an empty set. Hence $-u_0 \leq 0$, that is, $u_0 \geq 0$ and the proof is complete. □

Definition 2.4. A Gâtuax differentiable function I satisfies the Palais–Smale condition (in short, (PS)-condition) if any sequence $\{u_n\}$ such that

(a) $\{I(u_n)\}$ is bounded,
(b) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,

has a convergent subsequence.

Below, we will present a non-standard state of the Palais–Smale condition that is introduced in [5].

Definition 2.5. (see [5]) Fix $r \in]-\infty, +\infty]$. A Gâteaux differentiable function $I : X \to \mathbb{R}$ satisfies the Palais–Smale condition cut off upper at r (in short, $(PS)^{[r]}$-condition) if any sequence $\{u_n\} \subseteq X$ such that

(a) $\{I(u_n)\}$ is bounded,
(b) $\lim_{n\to+\infty} \|I'(u_n)\|_{X^*} = 0$,
(c) $\Phi(u_n) < r$, $\forall n \in \mathbb{N}$,

has a convergent subsequence.

Our main tool is the following critical point theorem.

Theorem 2.3 ([5, Theorem 7.3])**.** *Let X be a real Banach space and let $\Phi, \Psi : X \longrightarrow \mathbb{R}$ be two continuously Gâteaux differentiable functions with Φ bounded from below and convex such that* $\inf_X \Phi = \Phi(0) = \Psi(0) = 0$.

Assume that there are two positive constants r_1, r_2 and $\overline{u} \in X$, with $2r_1 < \Phi(\overline{u}) < \frac{r_2}{2}$ such that

(b$_1$) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} < \dfrac{2}{3}\dfrac{\Psi(\overline{u})}{\Phi(\overline{u})}$;

(b$_2$) $\dfrac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}{r_2} < \dfrac{1}{3}\dfrac{\Psi(\overline{u})}{\Phi(\overline{u})}$.

Assume also that for each

$$\lambda \in \Lambda = \left]\frac{3}{2}\frac{\Phi(\overline{u})}{\Psi(\overline{u})},\ \min\left\{\frac{r_1}{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)},\ \frac{\frac{r_2}{2}}{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)} \Psi(u)}\right\}\right[,$$

the functional $\Phi - \lambda\Psi$ satisfies the $(PS)^{[r_2]}$-condition and

$$\inf_{t\in[0,1]} \Psi(tu_1 + (1 - t)u_2) \geq 0,$$

for each $u_1, u_2 \in X$ *which are local minima for the functional* $\Phi - \lambda\Psi$ *and such that* $\Psi(u_1) \geq 0$ *and* $\Psi(u_2) \geq 0$.

Then, for each $\lambda \in \Lambda$*, the functional* $\Phi - \lambda\Psi$ *admits at least three critical points which lie in* $\Phi^{-1}(]-\infty, r_2[)$.

Now we present a proposition that will be needed to prove the main Theorem of this paper.

Proposition 2.1. *Take* $I_\lambda = \Phi - \lambda\Psi$ *as in Definition* 2.2. *Then* I_λ *satisfies the* $(PS)^{[r]}$*-condition for any* $r > 0$.

Proof. Consider the sequence $\{u_n\} \subseteq X$ such that $\{I_\lambda(u_n)\}$ is bounded, $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_{X^*} = 0$ and $\Phi(u_n) < r$, $\forall n \in \mathbb{N}$. Since $\Phi(u_n) < r$, from (2.2), we see that $\{u_n\}$ is bounded in X. Therefore passing to a subsequence, if necessary, we can assume that $u_n(x) \to u(x)$, and there is $s > 0$ such that $|u_n(x)| \leq s$ for all $x \in [a,b]$ and for all $n \in \mathbb{N}$ and also, $\{u_n\}$ weakly converges to u in $L^\infty([a,b])$ (see Remark 2.2). Now, according to Hölder's inequality and Lebesque's Dominated Convergence Theorem, since $\alpha(x)f(x,u_n(x)) \leq \alpha(x).\ \max_{|\xi|\leq s} f(x,\xi) \in L^1([a,b])$ for all $n \in \mathbb{N}$ and $f(x,u_n(x)) \to f(x,u(x))$ for a.e. $x \in [a,b]$ (f is L^1-Carathéodory function), one has $\alpha f(x,u_n)$ is strongly converging to $\alpha f(x,u)$ in $L^1([a,b])$. Now, since $u_n \rightharpoonup u$ in $L^\infty([a,b])$ and $\alpha f(x,u_n) \to \alpha f(x,u)$ in $L^1([a,b]) \subseteq (L^\infty([a,b]))^*$, from [6, Proposition 3.5(iv)], one has

$$\lim_{n\to+\infty} \int_a^b \alpha(x)f(x,u_n(x))(u_n(x)-u(x))dx = 0. \tag{2.3}$$

Similarly, we have

$$\lim_{n\to+\infty} \int_a^b h(x,u_n(x))(u_n(x)-u(x))dx = 0. \tag{2.4}$$

From $\lim_{n\to+\infty} \|I'_\lambda(u_n)\|_X = 0$, there exists a sequence $\{\varepsilon_n\}$, with $\varepsilon_n \to 0^+$ such that

$$\begin{aligned} \Bigg| K\bigg(\int_a^b |u'_n(x)|^2\bigg)\bigg(\int_a^b u'_n(x)v'(x)dx\bigg) - \lambda \int_a^b \alpha(x)f(x,u_n(x))v(x)dx \\ - \int_a^b h(x,u_n(x))v(x)dx\Bigg| \leq \varepsilon_n, \end{aligned} \tag{2.5}$$

for all $n \in \mathbb{N}$ and for all $v \in X$ with $\|v\| \leq 1$. Taking into account $v(x) = \frac{u_n(x)-u(x)}{\|u_n-u\|}$, from (2.5), one has

$$\begin{aligned} \Bigg| K\bigg(\int_a^b |u'_n(x)|^2\bigg) \int_a^b u'_n(x)(u'_n(x)-u'(x))dx - \lambda \int_a^b \alpha(x)f(x,u_n(x))(u_n(x)-u(x))dx \\ - \int_a^b h(x,u_n(x))(u_n(x)-u(x))dx\Bigg| \leq \varepsilon_n\|u_n-u\|, \end{aligned} \tag{2.6}$$

for all $n \in \mathbb{N}$. Now, according to the inequality $|a||b| \leq \frac{1}{2}|a|^2 + \frac{1}{2}|b|^2$, one has

$$\begin{aligned} &K\bigg(\int_a^b |u'_n(x)|^2\bigg) \int_a^b u'_n(x)(u'_n(x)-u'(x))dx \\ &= K\bigg(\int_a^b |u'_n(x)|^2\bigg)\bigg(\int_a^b |u'_n(x)|^2dx - \int_a^b u'_n(x)u'(x)dx\bigg) \end{aligned}$$

$$\geq m_0\Big(\|u_n\|^2 - \int\limits_a^b \Big(\frac{1}{2}|u_n'(x)|^2 + \frac{1}{2}|u'(x)|^2\Big)dx\Big)$$

$$= m_0\Big(\|u_n\|^2 - \frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2\Big) = m_0\Big(\frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2\Big).$$

Hence from (2.6), we have

$$m_0\Big(\frac{1}{2}\|u_n\|^2 - \frac{1}{2}\|u\|^2\Big) \leq \lambda \int\limits_a^b \alpha(x)f(x,u_n(x))(u_n(x)-u(x))dx$$

$$+ \int\limits_a^b h(x,u_n(x))(u_n(x)-u(x))dx + \varepsilon_n\|u_n - u\|,$$

that is,

$$\frac{m_0}{2}\|u_n\|^2 \leq \frac{m_0}{2}\|u\|^2 + \lambda \int\limits_a^b \alpha(x)f(x,u_n(x))(u_n(x)-u(x))dx$$

$$+ \int\limits_a^b h(x,u_n(x))(u_n(x)-u(x))dx + \varepsilon_n\|u_n - u\|. \tag{2.7}$$

Now, according to (2.3), (2.4) and (2.7), when $\varepsilon_n \to 0^+$, we have

$$\limsup_{n\to+\infty} \|u_n\| \leq \|u\|.$$

Thus [6, Proposition 3.32] ensures that $u_n \to u$ strongly in X, and the proof is complete. □

3. Main Results

In this section, we formulate our main results. Put

$$Z := \frac{1}{\gamma - a} + \frac{2(\beta-1)^2}{b-\gamma},$$

$$\tau := \sqrt{\frac{4m_0 - L(1+|\beta|)^2(b-a)^2}{4m_1 + L(1+|\beta|)^2(b-a)^2}}$$

and

$$\Theta := \frac{4m_0 - L(1+|\beta|)^2(b-a)^2}{2(1+|\beta|)^2(b-a)}.$$

Now, we express the main results.

Theorem 3.1. *Assume that there exist three positive constants δ, θ_1 and θ_2 with*

$$\frac{2\sqrt{2}\theta_1}{(1+|\beta|)\sqrt{Z(b-a)}} < \delta < \frac{\sqrt{2}\tau\theta_2}{(1+|\beta|)\sqrt{Z(b-a)}} \tag{3.1}$$

such that

(i) $$\frac{F^{\theta_1}}{{\theta_1}^2} < \frac{16}{3}\,\frac{\Theta \int\limits_\gamma^{\frac{b+\gamma}{2}} \alpha(x)F(x,\delta)dx}{\|\alpha\|_\infty\big(4m_1 + L(1+|\beta|)^2(b-a)^2\big)Z\delta^2},$$

(ii) $$\frac{F^{\theta_2}}{{\theta_2}^2} < \frac{8}{3}\,\frac{\Theta \displaystyle\int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x)F(x,\delta)dx}{\|\alpha\|_\infty\big(4m_1 + L(1+|\beta|)^2(b-a)^2\big)Z\delta^2},$$

where $\|\alpha\|_\infty = \|\alpha\|_{L^\infty}$. *Then, for each*

$$\lambda \in \Lambda := \left]\frac{3}{16}\frac{\big(4m_1 + L(1+|\beta|)^2(b-a)^2\big)Z\delta^2}{\displaystyle\int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x)F(x,\delta)dx}, \min\left\{\frac{\Theta\,{\theta_1}^2}{\|\alpha\|_\infty\ F^{\theta_1}}, \frac{\Theta\,{\theta_2}^2}{2\|\alpha\|_\infty\ F^{\theta_2}}\right\}\right[,$$

problem (1.1) *admits at least three non-negative weak solutions* u_i *for* $i = 1, 2, 3$, *in* X *such that* $0 \le u_i(x) < \theta_2$, $\forall x \in [a, b]$, $(i = 1, 2, 3)$.

Proof. Our aim is to apply Theorem 2.3 to problem (1.1). Fix λ, as in the conclusion. Take X, Φ and Ψ as in the previous section. We observe that the regularity assumptions of Theorem 2.3 on Φ and Ψ are satisfied. Also, according to Proposition 2.1, the functional I_λ satisfies the $(PS)^{[r]}$-condition for all $r > 0$.

Put

$$r_1 := \Theta\,\theta_1^2, \qquad r_2 := \Theta\,\theta_2^2$$

and

$$w(x) := \begin{cases} \frac{\delta}{\gamma-a}(x-a) & \text{if } x \in [a,\gamma) \\ \delta & \text{if } x \in [\gamma, \frac{b+\gamma}{2}] \\ \delta\big(\frac{2(\beta-1)}{b-\gamma}x - \frac{\beta(b+\gamma)-2b}{b-\gamma}\big) & \text{if } x \in (\frac{b+\gamma}{2}, b]. \end{cases}$$

We observe that $w \in X$ and

$$\|w\|^2 = \int_a^b |w'(x)|^2 dx = \Big(\frac{1}{\gamma-a} + \frac{2(\beta-1)^2}{b-\gamma}\Big)\delta^2 = Z\delta^2.$$

In particular, from (2.2), one has

$$\frac{\big(4m_0 - L(1+|\beta|)^2(b-a)^2\big)Z\delta^2}{8} \le \Phi(w) \le \frac{\big(4m_1 + L(1+|\beta|)^2(b-a)^2\big)Z\delta^2}{8}. \tag{3.2}$$

Therefore, from (3.1), one has $2r_1 < \Phi(w) < \frac{r_2}{2}$.

Now, for each $u \in X$ and bearing (2.1) in mind, we see that

$$\begin{aligned} \Phi^{-1}(]-\infty, r_i[) &= \{u \in X;\ \ \Phi(u) < r_i\} \\ &\subseteq \left\{u \in X;\ \frac{(4m_0 - L(1+|\beta|)^2(b-a)^2)}{8}\|u\|^2 < r_i\right\} \\ &\subseteq \{u \in X; \|u(x)| < \theta_i \quad \text{for each } x \in [a,b]\}, \end{aligned}$$

and it follows that

$$\begin{aligned} \sup_{u\in\Phi^{-1}(]-\infty,r_i[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r_i[)} \left(\int_a^b \alpha(x)\ F(x,u(x))dx\right) \\ &\le \int_a^b \alpha(x) \sup_{|\xi|\le\theta_i} F(x,\xi)dx \le \|\alpha\|_\infty\ F^{\theta_i}. \end{aligned}$$

Hence, we have

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)} \Psi(u)}{r_1} \le \frac{\|\alpha\|_\infty\ F^{\theta_1}}{\Theta\,{\theta_1}^2} < \frac{1}{\lambda}. \tag{3.3}$$

On the other hand, from (3.2) and since $\lambda \in \Lambda$, one has

$$\frac{2}{3}\frac{\Psi(w)}{\Phi(w)} \geq \frac{2}{3}\frac{\int_a^b \alpha(x)\ F(x,w(x))dx}{\frac{\left(4m_1+L(1+|\beta|)^2(b-a)^2\right)Z\delta^2}{8}} \geq \frac{16}{3}\frac{\int_\gamma^{\frac{b+\gamma}{2}} \alpha(x)F(x,\delta)dx}{\left(4m_1+L(1+|\beta|)^2(b-a)^2\right)Z\delta^2} > \frac{1}{\lambda}. \tag{3.4}$$

Now, from (3.3) and (3.4), we have

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)}{r_1} < \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.$$

Analogously, from (3.4), we get

$$\frac{2\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} \leq 2\frac{\|\alpha\|_\infty\ F^{\theta_2}}{\Theta\,{\theta_2}^2} < \frac{1}{\lambda} < \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}$$

which means that

$$\frac{\sup_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}{r_2} < \frac{1}{3}\frac{\Psi(w)}{\Phi(w)}.$$

Hence, (b_1) and (b_2) of Theorem 2.3 are established.

Now, if $u_1, u_2 \in X$ are the two local minima of the functional $I_\lambda = \Phi - \lambda\Psi$, with $\Psi(u_1) \geq 0$ and $\Psi(u_2) \geq 0$, then according to Lemma 2.2, u_1 and u_2 are nonnegative, and we get

$$\inf_{t\in[0,1]} \Psi(tu_1 + (1-t)u_2) \geq 0.$$

Finally, for every

$$\lambda \in \Lambda \subseteq \left]\frac{3}{2}\frac{\Phi(w)}{\Psi(w)},\ \min\left\{\frac{r_1}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_1[)}\Psi(u)},\ \frac{r_2/2}{\sup\limits_{u\in\Phi^{-1}(]-\infty,r_2[)}\Psi(u)}\right\}\right[,$$

since the weak solutions of problem (1.1) are exactly the solutions of the equation $I'_\lambda(u) = 0$, therefore Theorem 2.3 (with $\overline{u} = w$) and Lemma 2.2 will guarantee the conclusion. □

Remark 3.1. In Theorem 3.1, if $f(x,0) \neq 0$ or $h(x,0) \neq 0$, then problem (1.1) has at least three non-trivial and non-negative weak solutions.

Remark 3.2. According to $F(x,t) = \int_0^t f(x,\xi)d\xi$, for all $(x,t) \in [a,b]\times\mathbb{R}$, we can consider $F(t) = \int_0^t f(\xi)d\xi$ for all $t \in \mathbb{R}$. When f is a continuous and non-negative function, then $F \in C^1(\mathbb{R})$, and since $F'(t) = f(t) \geq 0$, for all $t \in \mathbb{R}$, we get that $F(t)$ is non-decreasing and so,

$$F^\theta = \int_a^b \sup_{|\xi|\leq\theta} F(\xi)dx = (b-a)\sup_{|\xi|\leq\theta} F(\xi) = (b-a)\,F(\theta) \quad \text{for all } \theta > 0.$$

Hence, when f does not depend on x, hypotheses (i) and (ii) in Theorem 3.1 take the following forms:

(I) $$\frac{F(\theta_1)}{{\theta_1}^2} < \frac{16}{3}\frac{\Theta\,F(\delta)\int_\gamma^{\frac{b+\gamma}{2}}\alpha(x)dx}{(b-a)\|\alpha\|_\infty\left(4m_1+L(1+|\beta|)^2(b-a)^2\right)Z\delta^2},$$

(II) $$\frac{F(\theta_2)}{{\theta_2}^2} < \frac{8}{3}\,\frac{\Theta\, F(\delta)\displaystyle\int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x)dx}{(b-a)\|\alpha\|_\infty\big(4m_1 + L(1+|\beta|)^2(b-a)^2\big)Z\delta^2},$$

and the interval becomes

$$\left]\frac{3}{16}\,\frac{\big(4m_1 + L(1+|\beta|)^2(b-a)^2\big)Z\delta^2}{F(\delta)\displaystyle\int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x)dx},\ \frac{1}{b-a}\min\left\{\frac{\Theta\,{\theta_1}^2}{\|\alpha\|_\infty F(\theta_1)}, \frac{\Theta\,{\theta_2}^2}{2\|\alpha\|_\infty F(\theta_2)}\right\}\right[.$$

As an example, we give the following consequence of Theorem 3.1.

Corollary 3.1. *Assume that* $\int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x)dx \neq 0$ *and let* $f : \mathbb{R} \to [0, +\infty[$ *be a continuous and nonzero function such that*

$$\lim_{\xi\to 0^+} \frac{f(\xi)}{\xi} = \lim_{\xi\to +\infty} \frac{f(\xi)}{\xi} = 0.$$

Then for each $\lambda > \lambda^*$, *where*

$$\lambda^* = \inf\left\{\frac{3}{16}\,\frac{\big(4m_1 + L(1+|\beta|)^2(b-a)^2\big)Z}{\displaystyle\int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x)\,dx}\,\frac{\delta^2}{\displaystyle\int_0^{\delta} f(\xi)\,d\xi} :\ \delta > 0,\ \int_0^{\delta} f\,(\xi)\,d\xi > 0\right\},$$

the problem

$$\begin{cases} -K\Big(\int_a^b |u'(x)|^2 dx\Big)u''(x) = \lambda\ \alpha(x)f(u(x)) + h(x, u(x)), & x \in (a, b),\\ u(a) = 0, & u(b) = \beta u(\gamma) \end{cases}$$

admits at least three distinct non-negative weak solutions.

Proof. Since $\int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x)dx \neq 0$ and $f \not\equiv 0$, we see that

$$\frac{3}{16}\,\frac{\big(4m_1 + L(1+|\beta|)^2(b-a)^2\big)Z}{\displaystyle\int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x)\,dx}\,\frac{\delta^2}{\displaystyle\int_0^{\delta} f(\xi)\,d\xi} < +\infty.$$

Suppose that $\lambda > \lambda^*$ is fixed. Let $\delta > 0$ such that $\int_0^{\delta} f\,(\xi)\,d\xi > 0$ and

$$\lambda > \frac{3}{16}\,\frac{\big(4m_1 + L(1+|\beta|)^2(b-a)^2\big)Z}{\displaystyle\int_{\gamma}^{\frac{b+\gamma}{2}} \alpha(x)\,dx}\,\frac{\delta^2}{\displaystyle\int_0^{\delta} f(\xi)d\xi}\,.$$

Also, we can consider $F(t)=\int_0^t f(\xi)d\xi$ for all $t\in\mathbb{R}$, and hence from Remark 3.2, we have

$$F^{\theta}=(b-a)F(\theta) \quad \text{for all } \ \theta>0.$$

Now, from $\lim_{\xi\to 0^+}\frac{f(\xi)}{\xi}=0$, we have $\lim_{\xi\to 0^+}\frac{\int_0^{\xi} f(t)dt}{\xi^2}=0$ and so, there is $\theta_1>0$ such that $\frac{2\sqrt{2}\,\theta_1}{(1+|\beta|)\sqrt{Z(b-a)}}<\delta$ and

$$\frac{\int_0^{\theta_1} f(t)dt}{{\theta_1}^2}=\frac{F(\theta_1)}{{\theta_1}^2}=\frac{F^{\theta_1}}{(b-a){\theta_1}^2}<\frac{\Theta}{\lambda\,(b-a)\,\|\alpha\|_{\infty}}.$$

Also, from $\lim_{\xi\to +\infty}\frac{f(\xi)}{\xi}=0$, we have $\lim_{\xi\to +\infty}\frac{\int_0^{\xi} f(t)dt}{\xi^2}=0$ and so, there is $\theta_2>0$ such that $\delta<\frac{\sqrt{2}\,\tau\,\theta_2}{(1+|\beta|)\sqrt{Z(b-a)}}$ and

$$\frac{\int_0^{\theta_2} f(t)dt}{{\theta_2}^2}=\frac{F(\theta_2)}{{\theta_2}^2}=\frac{F^{\theta_2}}{(b-a){\theta_2}^2}<\frac{\Theta}{2\lambda\,(b-a)\,\|\alpha\|_{\infty}}.$$

Now, we can apply Theorem 3.1 and the conclusion follows. □

In the following, we will present an example to illustrate Corollary 3.1.

Example 3.1. Let $f(t)=t^4e^{-t^5}$ and hence $\int_0^{\delta} f(\xi)d\xi=\frac{1}{5}(1-e^{-\delta^5})$ for all $\delta>0$. Also, suppose that

$$\alpha(x):=\begin{cases}\frac{2}{b-\gamma} & \text{if } x\in(\gamma,\frac{b+\gamma}{2}),\\ 1 & \text{otherwise}\end{cases}$$

and hence $\int_{\gamma}^{\frac{b+\gamma}{2}}\alpha(x)dx=1$. Put $K(t)=2+\tanh t$ for all $t\geq 0$ with $m_0=1$ and $m_1=3$.

Now, as an example, we can consider $h(x,t)=\frac{e^x}{2(b-a)^2e^b}|t|$ for each $x\in[a,b]$ and $t\in\mathbb{R}$ with $L=\frac{1}{2(b-a)^2}$. We see that the condition $4m_0>L(1+|\beta|)^2(b-a)^2$ is satisfied with $\beta=1$. Then, according to Corollary 3.1, for each

$$\lambda>\inf\left\{\frac{105\,\delta^2}{8(\gamma-a)(1-e^{-\delta^5})},\ \ \delta>0\right\},$$

the problem

$$\begin{cases} -\Big(2+\tanh\big(\int_a^b |u'(x)|^2dx\big)\Big)u''(x) \\ \qquad\qquad = \begin{cases} \dfrac{2\,\lambda}{b-\gamma}u(x)^4e^{-u(x)^5} + \dfrac{e^x}{2(b-a)^2e^b}|u(x)| & \text{if } x\in(\gamma,\frac{b+\gamma}{2}), \\ \lambda u(x)^4e^{-u(x)^5} + \dfrac{e^x}{2(b-a)^2e^b}|u(x)| & \text{if } x\in(a,\gamma]\cup[\frac{b+\gamma}{2},b), \end{cases} \\ u(a)=0, \quad u(b)=u(\gamma) \end{cases}$$

admits at least three non-negative weak solutions.

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