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ON THE L_∞-BIALGEBRA STRUCTURE OF THE RATIONAL HOMOTOPY GROUPS $\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}$

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Dedicated to the memory of Academician Nodar Berikashvili

Abstract. The notion of an L_∞-bialgebra structure on a vector space is introduced. It is shown that the rational homotopy groups $\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}$ admit such a structure for the loop space $\Omega\Sigma Y$ of a suspension ΣY that characterizes Y up to the rational homotopy equivalence.

1. Introduction

The homotopy groups $\pi_*(\Omega X)$ of the loops ΩX on a topological space X have no non-zero coproduct. Nevertheless, it may have non-trivial higher order cooperations that form an L_∞-coalgebra structure on $\pi_*(\Omega X)$. The Samelson product is compatible with this structure in a sense that leads to the notion of an L_∞-bialgebra. Let $H_*(X)$ denote the homology with rational coefficients $\mathbb{Q}$. It admits an A_∞-coalgebra structure, more precisely, a C_∞-coalgebra structure, dual to the A_∞-algebra and C_∞-algebra structures on the cohomology $H^*(X)$ (cf. [4]). In [7], the notion of an A_∞-bialgbera is introduced on a vector space V and it is proved that the loop homology $H_*(\Omega X)$ admits such a structure for a simply connected space X. The motivation of the paper is Theorem 12.2 in [7] asserting that the Bott–Samelson bialgebra isomorphism $T^a\tilde{H}_*(Y)\approx H_*(\Omega\Sigma Y)$ extends to an isomorphism of A_∞-bialgebras, where the A_∞-bialgebra structure on the left-hand side consists of the tensor multiplication and of the A_∞-coalgebra structural cooperations extended from $H_*(Y)$. There is the (anti)symmetrization functor from the category of A_∞-algebras to the category of L_∞-algebras (cf. [2,5,6]), and dually from the category of A_∞-coalgebras to the category of L_∞-coalgebras. Here, we have to modify the above extension rule for the A_∞-coalgebra structure of $H_*(Y)$ so that the obtained A_∞-coalgebra structural cooperations of $T^a\tilde{H}_*(Y)$ preserve the primitives $PT^a\tilde{H}_*(Y)\subset T^a\tilde{H}_*(Y)$, i.e., the rational homotopy groups $\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}$. Then the L_∞-bialgebra structure on $\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}$ is obtained by the symmetrization of the A_∞-coalgebra structure.

Furthermore, the L_∞-bialgebra structure on $\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}$ characterizes Y up to the rational homotopy equivalence.

The rational homotopy groups admit L_∞-algebra structures, but for $\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}$, these structures are degenerated and consist only of the Samelson binary product, since $\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}$ is a free Lie algebra for an arbitrary Y. In general, it may make sense to establish a compatibility relation between an L_∞-algebra and an L_∞-coalgebra structures on a vector space V by using the symmetrization of the aforementioned A_∞-bialgebra structure on V.

2. A_∞-coalgebras

A differential graded coalgebra (dgc) $(C_*, d, \Delta : C_*\to C_*\otimes C_*)$ is non-negatively graded. It is *connected* if $C_0=\mathbb{Q}$. A dgc may or may not be coassociative. The *reduced* coalgebra $\tilde{C}_*$ is defined by $\tilde{C}_*=C_{>0}$, and $PC\subset C$ denotes the vector subspace of the primitives, $PC=\{c\in C\mid \Delta(c)=1\otimes c+c\otimes 1\}$ for $1\in C_0$. We assume dgc's are connected and of finite types unless otherwise is stated explicitly. The cellular chains $(C_*(K_n), d, \Delta_K)$ of the associahedron K_n, $n\geq 2$, is a non-connected,

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non-coassociative dgc with the coproduct

$$\Delta_K : C_*(K_n) \to C_*(K_n) \otimes C_*(K_n) \tag{2.1}$$

defined as follows (cf. [8]). Recall the partial (Tamari) ordering on the vertices of K_n defined by $u \leq v$ if there exists an oriented edge-path from u to v in K_n. Denote the minimal and maximal vertices of a cell a of K_n by $\min a$ and $\max a$ respectively, and extend the partial ordering on the cells of K_n by $a \leq b$ if $\max a \leq \min b$. Then

$$\Delta_K(e) = \sum_{\substack{|a|+|b|=|e| \\ a\leq b}} \operatorname{sgn}(a,b)\, a\otimes b, \quad a\times b \subset e\times e,$$

where $e \subset K_n$ is a cell of K_n in which the top cell is denoted by e^{n-2} for $n \geq 2$.

Given a dg vector space (C,d), an A_∞-coalgebra structure

$$(C, d, \{\psi_n : C \to C^{\otimes n}\}_{n\geq 2}) \ \text{ on } \ C$$

is defined by a chain (operadic) map

$$\psi : C_*(K_n) \to \operatorname{Hom}(C, C^{\otimes n}) \ \text{ with } \ \psi(e^{n-2}) = \psi_n, \ \text{ a map of degree } \ n-2.$$

In particular, $(C, d, \Delta := \psi_2)$ is a dgc. Denoting $\psi_1 := d$, for each $n \geq 1$, the cooperations ψ_n satisfy the following quadratic relations in $\operatorname{Hom}(C, C^{\otimes n})$:

$$\sum_{\substack{0\leq k\leq n-1 \\ 0\leq i\leq n-k-1}} (-1)^{k(n+i+1)} \left(id^{\otimes i} \otimes \psi_{k+1} \otimes id^{\otimes n-k-1-i}\right) \circ \psi_{n-k} = 0. \tag{2.2}$$

A C_∞**-coalgebra** $(C, d, \{\psi_r\}_{r\geq 2})$ consists of the data similar to those of an A_∞-coalgebra, but specified by the condition that each dual operation $\psi_r^* : (C^*)^{\otimes r} \to C^*$ vanishes on the decomposables under the shuffle product on $TC^* = \bigoplus_{k\geq 1} (C^*)^{\otimes k}$.

Given two A_∞-coalgebras $(A, d, \{\psi_r^A\}_{r\geq 2})$ and $(B, d, \{\psi_r^B\}_{r\geq 2})$, the definition of their tensor product $(A\otimes B, d_\otimes, \{\Psi_r\}_{r\geq 2})$ relies on the coproduct (2.1) as follows. Let the map

$$\chi : \operatorname{Hom}(A, A^{\otimes r}) \otimes \operatorname{Hom}(B, B^{\otimes r}) \to \operatorname{Hom}(A\otimes B, (A\otimes B)^{\otimes r})$$

be defined by the composition $\chi := \sigma_{r,2}^* \circ \iota$, where

$$\iota : \operatorname{Hom}(A, A^{\otimes r}) \otimes \operatorname{Hom}(B, B^{\otimes r}) \to \operatorname{Hom}(A\otimes B, A^{\otimes r} \otimes B^{\otimes r})$$

is the standard map and

$$\sigma_{r,2}^* : \operatorname{Hom}(A\otimes B, A^{\otimes r} \otimes B^{\otimes r}) \to \operatorname{Hom}(A\otimes B, (A\otimes B)^{\otimes r})$$

is induced by the standard permutation $\sigma_{r,2} : A^{\otimes r} \otimes B^{\otimes r} \to (A\otimes B)^{\otimes r}$. For each $r \geq 2$, the tensor cooperation

$$\Psi_r : A\otimes B \to (A\otimes B)^{\otimes r}$$

satisfying (2.2) is given by

$$\Psi_r = \chi \circ (\psi^A \otimes \psi^B) \circ \Delta_K(e^{r-2}).$$

Let now C be a dg algebra (C, d, μ) and an A_∞-coalgebra $(C, d, \{\psi_r\}_{r\geq 2})$ simultaneously, and $(C\otimes C, d_\otimes, \{\Psi_r\}_{r\geq 2})$ be the tensor A_∞-coalgebra. Then $(C, d, \mu, \{\psi_r\}_{r\geq 2})$ is **an** A_∞**-bialgebra** if the following diagram

$$\begin{array}{ccc} C\otimes C & \xrightarrow{\Psi_r} & (C\otimes C)^{\otimes r} \\ \mu\downarrow & & \downarrow \mu^{\otimes r} \\ C & \xrightarrow{\psi_r} & C^{\otimes r} \end{array}$$

commutes, i.e., the following equation:

$$\psi_r \circ \mu = \mu^{\otimes r} \circ \Psi_r, \quad r \geq 2, \tag{2.3}$$

holds. In small dimensions, equality (2.3) reads as

$$\begin{aligned}
\psi_2\circ\mu &=(\mu\otimes\mu)\circ\chi\circ(\psi_2\otimes\psi_2)\\
\psi_3\circ\mu &=\mu^{\otimes 3}\circ\chi\circ((\psi_2\otimes 1)\psi_2\otimes\psi_3+\psi_3\otimes(1\otimes\psi_2)\psi_2)\\
\psi_4\circ\mu &=\mu^{\otimes 4}\circ\chi\circ\\
&\quad((\psi_2\otimes 1\otimes 1)(\psi_2\otimes 1)\psi_2\otimes\psi_4+\psi_4\otimes(1\otimes 1\otimes\psi_2)(1\otimes\psi_2)\psi_2\\
&\quad+(\psi_3\otimes 1)\psi_2\otimes((1\otimes\psi_2\otimes 1)\psi_3+(1\otimes\psi_3)\psi_2)\\
&\quad+(1\otimes\psi_2\otimes 1)\psi_3\otimes(1\otimes\psi_3)\psi_2\\
&\quad-(\psi_2\otimes 1\otimes 1)\psi_3\otimes(1\otimes 1\otimes\psi_2)\psi_3).
\end{aligned}$$

We rewrite equation (2.3) as follows. Denote $xy:=\mu(x,y)$ and

$$(x_1\otimes\cdots\otimes x_r)\cdot(y_1\otimes\cdots\otimes y_r):=x_1y_1\otimes\cdots\otimes x_ry_r,$$

and for cells a,b of K_r and for $x,y\in C$,

$$\psi_r(xy)=\sum_{\substack{|a|+|b|=r-2\\ a\le b}}\operatorname{sgn}(a,b)\,\psi(a)(x)\cdot\psi(b)(y). \tag{2.4}$$

In particular, (C,d,μ,ψ_2) is a bialgebra (Hopf algebra). Furthermore, given an A_∞-coalgebra $(C,d,\{\Delta_r:C\to C^{\otimes r}\}_{r\geq 2})$, consider the tensor algebra $(T^a(\tilde{C}),d,\mu)$. Use the freeness of $T^a(\tilde{C})$, and by induction on the tensor wordlength apply to formula (2.4) to extend each cooperation Δ_r to the cooperation $\psi_r\colon T^a(\tilde{C})\to T^a(\tilde{C})^{\otimes r}$, and, hence, to obtain the A_∞-bialgebra $(T^a(\tilde{C}),d,\mu,\{\psi_r\}_{r\geq 2})$.

Remark 2.1. The coproduct Δ_K in (2.1) is not coassociative, so we fix the left most association by iterative application of (2.4).

The homology $H_*(\Omega X)$ admits an A_∞-bialgebra structure [7]. However, for a suspension $X=\Sigma Y$, this structure is specified by the fact that the Bott–Samelson isomorphism $T^a\tilde{H}_*(Y)\approx H_*(\Omega\Sigma Y)$ induced by the inclusion $Y\hookrightarrow\Omega\Sigma Y$ extends to that of the A_∞-bialgebras. In particular, the A_∞-algebra substructure on $H_*(\Omega\Sigma Y)$ reduces to the loop (Pontryiagin) multiplication because $H_*(\Omega\Sigma Y)$ is a free algebra.

However, ψ_r given by (2.4) does not preserve the primitives $PT^a(\tilde{C})\subset T^a(\tilde{C})$, so we have to modify (2.4) as follows. Given a dg coalgebra (C,d,Δ), bearing in mind the primitive subcoalgebra (PC,d,Δ) as a degenerated A_∞-coalgebra, we consider two tensor A_∞-coalgebras:

$$\big(A\otimes B,\{{}^P\Psi_r\}_{r\geq 2}\big)=(PA,d_A,\Delta)\otimes(B,\{\psi_r^B\}_{r\geq 2})$$

and

$$\big(A\otimes B,\{\Psi_r^P\}_{r\geq 2}\big)=(A,\{\psi_r^A\}_{r\geq 2})\otimes(PB,d_B,\Delta).$$

In fact, ${}^P\Psi_r$ and Ψ_r^P, referred to as *primitive tensor cooperations*, are of the form

$${}^P\Psi_r=\Delta^{(r-1)}\otimes\psi_r\quad\text{and}\quad\Psi_r^P=\psi_r\otimes\Delta^{(r-1)},$$

respectively, where $\Delta^{(r-1)}:PC\to PC^{\otimes r}$ denotes the $(r-1)$-iteration of $\Delta=\Delta^1$ for $r\geq 2$.

Given an A_∞-coalgebra $(C,d,\{\Delta_r\}_{r\geq 2})$, for each $r\geq 2$, define the cooperation

$$\varrho_r:T^a(\tilde{C})\to T^a(\tilde{C})^{\otimes r}$$

with $\varrho_r|_C=\Delta_r$ as follows. Set $C=A=B$ above, and form the sum

$$\varrho_r|_{C^{\otimes 2}}:={}^P\Psi_r+\Psi_r^P\ \text{ on }\ C\otimes C. \tag{2.5}$$

Then set $A=C\otimes C$ with $\psi_r^A=P\Psi_r|_{C^{\otimes 2}}$ and form the sum

$$\varrho_r|_{C^{\otimes 3}}:={}^P\Psi_r+\Psi_r^P\ \text{ on }\ C^{\otimes 3},$$

and so on. Obviously, the cooperations ϱ_r are related to the product on $T^a(\tilde{C})$ by the following formula:

$$\varrho_r(xy)=\sum_{1\leq i\leq r}y_1\otimes\cdots\otimes xy_i\otimes\cdots\otimes y_r+x_1\otimes\cdots\otimes x_iy\otimes\cdots\otimes x_r, \tag{2.6}$$

where $\varrho_r(x) := x_1 \otimes \cdots \otimes x_r$ (the Sweedler type notation). The following proposition is immediate.

Proposition 2.1. *The cooperations ϱ_r given by (2.6) preserve the vector subspace $PT(\tilde{C}) \subset T^a(\tilde{C})$*

$$\varrho_r : PT(\tilde{C}) \to PT(\tilde{C})^{\otimes r}.$$

Definition 2.1. An A_∞-coalgebra $(C, d, \{\Delta_r\}_{r\geq 2})$ is **primitive** if the cooperations

$$\varrho_r : T^a(\tilde{C}) \to T^a(\tilde{C})^{\otimes r}, \quad r \geq 2,$$

satisfy (2.2) and, hence, form an A_∞-coalgebra structure on $(T^a(\tilde{C}), d)$.

Denoting $[x, y] = xy - (-1)^{|x||y|} yx$ and taking into account (2.6), we immediately obtain

Proposition 2.2. *A primitive A_∞-coalgebra $(C, d, \{\Delta_r\}_{r\geq 2})$ induces the A_∞-coalgebra structure on $(PT^a(\tilde{C}), d)$ satisfying the equality*

$$\varrho_r[x, y] = \sum_{1\leq i\leq r} y_1 \otimes \cdots \otimes [x, y_i] \otimes \cdots \otimes y_r + x_1 \otimes \cdots \otimes [x_i, y] \otimes \cdots \otimes x_r. \tag{2.7}$$

When $(C = PC, d, \Delta_2)$ is a primitive dgc, compare the two induced A_∞-coalgebra structures

$$(T^a(\tilde{C}), d, \{\psi_r\}_{r\geq 2}) \ \text{ and } \quad (T^a(\tilde{C}), d, \{\varrho_r\}_{r\geq 2}) \ \text{ on } \ T^a(\tilde{C})$$

to deduce

$$2\psi_2 = \varrho_2 \ \text{ and } \ \psi_3 = \varrho_3, \tag{2.8}$$

while on the decomposables

$$\psi_r = \varrho_r + \bar{\psi}_r \ \text{ for } \ r \geq 4,$$

where $\bar{\psi}_r$ is the non-primitive summand component of ψ_r in (2.4). In particular, (2.8) implies

Proposition 2.3. *An A_∞-coalgebra of the form $(C = PC, d, \{\Delta_2, \Delta_3, 0, \ldots\})$ is primitive.*

3. L_∞-coalgebras

The notion of an L_∞-coalgebra is dual to that of an L_∞-algebra [5,6]. Let

$$S(n) : C^{\otimes n} \to C^{\otimes n}$$

be a map defined for $a_1 \otimes \cdots \otimes a_n \in C^{\otimes n}$ by

$$S(n)(a_1 \otimes \cdots \otimes a_n) = \sum_{\sigma\in S_n} \operatorname{sgn}(\sigma)\varepsilon(\sigma)\, a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)},$$

where $\operatorname{sgn}(\sigma)$ is the standard sign of a permutation σ, and $\varepsilon(\sigma)$ is determined by the Koszul sign rule. Let $S_{i,n-i} \subset S_n$ denote the subset of $(i, n-i)$-shuffles with $S_{n,0} = 1 \in S_n$, and let $S(i, n-i) : C^{\otimes n} \to C^{\otimes n}$ be a map defined for $a_1 \otimes \cdots \otimes a_n \in C^{\otimes n}$ by

$$S(i, n-i)(a_1 \otimes \cdots \otimes a_n) = \sum_{\sigma\in S_{i,n-i}} \operatorname{sgn}(\sigma)\varepsilon(\sigma)\, a_{\sigma(1)} \otimes \cdots \otimes a_{\sigma(n)}.$$

An L_∞-coalgebra is a dg vector space (L, d) together with the linear maps

$$\{\ell^r : L \to L^{\otimes r}\}_{r\geq 1} \ \text{ of degree } \ r-2 \ \text{ with } \ \ell^1 := d$$

such that

(i) $\ell^r = S(r) \circ \ell^r$, $r \geq 1$;

(ii) $\sum_{1\leq i\leq n} (-1)^{i(n-i)} S(i, n-i) \circ (\ell^i \otimes 1^{\otimes n-i}) \circ \ell^{1+n-i} = 0.$

In particular, (L, ℓ^2) is a (graded) Lie coalgebra, when $d = 0$, or (L, d, ℓ^2) is a dg Lie coalgebra, when $\ell_3 = 0$. Denote $\ell^r(x) := x_1 \otimes \cdots \otimes x_r$ (the Sweedler type notation), and $\ell_2(x, y) := [x, y]$.

Definition 3.1. Let L be a dg Lie algebra (L, d, ℓ_2) and an L_∞-coalgebra $(L, d, \{\ell^r\}_{r\geq 2})$ simultaneously. Then $(L, d, \ell_2, \{\ell^r\}_{r\geq 2})$ is **an L_∞-bialgebra** if for each $r \geq 2$,

$$\ell^r[x, y] = \sum_{1\leq i\leq r} y_1 \otimes \cdots \otimes [x, y_i] \otimes \cdots \otimes y_r + x_1 \otimes \cdots \otimes [x_i, y] \otimes \cdots \otimes x_r. \tag{3.1}$$

In particular, (L, ℓ_2, ℓ^2) is a (graded) Lie bialgebra, when $d = 0$, or (L, d, ℓ_2, ℓ^2) is a dg Lie bialgebra, when $\ell_3 = 0$ in a sense [1]. A motivated example of an L_∞-bialgebra is given by Theorem 3.1 below.

3.1. **Symmetrization.** Given an A_∞-coalgebra $(C, d, \{\psi_r : C \to C^{\otimes r}\}_{r\geq 2})$, there is the associated L_∞-coalgebra $(L, d, \{\ell^r : L \to L^{\otimes r}\}_{r\geq 2})$, where $(L, d) = (C, d)$, and for each $r \geq 2$, the structural cooperation $\ell^r : L \to L^{\otimes r}$ is obtained by the symmetrization

$$\ell^r = \psi_r^{sym} \quad \text{for} \quad \psi_r^{sym} := S(r) \circ \psi_r.$$

Theorem 3.1. *If the structural cooperations of an A_∞-coalgebra $(C, d, \{\Delta_r\}_{r\geq 2})$ restrict to $\Delta_r : PC \to PC^{\otimes r}$ for all r and form a primitive A_∞-coalgebra structure on PC, then the free Lie algebra $(L(PC), \ell_2)$ admits a canonical L_∞-bialgebra structure $(L(PC), d, \ell_2, \{\ell^r\}_{r\geq 2})$ with $\ell^2 = 0$.*

Proof. First, recall that $PT^a(\tilde{C}) = L(PC)$ and the free Lie algebra is generated by PC. Applying to Proposition 2.2, we obtain $\ell^r : L(PC) \to L(PC)^{\otimes r}$ as $\ell^r = \varrho_r^{sym}$ on $L(PC)$. Then (2.7) implies (3.1). Since $\Delta_2 : PC \to PC \otimes PC$ is cocommutative, $\ell^2 = 0$. □

Denoting $L := L(PC)$, equality (3.1) is equivalent to the following commutative diagram:

$$\begin{array}{ccc} L \otimes L & \xrightarrow{\Phi_r^{sym}} & (L \otimes L)^{\otimes r} \\ \ell_2 \downarrow & & \downarrow \bar{\ell}_2 \\ L & \xrightarrow{\ell^r} & L^{\otimes r}, \end{array}$$

where $\Phi_r := {}^P\Psi_r + \Psi_r^P$ in which the right-hand side is defined by (2.5) for $C = L$, $\psi_r = \varrho_r$, and $\bar{\ell}_2 := \sum_{1\leq i\leq r} \mu^{\otimes i-1} \otimes \ell_2 \otimes \mu^{\otimes r-i-1}$. In fact, $\bar{\ell}_2$ consists of only one non-trivial monomial because of $[1, -] = [-\,, 1] = 0$ for $1 \in T^a(\tilde{C})$.

Let $C = H_*(Y)$. Then $H_*(Y)$ admits an A_∞-coalgebra structure, or more precisely, a C_∞-coalgebra structure [4]. Taking into account the Milnor–Moore theorem, we have

$$\pi_*(\Omega\Sigma Y) \otimes \mathbb{Q} = PH_*(\Omega\Sigma Y) = PT^a(\tilde{H}_*(Y)) = L(PH_*(Y)),$$

and then Theorem 3.1 implies

Theorem 3.2. *If the cooperations $\Delta_r : H_*(Y) \to H_*(Y)^{\otimes r}$ for $r \geq 3$ preserve the primitives $PH_*(Y)$ and form a primitive A_∞-coalgebra structure on $PH_*(Y)$, then the rational homotopy groups $(\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}, \ell_2)$, being the Lie algebra with the Samelson product ℓ_2, admit a canonical L_∞-bialgebra structure $(\pi_*(\Omega\Sigma Y) \otimes \mathbb{Q}, \ell_2, \{\ell^r\}_{r\geq 2})$ with $\ell^2 = 0$.*

Corollary 3.1. *For $PH_*(Y) = H_*(Y)$ and $(H_*(Y), \{\Delta_r\}_{r\geq 2})$ to be primitive, the rational homotopy groups $\pi_*(\Omega\Sigma Y) \otimes \mathbb{Q}$ admit a canonical L_∞-bialgebra structure.*

In general, the rational homotopy groups admit an L_∞-algebra structure with the higher order operations ℓ_r rather than ℓ_2 (cf. [2]), but in the case of $\pi_*(\Omega\Sigma Y) \otimes \mathbb{Q}$, it reduces to the Samelson bracket ℓ_2 because $(\pi_*(\Omega\Sigma Y) \otimes \mathbb{Q}, \ell_2)$ is a free Lie algebra. Furthermore, note that although the Lie coalgebra structure of $\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}$ is *abelian*, the higher order cooperations ℓ^r, $r \geq 3$, on $\pi_*(\Omega\Sigma Y)\otimes\mathbb{Q}$ may be non-trivial (cf. Example 3.1 below).

Theorem 3.3. *If there is an isomorphism $\pi_*(\Omega\Sigma Y) \otimes \mathbb{Q} \approx \pi_*(\Omega\Sigma Y') \otimes \mathbb{Q}$ of the L_∞-bialgebras, then Y and Y' are equivalent in a rational homotopy category.*

Proof. The isomorphism of the L_∞-bialgebras of the theorem implies the isomorphism

$$(H_*(Y), \{\Delta_r\}) \approx (H_*(Y'), \{\Delta'_r\})$$

of C_∞-coalgebras. On the other hand, the C_∞-coalgebra structure of $H_*(Y)$ uniquely characterizes Y in the rational homotopy category [4], so the proof of the theorem follows. □

Example 3.1. 1. Let a space $Y = S^2 \vee S^2 \vee S^2 \cup_f e^5$ be obtained from the wedge of three 2-spheres by attaching the 5-cell e^5 via a map $f : S^4 \to S^2 \vee S^2 \vee S^2$, being a representative of the element $\big[i_1, [i_2, i_3]\big] \in \pi_4(S^2 \vee S^2 \vee S^2)$, the iterated Whitehead product, where $i_j : S^2 \to S^2 \vee S^2 \vee S^2$ denotes the standard inclusion at the j^{th}-component $j = 1, 2, 3$. Then for $H := H_*(Y)$, we have $H_0 = \mathbb{Q}$, $H_2 = \mathbb{Q} \oplus \mathbb{Q} \oplus \mathbb{Q}$ and $H_5 = \mathbb{Q}$. Although $PH = H$, the C_∞-coalgebra structure on H is non-trivial: namely, there is a representative $\Delta_3 : H \to H \otimes H \otimes H$ with $\Delta_3(w_5) = x_2 \otimes y_2 \otimes z_2$ for $(x_2, y_2, z_2) \in H_2$

and $w_5 \in H_5$ (compare [3, Example 6.6]). In particular, the C_∞-coalgebra (H, Δ_3) is primitive. We have $H_*(Y) \subset \pi_*(\Omega\Sigma Y) \otimes \mathbb{Q}$ with identifications $H_2(Y) = \pi_2(\Omega\Sigma Y) \otimes \mathbb{Q}$ and $H_5(Y) = \pi_5(\Omega\Sigma Y) \otimes \mathbb{Q}$. Consequently, for $\ell^3 = \Delta_3^{sym}$ on H, the cooperation

$$\ell^3 : \pi_5(\Omega\Sigma Y) \otimes \mathbb{Q} \to (\pi_2(\Omega\Sigma Y) \otimes \mathbb{Q}) \otimes (\pi_2(\Omega\Sigma Y) \otimes \mathbb{Q}) \otimes (\pi_2(\Omega\Sigma Y) \otimes \mathbb{Q})$$

defined for $w_5 \in \pi_5(\Omega\Sigma Y) \otimes \mathbb{Q}$ by

$$\ell^3(w_5) = x_2 \otimes y_2 \otimes z_2 - y_2 \otimes x_2 \otimes z_2 + y_2 \otimes z_2 \otimes x_2 - x_2 \otimes z_2 \otimes y_2 + z_2 \otimes x_2 \otimes y_2 - z_2 \otimes y_2 \otimes x_2,$$

is non-trivial.

2. Let $Y' = S^2 \vee S^2 \vee S^2 \vee S^5$. Since Y' is a suspension, the C_∞-coalgebra structure on $H_*(Y')$ is degenerated. Hence $H_*(Y)$ and $H_*(Y')$ are isomorphic as coalgebras, but not as C_∞-coalgebras. Consequently, $\pi_*(\Omega\Sigma Y) \otimes \mathbb{Q}$ and $\pi_*(\Omega\Sigma Y') \otimes \mathbb{Q}$ are isomorphic as Lie algebras, but not as L_∞-bialgebras. In fact, there are only two rational homotopy types determined by these spaces in question.

Finally, remark that the above method applies to introduce an L_∞-bialgebra structure on the homotopy groups $\pi_*(\Omega\Sigma Y)$ whenever the Hurewitz homomorphism $\pi_*(\Omega\Sigma Y) \to H_*(\Omega\Sigma Y; \mathbb{Z})$ is an inclusion.

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