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THE BOUNDARY-CONTACT PROBLEM OF DYNAMICAL VISCOELASTICITY

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Abstract. The dynamical contact problem for a viscoelastic half-plane is considered. The half-plane is reinforced along its boundary by thin elastic cover plate (patch), which in general excited by harmonic tangential and normal forces. Using the methods of integral transformations the problems is reduced to the integro-differential equations with respect unknown contact stresses. The properties and method of orthogonal polynomials the integro-differential equation reduces to an infinite system of linear algebraic equations. The quasi-completely regularity of the obtained systems is proved.

1. Statement of the Problem

We consider the dynamical contact problem for a viscoelastic half-plane which is reinforced along its boundary by an elastic cover plate (patch) with small thickness. The cover plate is under the action of harmonic horizontal $-\tau_0 e^{-i\omega t}\delta(x+1)$ and vertical $p_0 e^{-i\omega t}\delta(x)$ forces with oscillation frequency ω, $\delta(x)$ is the Dirac function. In the linear theory of viscoelasticity for Kelvin–Voigt materials it is required to find the unknown contact stresses along of contact line, $\tau(x)e^{-i\omega t}$ and $p(x)e^{-i\omega t}$, where t is time parameter.

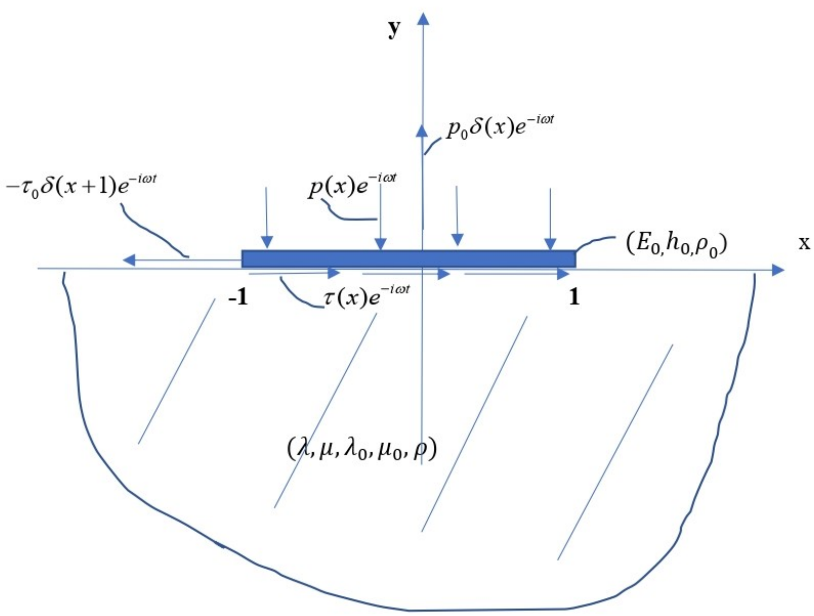

Figure 1

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The problem is formulated in the form of the Lame's differential equations [2, 5–7]

$$\begin{aligned} &\mu\Delta u+\mu_0\Delta\frac{\partial u}{\partial t}+(\lambda+\mu)\frac{\partial\theta}{\partial x}+(\lambda_0+\mu_0)\frac{\partial^2\theta}{\partial x\partial t}=\rho\frac{\partial^2 u}{\partial t^2},\\ &\mu\Delta v+\mu_0\Delta\frac{\partial v}{\partial t}+(\lambda+\mu)\frac{\partial\theta}{\partial y}+(\lambda_0+\mu_0)\frac{\partial^2\theta}{\partial y\partial t}=\rho\frac{\partial^2 v}{\partial t^2} \end{aligned} \tag{1.1}$$

with the boundary condition

$$\begin{aligned} &\sigma_y=\Big(\lambda\theta+2\mu\frac{\partial v}{\partial y}+\lambda_0\frac{\partial\theta}{\partial t}+2\mu_0\frac{\partial^2 v}{\partial y\partial t}\Big)_{y=0}=p(x)e^{-i\omega t},\\ &\tau_{xy}=\mu\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)_{y=0}+\mu_0\frac{\partial}{\partial t}\Big(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\Big)_{y=0}=-\tau(x)e^{-i\omega t}, \end{aligned} \tag{1.2}$$

where $u(x,y,t)$ and $v(x,y,t)$ are the components of the displacement vector, σ_y, τ_{xy} are stresses components, λ, μ and λ_0, μ_0 are elastic and viscoelastic Lame's constants, respectively. ρ is density of the material of plate, $p(x)$ and $\tau(x)$ are amplitudes of normal and tangential contact stress, respectively. $\theta(x,y,t)=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$.

2. Solution of the Problem

Considering stationary oscillation of the elastic half-plate and assuming that

$$u(x,y,t)=u_0(x,y)e^{-i\omega t},\qquad v(x,y,t)=v_0(x,y)e^{-i\omega t}$$

from (1.1), (1.2), we obtain the following boundary value problem

$$\begin{aligned} &(\Delta+p_2^2)u_0+\Big(\frac{C_1}{C_2}-1\Big)\frac{\partial\theta_0}{\partial x}=0,\qquad (\Delta+p_2^2)v_0+\Big(\frac{C_1}{C_2}-1\Big)\frac{\partial\theta_0}{\partial y}=0,\\ &\Big(\lambda^*\theta_0+2\mu^*\frac{\partial v_0}{\partial y}\Big)_{y=0}=p(x),\qquad \mu^*\Big(\frac{\partial u_0}{\partial y}+\frac{\partial v_0}{\partial x}\Big)_{y=0}=-\tau(x), \end{aligned} \tag{2.1}$$

where

$$\theta_0(x,y)=\frac{\partial u_0}{\partial x}+\frac{\partial v_0}{\partial y},\quad \lambda^*=\lambda-i\lambda_0\omega,\quad \mu^*=\mu-i\mu_0\omega,$$
$$C_1=\frac{\lambda^*+2\mu^*}{\rho}=c_1^2-\frac{i(\lambda_0+2\mu_0)\omega}{\rho},\quad C_2=\frac{\mu^*}{\rho}=c_2^2-\frac{i\mu_0\omega}{\rho},$$
$$p_1^2=\frac{\omega^2}{C_1},\quad p_2^2=\frac{\omega^2}{C_2},$$

$c_1=\big(\frac{\lambda+2\mu}{\rho}\big)^{\frac12}$, $c_2=\big(\frac{\mu}{\rho}\big)^{\frac12}$ are velocity propagation of extension and distortion waves, respectively. $p_1=k_1+ik_1'$, $p_2=k_2+ik_2'$, $k_1>0$, $k_2>0$, $k_1'>0$, $k_2'>0$.

To solve the boundary value problem (2.1), we use the method of the complex Fourier integral transformation with respect to the variable x and we obtain a system of ordinary differential equations under the boundary conditions [3, 12]

$$\begin{aligned} &\frac{d^2u_0^*}{dy^2}+\frac{C_1}{C_2}(p_1^2-\alpha^2)u_0^*-i\alpha\Big(\frac{C_1}{C_2}-1\Big)\frac{dv_0^*}{dy}=0,\\ &\frac{C_1}{C_2}\frac{d^2v_0^*}{dy^2}+(p_2^2-\alpha^2)v_0^*-i\alpha\Big(\frac{C_1}{C_2}-1\Big)\frac{du_0^*}{dy}=0, \end{aligned} \tag{2.2}$$

$$\Big[-\lambda^*i\alpha u^*+(\lambda^*+2\mu^*)\frac{dv^*}{dy}\Big]_{y=0}=p^*(\alpha),\qquad \mu^*\Big[\frac{du^*}{dy}-i\alpha v^*\Big]_{y=0}=-\tau^*(\alpha), \tag{2.2$_0$}$$

where

$$u_0{}^*(\alpha,y)=\int\limits_{-\infty}^{\infty}u_0(x,y)e^{i\alpha x}dx,\qquad v_0{}^*(\alpha,y)=\int\limits_{-\infty}^{\infty}v_0(x,y)e^{i\alpha x}dx,$$

$$p^*(\alpha)=\int\limits_{-\infty}^{\infty}p(x)e^{i\alpha x}dx,\qquad \tau^*(\alpha)=\int\limits_{-\infty}^{\infty}\tau(x)e^{i\alpha x}dx,\quad \alpha=\sigma+i\tau.$$

The general solutions of the system (2.2) have the form

$$u_0^*(\alpha,y)=i\alpha Ae^{-\gamma_1 y}-\gamma_2 Be^{-\gamma_2 y}+Ce^{\gamma_1 y}+De^{\gamma_2 y},$$
$$v_0^*(\alpha,y)=\gamma_1 Ae^{-\gamma_1 y}+i\alpha Be^{-\gamma_2 y}+Ee^{\gamma_1 y}+Fe^{\gamma_2 y}.$$

The boundary values ($y=0$) of solutions of the problem (2.2)–(2.2$_0$) vanishing at infinity ($y\to\infty$) are represented as follows

$$\begin{aligned}u_0{}^*(\alpha,0)&=\frac{\gamma_2 p_2^2}{\mu^*\Delta(\alpha)}\tau^*(\alpha)+\frac{i\alpha(2\alpha^2-p_2^2-2\gamma_1\gamma_2)}{\mu^*\Delta(\alpha)}p^*(\alpha),\\ v_0{}^*(\alpha,0)&=\frac{i\alpha(2\alpha^2-p_2^2-2\gamma_1\gamma_2)}{\mu^*\Delta(\alpha)}\tau^*(\alpha)-\frac{\gamma_1 p_2^2}{\mu^*\Delta(\alpha)}p^*(\alpha),\end{aligned}\tag{2.3}$$

where $\Delta(\alpha)=4\alpha^2\gamma_1(\alpha)\gamma_2(\alpha)-(2\alpha^2-p_2^2)^2$, $\gamma_1(\alpha)=\sqrt{\alpha^2-p_1^2}$, $\gamma_2(\alpha)=\sqrt{\alpha^2-p_2^2}$.

The function $\Delta(\alpha)$ does not have roots on the real axis. According (2.3), the amplitudes $u_0(x)=u_0(x,0)$ and $v_0(x)=v_0(x,0)$ of horizontal and vertical displacements of boundary points of the viscoelastic half-plane from the amplitudes of contact stresses are given by the formulas

$$u_0(x)=\int\limits_{-1}^{1}k_1(|x-s|)\tau(s)ds+\int\limits_{-1}^{1}k_2(|x-s|)p(s)ds,$$

$$v_0(x)=\int\limits_{-1}^{1}k_2(|x-s|)\tau(s)ds+\int\limits_{-1}^{1}k_3(|x-s|)p(s)ds,$$

where

$$k_1(x)=\frac{1}{2\pi\mu^*}\int\limits_{-\infty}^{\infty}\frac{\gamma_2(\sigma)p_2^2e^{-i\sigma x}d\sigma}{\Delta(\sigma)},\quad k_2(x)=\frac{1}{2\pi\mu^*}\int\limits_{-\infty}^{\infty}\frac{\sigma(2\sigma^2-p_2^2-2\gamma_1(\sigma)\gamma_2(\sigma))e^{-i\sigma x}d\sigma}{\Delta(\sigma)},$$

$$k_3(x)=\frac{1}{2\pi\mu^*}\int\limits_{-\infty}^{\infty}\frac{\gamma_1(\sigma)p_2^2e^{-i\sigma x}d\sigma}{\Delta(\sigma)}.$$

Now let's consider the dynamical boundary value problem of an viscoelastic half-plane whose boundary on the segment $[-1,1]$ is only under the action of horizontal harmonic forces. Based on the latest formulas, we have

$$u_0(x,0)=\frac{1}{\mu^*}\int\limits_{-1}^{1}K(p_2|x-s|)\tau(s)ds,\tag{2.4}$$

where

$$K(z)=\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{\sqrt{s^2-1}e^{-izs}ds}{(2s^2-1)^2-4s^2\sqrt{(s^2-1)(s^2-q^2)}},\quad q^2=\frac{p_1^2}{p_2^2}.$$

The function $k(s)\equiv\dfrac{\sqrt{s^2-1}}{(2s^2-1)^2-4s^2\sqrt{(s^2-1)(s^2-q^2)}}$ in the neighbourhood of infinity has the following representation $k(s)=-\dfrac{1}{2(1-q^2)}\frac{1}{|s|}+O(|s|^{-1})$, $|s|\to\infty$. Therefore, its inverse Fourier transform $K(p_2x)$ is represented as a sum of principal and regular part by the formula

$$K(p_2x) = -\frac{1}{2\pi(1-q^2)} \ln\frac{1}{|p_2x|} + R(|p_2x|). \tag{2.5}$$

The equation of stationary oscillation of the elastic patch have the form

$$\Big(\frac{d^2}{dx^2} + \frac{\rho_0}{E_0}\omega^2\Big)u^{(1)}(x) = -\frac{1}{h_0E_0}\tau(x) - \frac{1}{h_0E_0}\tau_0\delta(x+1), \quad -1 < x < 1$$

where $u^{(1)}(x)$ is horizontal displacement of the patch points, E_0 and ρ_0 are the elasticity modulus and the density of the patch material, respectively, h_0 is its thickness [10, 11].

Based of the contact condition: $u^{(1)}(x) = u_0(x,0)$ and formulas (2.4), (2.5) we obtain the following integro-differential equation

$$\Big(\frac{d^2}{dx^2} + \frac{\rho_0}{E_0}\omega^2\Big)\Big[\frac{1}{2\pi\mu^*(1-q^2)}\int_{-1}^{1}\ln\frac{1}{|p_2|\,|x-s|}\tau(s)ds + \int_{-1}^{1}R(|p_2|\,|x-s|)\tau(s)ds\Big]$$
$$= -\frac{1}{h_0E_0}\tau(x) - \frac{1}{h_0E_0}\tau_0\delta(x+1), \quad -1 < x < 1 \tag{2.6}$$

with following condition

$$\int_{-1}^{1}\tau(s)ds = -\tau_0. \tag{2.7}$$

A solution of the problem (2.6), (2.7) will be sought in the form

$$\tau(x) = \frac{a_0}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}\sum_{m=1}^{\infty} a_mT_m(x), \tag{2.8}$$

where $T_m(x)$ are the first kind Chebyshev's orthogonal polynomials, $\{a_n\}_{n\geq 1}$ is unknown sequences.

By virtue of the equilibrium conditions (2.7) of the patch, we obtain $a_0 = -\frac{\tau_0}{\pi}$.

Using the Rodrigue's formula for the Jacobi's polynomials and following spectral relation [13]

$$\frac{1}{\pi}\int_{-1}^{1}\ln\frac{1}{|x-y|}\frac{T_m(y)dy}{\sqrt{1-y^2}} = \mu_mT_m(x), \qquad \mu_m = \begin{cases}\ln 2, & m = 0\\ \dfrac{1}{m}, & m \neq 0\end{cases}$$

from integro-differential equation (2.6) and formul (2.8) we have

$$\frac{\sqrt{\pi}}{4}\sum_{m=2}^{\infty} a_m\frac{\Gamma(m+2)}{\Gamma(m+0.5)}P_{m-2}^{(\frac{3}{2},\frac{3}{2})}(x) + \frac{\rho_0\omega^2}{E_0}\sum_{m=0}^{\infty}\frac{a_m}{m}T_m(x) + \sum_{m=0}^{\infty} a_m\int_{-1}^{1}K_0(|p_2|\,|x-s|)\frac{T_m(s)}{\sqrt{1-s^2}}\,ds$$
$$+\frac{r_0}{\sqrt{1-x^2}}\sum_{m=0}^{\infty} a_mT_m(x) = \tau_0g(x), \tag{2.9}$$

where

$$K_0(|p_2|\,|x-s|) = \frac{\partial^2R(|p_2|\,|x-s|)}{\partial x^2} + \frac{\rho_o\omega^2}{E_0}R(|p_2|\,|x-s|), \quad r_0 = \frac{2\pi\mu^*(1-q^2)}{h_0E_0},$$

$$g(x) = \frac{1}{\pi}\int_{-1}^{1}\frac{K_0(|p_2||x-s|)T_m(s)ds}{\sqrt{1-s^2}} + \frac{r_0}{\pi\sqrt{1-x^2}} - r_0\delta(x+1) + \frac{\rho_0\omega^2}{E_0}\Big(\frac{\ln 2}{\pi} - \ln|p_2|\Big),$$

$p_{m-2}^{(\frac{3}{2},\frac{3}{2})}$ are Jacob's orthogonal polynomials.

Multiplying both parts of the relation (2.9) by $(1-x^2)^{\frac{3}{2}}\, P_{n-2}^{(\frac{3}{2},\frac{3}{2})}(x)$ integrating in the interval $(-1,1)$ and based on orthogonality of the Jacobi's polynomials, we obtain the infinite system of linear algebraic equations

$$a_n + \sum_{m=1}^{\infty} R_{nm} a_m = \tau_0 g_n, \quad n = 2, 3, \ldots \tag{2.10}$$

where

$$R_{nm} = \frac{\rho_0 \omega^2}{E_0} R_{nm}^{(1)} + R_{nm}^{(2)} + r_0 R_{nm}^{(3)}, \qquad R_{nm}^{(1)} = \frac{1}{\gamma_n m} \int_{-1}^{1} (1-x^2)^{\frac{3}{e2}}\, P_{n-2}^{(\frac{3}{2},\frac{3}{2})}(x) T_m(x)dx,$$

$$R_{nm}^{(2)} = \frac{1}{\gamma_n} \int_{-1}^{1} (1-x^2)^{\frac{3}{2}}\, P_{n-2}^{(\frac{3}{2},\frac{3}{2})}(x) \bigg(\int_{-1}^{1} K(|p_2|\,|x-s| \frac{T_m(s)ds}{\sqrt{1-s^2}} \bigg) dx,$$

$$R_{nm}^{(3)} = \frac{1}{\gamma_n} \int_{-1}^{1} (1-x^2)\, P_{n-2}^{(\frac{3}{2},\frac{3}{2})}(x) T_m(x)dx,$$

$$g_n = \frac{1}{\gamma_n} \int_{-1}^{1} g(x)(1-x^2)^{\frac{3}{2}}\, P_{n-2}^{(\frac{3}{2},\frac{3}{2})}(x)dx, \quad \gamma_n = \frac{2\sqrt{\pi}\Gamma(n+\frac{1}{2})}{n\Gamma(n-1)}.$$

Now one investigates the regularity of the infinite system (2.10).

According to the Stirling's formula for $\Gamma(z)$ function [1] we have:

$$\gamma_n = O(n^{\frac{1}{2}}), \quad n \to \infty. \tag{2.11}$$

Using now the Rodrigue's formula and Darboux asymptotic formula for the Jacobi's polynomials [1, 13], after some calculations, we get

$$R_{nm}^{(1)} = \frac{1}{\gamma_n \sqrt{\pi(n-2)} m} \begin{cases} 0, & m \neq n, \quad m \neq n \pm 2, \\ \pi, & m = n, \\ -\dfrac{\pi}{2}, & m = n \pm 2, \end{cases} + O(n^{-\frac{3}{2}}) \frac{1}{\gamma_n m} \begin{cases} 0, & m \neq 2, 4, \\ -\dfrac{\pi}{4}, & m = 2, \\ -\dfrac{\pi}{16}, & m = 4, \end{cases} \quad n \to \infty,$$

$$R_{nm}^{(2)} = -\frac{\sqrt{\pi}\Gamma(m+1)}{8\Gamma\big(m+\dfrac{1}{2}\big)(n-2)\gamma_n m(m-1)} \int_{-1}^{1} \frac{d}{dx}(1-x^2)^{\frac{5}{2}} P_{n-3}^{(\frac{5}{2},\frac{5}{2})}(x)$$

$$\times \bigg(\int_{-1}^{1} K_0(|p_2|\,|x-s|) \frac{d^2}{ds^2}(1-s^2)^{\frac{3}{2}} P_{m-2}^{(\frac{3}{2},\frac{3}{2})}(s)ds \bigg) dx$$

$$= \frac{\sqrt{\pi}\Gamma(m+1)}{8\Gamma\big(m+\dfrac{1}{2}\big)(n-2)\gamma_n m(m-1)} \int_{-1}^{1} (1-x^2)^{\frac{5}{2}} P_{n-3}^{(\frac{5}{2},\frac{5}{2})}(x)$$

$$\times \bigg(\int_{-1}^{1} (1-s^2)^{\frac{3}{2}} P_{m-2}^{(\frac{3}{2},\frac{3}{2})}(s) \frac{\partial^3 K_0(|p_2|\,|x-s|)}{\partial x \partial s^2} ds \bigg) dx,$$

$$g_n = O\big(n^{-\frac{3}{2}}\big), \quad n \to \infty.$$

For $R_{nm}^{(3)}$ we obtain the following estimate

$$R_{nm}^{(3)} \sim \frac{2[(-1)^{n+m}+1]}{\sqrt{\pi(n-2)}\gamma_n}\left(\frac{1}{(n+m)^2-1}+\frac{1}{(n-m)^2-1}\right)$$

$$+O(n^{-\frac{3}{2}})\frac{6[(-1)^m+1]}{\gamma_n(m^2-1)(m^2-9)}, \quad n\neq m\pm 1, \quad n\neq 1;3, \quad n\to\infty, \quad m\to\infty$$

and the corresponding double series satisfies the following estimate

$$\sum_{n=3,m=1}^{\infty}\left|R_{nm}^{(3)}\right|^2 = \sum_{n=3,m=1}^{\infty}\frac{[(-1)^{n+m}+1]^2}{(n-2)\gamma_n^2}\left\{\frac{1}{(n+m)^2-1}+\frac{1}{(n-m)^2-1}\right\}^2$$

$$\leq 4\sum_{n=3,m=1}^{\infty}\frac{1}{n(n-2)}\frac{1}{[(n+m)^2-1]^2}+4\sum_{n=3,m=1}^{\infty}\frac{1}{n(n-2)}\frac{1}{[(n-m)^2-1]^2}.$$

We study the convergence of double series:

$$\sum_{n=3,m=1}^{\infty}\frac{1}{n(n-2)}\frac{1}{[(n+m)^2-1]^2} \leq \sum_{n=3,m=1}^{\infty}\frac{1}{[(n+m)^2-1]^2}.$$

To do this, let's present it in the form of a simple series, lying its terms along the diagonals, $n+m=2p$, $p=2,3,4,\dots$

$$\begin{array}{ccccccc}
0 & \frac{1}{15^2} & 0 & \frac{1}{35^2} & 0 & \frac{1}{63^2} & 0 \\
\frac{1}{15^2} & 0 & \frac{1}{35^2} & 0 & \frac{1}{63^2} & 0 & \\
0 & \frac{1}{35^2} & 0 & \frac{1}{63^2} & 0 & & \\
\frac{1}{35^2} & 0 & \frac{1}{63^2} & 0 & & & \\
0 & \frac{1}{63^2} & 0 & & & & \\
\frac{1}{63^2} & 0 & & & & & \\
0. & & & & & &
\end{array}$$

Since the term lying on the same diagonal are equal, combining them for ease of calculation, we obtain the series

$$\sum_{p=3}^{\infty}\frac{2p-2}{(4p^2-1)^2}$$

and this series is convergent.

The convergence of double series

$$\sum_{n=3,m=1}^{\infty}\frac{1}{n(n-2)}\frac{1}{[(n-m)^2-1]^2}$$

follows from following estimate

$$\sum_{n=3,m=1}^{\infty}\frac{1}{n(n-2)}\frac{1}{[(n-m)^2-1]^2} \leq \sum_{n=3,p=0}^{\infty}\frac{1}{n(n-2)}\frac{1}{(4p^2-1)^2}$$

$$=\sum_{p=0}^{\infty}\frac{1}{(4p^2-1)^2}\sum_{n=3}^{\infty}\frac{1}{n(n-2)}<\infty, \quad n-m=2p, \quad p=0,1,2,\dots.$$

Taking into account the obtained estimations including (2.11), for the system (2.10) we obtain the following conditions

$$\sum_{m=1,n=2}^{\infty} |R_{nm}|^2 < \infty, \qquad \sum_{n=2}^{\infty} |g_n|^2 < \infty.$$

These conditions prove that the infinite system is quasi-completely regular in space l_2 that is, their solutions satisfy the conditions $\sum_{n=1}^{\infty} |a_n|^2 < \infty$ [8,9].

The results of [8, p. 534] are applicable to an infinite system (2.10). On the basis of this fact, the system

$$a_n^N + \sum_{m=1}^{N} R_{nm} a_m^N = \tau_0 g_n, \quad n = 1, 2, \dots N, \tag{2.12}$$

is solvable for sufficiently large N and convergence of approximate solutions $\{a_n^N\}_{n=1,\dots,N}$ to exact solution $\{a_n\}_{n\geq 1}$ is valid in the sense of the norm of the space l_2.

The convergence rate is determined by the inequality

$$\left\| a - \varphi_0^{-1} \bar{a}^N \right\|_{l_2} \leq C_1 \left[\sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} |R_{nm}|^2 \right]^{\frac{1}{2}} + C_2 \left(\frac{\sum\limits_{n=N+1}^{\infty} g_n^2}{\sum\limits_{n=1}^{\infty} g_n^2} \right)^{\frac{1}{2}},$$

where $a = \{a_n\}_{n\geq 1} = (a_1, a_2, \dots a_n, \dots)$ is the solution of the system (2.10), $\bar{a}^N = (a_1^N, a_2^N, \dots, a_N^N)$ is the solution of the system (2.12), $\varphi_0^{-1}\ \bar{a}^N = (a_1^N, a_2^N, \dots, a_N^N, 0, 0, \dots)$.

Considering the expression for R_{nm}, we have

$$C_1 \left[\sum_{n=N+1}^{\infty} \sum_{m=1}^{\infty} |R_{nm}|^2 \right]^{\frac{1}{2}} \leq C_1^* \left[\sum_{n=1}^{\infty} \frac{1}{(n+N)^4} \right]^{\frac{1}{2}} = C_1^* \left[\zeta(4, N)\right]^{\frac{1}{2}},$$

$$C_2 \left(\frac{\sum\limits_{n=N+1}^{\infty} g_n^2}{\sum\limits_{n=1}^{\infty} g_n^2} \right)^{\frac{1}{2}} < C_2^* \left[\sum_{n=1}^{\infty} \frac{1}{(n+N)^4} \right]^{\frac{1}{2}} = C_2^* \left[\zeta(4, N)\right]^{\frac{1}{2}}$$

where $\zeta(s, N)$ is known generalized Zeta function.

Using the asymptotic of the Zeta function [4, p. 62]

$$\zeta(2m, N) \equiv \sum_{n=1}^{\infty} \frac{1}{(n+N)^{2m}} = \frac{N^{-2m+1}}{2m-1} + \frac{1}{2} N^{-2m} + \sum_{k=1}^{\infty} B_{2k} \frac{\Gamma(N+2k+1)}{(2k)! N^{2k+2m+1}} + O(N^{-2m-2N-1}),$$

we obtain

$$\left\| a - \varphi_0^{-1} \bar{a}^N \right\|_{l_2} \leq C N^{-\frac{3}{2}}.$$

Thus, the solutions of the system (2.10) can be constructed by the reduction method with any accuracy [8,9].

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