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BERIKASHVILI'S FUNCTOR D FOR HOMOTOPY G-ALGEBRAS AND DEFORMATION OF ASSOCIATIVE ALGEBRAS

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Dedicated to the memory of Nodar Berikashvili

Abstract. In [16], we described a generalization of Berikashvili's equivalence of Edgar Brown twisting cochains $da = a \smile a$ and the corresponding functor D for A_∞-algebras.

In the present work dedicated to the memory of Nodar Berikashvili, we are going to present a similar generalization, but now for a different type of algebra, namely, for hGa, the homotopy Gerstenhaber algebra. We use this version to describe the Gerstenhaber deformations of associative algebras.

1. Brown's Twisting Cochains and Berikashvili's Functor D

Let $(A^*, d : A^* \to A^{*+1}, \cdot : A^* \otimes A^* \to A^*)$ be a dg-algebra with differential d and multiplication $a \cdot b$. A *twisting element* (Ed. Brown [5]) is defined as $a \in A^1,\ da = a \cdot a$.

Later, N. Berikashvili [4] introduced the notion of *perturbation* of twisting elements for an invertible element $g \in A^0$, the combination $a' = g \cdot a \cdot g^{-1} + dg \cdot g^{-1}$ is likewise a twisting element. Actually, this is an action of the group of units $G = \{g \in A^0,\ \exists g^{-1}\}$ on the set of all twisting elements of A. So, this action $g * a = g \cdot a \cdot g^{-1} + dg \cdot g^{-1}$ induces the equivalence relation $a' \sim a$ on the set of all twisting elements $Tw(A)$. The factor set $Tw(A)/\sim$ is Berikashvili's functor $D : DG\mathrm{Alg} \to \mathrm{Sets}$.

The possibility of perturbing twisting elements has various applications in different directions. Suppose $(M, d_M : M \to M, A \otimes M \to M)$ is a differential graded A-module. Any twisting element $a \in A$ induces the perturbed differential $d_a : M \to M,\ d_a(m) = d_M(m) + a \cdot m$: Brown's condition $da = a \cdot a$ guarantees that $d_a d_a = 0$, and this *twisted* dg-module (M, d_a) is used for various calculations. If $a' \sim a$, then $f_g : (M, d_a) \to (M, d_{a'})$, given by $f_g(m) = g \cdot m$, is an isomorphism of dg-A-modules: Berikashvili's condition $a' = g \cdot a \cdot g^{-1} + dg \cdot g^{-1}$ guarantees that f_g is a chain map which is an isomorphism since $g \in A^0$ is invertible.

These notions have many applications in the homology theory of fibrations, as well as in differential geometry and physics. Let us briefly touch upon this subject-matter. A *connection* $a \in A^1$ determines the *curvature* $\Omega = da - a \cdot a$, so a twisting element is a *flat* $(\Omega = 0)$ connection. Take an invertible $g \in A^0$ and perturb the connection a as $a' = g \cdot a \cdot g^{-1} + dg \cdot g^{-1}$ (the gauge transformation). Then it is easy to see that $\Omega' = g \cdot \Omega \cdot g^{-1}$.

In the non-geometrical (d = 0) situation, a twisting element is nilpotent $a \cdot a = 0$ and the perturbation means a similarity $a' = g \cdot a \cdot g^{-1}$.

As we have mentioned above, in the previous article [16], we generalized the notions of the Brown twisting elements and Berikashvili perturbations from the case of dg-algebras to the case of Stasheff A_∞-algebras.

In this article, our aim is to modify the notions of twisting element and perturbation for the Steenrod $\smile_1$ product instead of $a \cdot b = a \smile b$. It is easy to formulate the notion of $\smile_1$-twisting element, this is, $a \in A^2,\ da = a \smile_1 a$. But since $\smile_1$ is not associative and has some more sophisticated properties than $\smile$, the concept of perturbing such twisting elements requires some additional structure, namely, the structure of a homotopy G-algebra which is, in fact, a dg-algebra with "good" $a \smile_1 b$ product and also some subsequent higher operations $a \smile_1 \{b_1, \dots, b_n\}$.

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The generalization of the notion of a twisting element to the case of $\smile_1$ product is aimed at some particular problems, namely, $\smile_1$-twisting elements control Stasheff's minimal $A(\infty)$-algebras, on the one hand (see, [15]), and Gerstenhaber's deformations of algebras, on the other hand (see, below).

2. Homotopy G-algebras

A *homotopy G-algebra* (hGa, in short) is a dg-algebra with a "good" $\smile_1$ product. The general notion was introduced by Gerstenhaber and Voronov in [20] (see also [19]).

Specific objects appeared earlier, in particular, the hGa structure appeared on the cochain complex of 1-reduced simplicial set $C^*(X)$ ([2]). This hGa structure is a consequence of the dualization of the diagonal constructed on the cobar construction $\Omega C_*(X)$ by Baues in [2]. The starting operation $E_{1,1}$ is the classical Steenrod $\smile_1$ product.

The second example is the complex of Hochschild cochains $C^*(U,U)$ of the associative algebra U. The operations $E_{1,k}$ here were defined in [12] with the purpose to describe $A(\infty)$-algebras in terms of the Hochschild cochains, although the properties of those operations which were used as defining ones for the notion of homotopy G-algebra in [20] did not appear there. These operations were defined also in [9]. Again, the starting operation $E_{1,1}$ is the classical Gerstenhaber circle product which is a sort of the $\smile_1$-product.

These examples will be presented in this section later.

Note that here and in the sequel, for the sake of simplicity, we ignore the signs (work in Z_2!).

Definition 2.1. A homotopy G-algebra (hGa, in short) is defined as a dg-algebra $(A,d,\cdot)$ with the given sequence of operations

$$E_{1,k}: A\otimes(A^{\otimes k})\to A,\quad k=0,1,2,3,\dots$$

(the value of the operation $E_{1,k}$ on $a\otimes b_1\otimes\cdots\otimes b_k\in A\otimes(A\otimes\cdots\otimes A)$ we write as $E_{1,k}(a;b_1,\dots,b_k)$) which satisfies the conditions $\deg E_{1,k}=k$, $E_{1,0}=id$ and

$$\begin{aligned}dE_{1,k}(a;b_1,\dots,b_k)+E_{1,k}(da;b_1,\dots,b_k)+\sum_i E_{1,k}(a;b_1,\dots,db_i,\dots,b_k)\\ =b_1\cdot E_{1,k-1}(a;b_2,\dots,b_k)+E_{1,k-1}(a;b_1,\dots,b_{k-1})\cdot b_k\\ +\sum_i E_{1,k-1}(a;b_1,\dots,b_i\cdot b_{i+1},\dots,b_k),\end{aligned}\tag{2.1}$$

$$\begin{aligned}E_{1,k}(a_1\cdot a_2;b_1,\dots,b_k)=a_1\cdot E_{1,k}(a_2;b_1,\dots,b_k)+E_{1,k}(a_1;b_1,\dots,b_k)\cdot a_2\\ +\sum_{p=1}^{k-1}E_{1,p}(a_1;b_1,\dots,b_p)\cdot E_{1,m-p}(a_2;b_{p+1},\dots,b_k),\end{aligned}\tag{2.2}$$

$$\begin{aligned}&E_{1,n}(E_{1,m}(a;b_1,\dots,b_m);c_1,\dots,c_n)\\ &=\sum_{0\le i_1\le j_1\le\cdots\le i_m\le j_m\le n}E_{1,n-(j_1+\cdots+j_m)+(i_1+\cdots+i_m)+m}(a;c_1,\dots,c_{i_1},E_{1,j_1-i_1}(b_1;c_{i_1+1},\dots,c_{j_1}),\\ &\qquad c_{j_1+1},\dots,c_{i_2},E_{1,j_2-i_2}(b_2;c_{i_2+1},\dots,c_{j_2}),c_{j_2+1},\dots,\\ &\qquad c_{i_m},E_{1,j_m-i_m}(b_m;c_{i_m+1},\dots,c_{j_m}),c_{j_m+1},\dots,c_n).\end{aligned}\tag{2.3}$$

The meanings of these conditions will be explained later. Here, we present these conditions in low dimensions.

Condition (2.1) for $k=1$ looks as

$$dE_{1,1}(a;b)+E_{1,1}(da;b)+E_{1,1}(a;db)=a\cdot b-b\cdot a.$$

So, the operation $E_{1,1}$ is a sort of the *Steenrod* $\smile_1$ product: it is the chain homotopy which measures the noncommutativity of A, the above condition coincides with the classical $d(a\smile_1 b)+da\smile_1 b+a\smile_1 db=a\cdot b-b\cdot a$. Below, we denote $a\smile_1 b=E_{1,1}(a;b)$. The higher operations $E_{1,k}$ in [12] are denoted as $E_{1,k}(a;b_1,\dots,b_k)=a\smile_1\{b_1,\dots,b_k\}$ and in [9] as $E_{1,k}(a;b_1,\dots,b_k)=a\{b_1,\dots,b_k\}$. Likely, that is why the hGa operations $E_{1,k}$ are sometimes called the Getzler-Kadeishvili *brace* operations.

Condition (2.2) for $k=1$ looks as

$$(a\cdot b)\smile_1 c+a\cdot(b\smile_1 c)+(a\smile_1 c)\cdot b=0, \tag{2.4}$$

which implies that the operation $E_{1,1}=\smile_1$ satisfies the *left Hirsch formula.*

Condition (2.1) for $k=2$ has the form

$$\begin{aligned} &a\smile_1(b\cdot c)+(a\smile_1 b)\cdot c+b\cdot(a\smile_1 c)\\ &=dE_{1,2}(a;b,c)+E_{1,2}(da;b,c)+E_{1,2}(a;db,c)+E_{1,2}(a;b,dc), \end{aligned} \tag{2.5}$$

implying that this $\smile_1$ satisfies the *right Hirsch formula* just up to the homotopy, and an appropriate chain homotopy is the operation $E_{1,2}$.

Condition (2.3) for $n=m=2$ is written as

$$(a\smile_1 b)\smile_1 c+a\smile_1(b\smile_1 c)=E_{1,2}(a;b,c)+E_{1,2}(a;c,b), \tag{2.6}$$

this means that the same operation $E_{1,2}$ measures also the deviation from the associativity of the operation $E_{1,1}=\smile_1$.

From this condition follows

$$(a\smile_1 b)\smile_1 c+a\smile_1(b\smile_1 c)=(a\smile_1 c)\smile_1 b+a\smile_1(c\smile_1 b), \tag{2.7}$$

which is the *pre-Lie condition* ensuring that the commutator $[a,b]=a\smile_1 b-b\smile_1 a$ satisfies the Jacobi identity, so produces a Lie algebra even if the $\smile_1$ is not strictly associative. That is why condition (2.7), introduced by Gerstenhaber, is called a pre-Lie condition.

2.1. **Three Aspects of hGa. 1. hGa Structure and Multiplication in the Bar Construction.** For a dga $(A,d,\cdot)$, its bar construction BA is a dg-coalgebra cogenerated by its desuspension $BA=T^c(A)=\sum_{i=0}^{\infty}A\otimes\cdots(i-times)\cdots\otimes A$ with grading $dim(a_1\otimes\cdots\otimes a_n)=\sum_i dim\ a_i-n$. This is a cofree object in the category of graded coalgebras, and this implies that a multiplication $\mu_E:BA\otimes BA\to BA$, which first of all must be a coalgebra map, is induced by a homomorphism $E_{*,*}$,

$$\begin{array}{ccc} A & \longleftarrow & T^c(A)\\ & \nwarrow\ E_{*,*} & \uparrow\mu_E\\ & & T^c(A)\otimes T^c(A), \end{array}$$

consisting, in fact, of the components $E_{01}=id,\ E_{10}=id$ and

$$\begin{array}{ccccccc} & & & E_{11} & & & \\ & & E_{12} & & E_{21} & & \\ & E_{13} & & E_{22} & & E_{31} & \\ 14 & & E_{23} & E_{32} & & E_{41} & \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \end{array} \tag{2.8}$$

where $E_{pq}:A^{\otimes^p}\otimes A^{\otimes^p}\to A$. In order μ_E to have unit, be associative and be a chain map, this collection must satisfy certain conditions, such objects $(A,d,\cdot,\{E_{pq}\})$ is a particular case of the so-called B_∞ algebra of Getzler [9].

But for the hGa $(A,d,\cdot,\{E_{1,k}\})$, it turns out that $E_{p,q>1}=0$, so the above triangle (2.8) is degenerated into the line of operations

$$\begin{array}{ccccccc} & & & E_{11} & & & \\ & & E_{12} & & 0 & & \\ & E_{13} & & 0 & & 0 & \\ E_{14} & & 0 & & 0 & & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots, \end{array} \tag{2.9}$$

and the defining conditions of hGa [2.1] and [2.2] guarantee that $E_{*,*}:BA\otimes BA\to A$ is a classical twisting cochain, which in turn guarantees that $\mu_{E_{*,*}}:BA\otimes BA\to BA$ is a chain map, and condition [2.3] induces that $\mu_{E_{*,*}}$ is associative.

Thus the hGa structure $\{E_{1,k}\}$ defines on the bar a construction BA of a dga $(A, d, \cdot)$ is a multiplication $\mu_{E_{*,*}}$ turning the bar construction BA into a dg-bialgebra $(BA, d_{BA}, \Delta_{BA}, \mu_{E_{*,*}})$. In fact, this means that the hGa is a B_∞-algebra in the sense of Getzler [9].

Note that if $(A, d, \cdot)$ is a commutative dga, then its bar construction BA is a dg-bialgebra with respect to the shuffle product. This situation corresponds to the hGa structure $(A, d, \cdot, \{E_{1,k}\})$ with all $E_{1,k} = 0$ for all $k \geq 1$.

2. Gerstenhaber Algebra Structure in Homology of the hGa.

A Gerstenhaber algebra is defined as a graded commutative algebra $(H, \cdot)$ equipped additionally with a Lie bracket of degree -1 $[\ ,\ ] : H^p \otimes H^q \to H^{p+q-1}$ satisfying the Leibniz rule $[a, b \cdot c] = b \cdot [a, c] + [a, b] \cdot c$.

For the hGa $(A, d, \cdot, \{E_{1,k}\})$, the operation $E_{1,1} = \smile_1$ is not associative, but as is mentioned above, from condition (2.6) follows the so-called "pre-Lie condition" (2.7) which guarantees that the commutator $[a, b] = a \smile_1 b - b \smile_1 a$ satisfies the Jacobi identity. Thus on the desuspension $s^{-1}A$, it forms a structure of the dg-Lie algebra. So, it induces the Lie bracket on the homology $H(A)$. Besides, conditions (2.4) and (2.7) guarantee that the Lie bracket induced on $H(A)$ satisfies the Leibniz rule. Thus $H(A)$ is a Gerstenhaber algebra.

3. hGa as a "Strong Homotopy Commutative" dga. There is the third aspect of the hGa [13]: it measures the noncommutativity of A. The Steenrod $\smile_1$ product is the classical tool which measures the noncommutativity of the dg-algebra: $d(a \smile_1 b) - da \smile_1 b - a \smile_1 db = a \cdot b - b \cdot a$. The existence of $\smile_1$ in a dga $(A, d, \cdot)$ guarantees the commutativity of $H(A)$, but a $\smile_1$ product satisfying just this condition is too poor for the most of applications, whereas a $\smile_1$ product, which is a starting operation of some hGa structure, is much more powerful: it satisfies the left Hirsch formula (2.4) up to the homotopy right Hirsch formula (2.5), pre-Lie condition (2.7), etc.

2.2. **Three Examples of hGa.** Here, we present three main examples of homotopy G-algebras.

1. Cochain complex of 1-reduced simplicial set $C^*(X)$. In [2] (1981), Baues constructed a diagonal on the cobar construction $\Omega C_*(X)$. Dualization of this structure gives multiplication μ_E on the bar construction $BC^*(X)$ and this multiplication is induced not by triangle (2.8), but by a line (2.9) of operations, that is, by a certain hGa structure $(C^*(X), \delta, \smile, \{E_{1,k}\})$, so there appears the corresponding dg-bialgebra $(BC^*(X), d_B, \Delta_B, \mu_E)$. The starting operation $E_{1,1}$ is the classical Steenrod $\smile_1$ product.

This multiplication has the following application: Homologies of just bar construction $BC^*(X)$ give cohomology *modules* of the loop space $H^*(\Omega X)$, but this additional hGa structure $\{E_{1,k}\}$ produces the multiplication μ_E that describes the multiplicative structure on the cohomology of loop space $H^*(\Omega X) = H((BC^*(X), d_B, \Delta_B, \mu_E))$. So, it allows to produce the second bar construction $BBC^*(X)$ which determines cohomology modules of the second loop space $H^*(\Omega^2 X)$.

2. Hochschild Cochain Complex as the hGa. Let A be an algebra and M be a two-sided module on A. The Hochschild cochain complex $C^*(A; M)$ is defined as $C^n(A; M) = \mathrm{Hom}(A^{\otimes^n}, M)$ with differential $\delta : C^{n-1}(A; M) \to C^n(A; M)$ given by

$$\begin{aligned}\delta f(a_1 \otimes \cdots \otimes a_n) =& a_1 \cdot f(a_2 \otimes \cdots \otimes a_n) + f(a_1 \otimes \cdots \otimes a_{n-1}) \cdot a_n \\ &+ \sum_{k=1}^{n-1} f(a_1 \otimes \cdots \otimes a_{k-1} \otimes a_k \cdot a_{k+1} \otimes \cdots \otimes a_n).\end{aligned}$$

We focus on the case $M = A$. The Hochschild complex $C^*(A; A)$ becomes a dg-algebra with respect to the $\smile$ product defined in [7] by

$$f \smile g(a_1 \otimes \cdots \otimes a_{n+m}) = f(a_1 \otimes \cdots \otimes a_n) \cdot g(a_{n+1} \otimes \cdots \otimes a_{n+m}).$$

In [12] (see also [9, 20]), the operations

$$E_{1,i} : C^n(A; A) \otimes C^{n_1}(A; A) \otimes \cdots \otimes C^{n_i}(A; A) \to C^{n+n_1+\cdots+n_i-i}(A; A)$$

given by

$$E_{1,i}(f;g_1,\dots,g_i)(a_1\otimes\cdots\otimes a_{n+n_1+\cdots+n_i-i})$$
$$=\sum_{k_1,\dots,k_i} f(a_1\otimes\cdots\otimes a_{k_1}\otimes g_1(a_{k_1+1}\otimes\cdots\otimes a_{k_1+n_1})\otimes a_{k_1+n_1+1}$$
$$\otimes\cdots\otimes a_{k_2}\otimes g_2(a_{k_2+1}\otimes\cdots\otimes a_{k_2+n_2})\otimes a_{k_2+n_2+1}$$
$$\otimes\cdots\otimes a_{k_i}\otimes g_i(a_{k_i+1}\otimes\cdots\otimes a_{k_i+n_i})\otimes a_{k_i+n_i+1}\otimes\cdots\otimes a_{n+n_1+\cdots+n_i-i})$$

are defined. The straightforward verification shows that the collection $\{E_{1,k}\}$ satisfies conditions (2.1), (2.2) and (2.3) (in fact, it seems that the properties of these operations served as a source of these conditions), thus it forms on the Hochschild complex $C^*(A;A)$ a structure of homotopy G-algebra.

We remark here that for any $f\in C^n(A,A)$, we have

$$E_{1,k}(f;g_1,\dots,g_k)=0 \quad\text{for}\quad k>n,$$

since there will be no space to substitute in f more than n g-s.

We note that the operation $E_{1,1}$ coincides with the circle product defined by Gerstenhaber in [7], and the the operation $E_{1,2}$ satisfying (2.5) and (2.6) is also defined therein.

One of the interesting applications of this hGa structure on the Hochschild complex is *Deligne's Conjecture* which states that the little square operad acts on the Hochschild cochain complex $C^*(A,A)$ of an associative algebra.

This hGa structure on $C^*(A,A)$ constructed in [12] and [9] was used for proving Deligne's conjecture in [17]. The elements of surjection operad $E_{1,k}=(1,2,1,\dots,1,k+1,1)$ together with the element $(1,2)$ generate the suboperad $F_2\chi$ which is equivalent to the little square operad [3,17]. Some authors call these hGa operations on the Hochschild complex as the Getzler-Kadeishvili's braces.

3. The cobar construction of adg *bialgebra* ([14]). By definition, the cobar construction ΩC of a dg coalgebra $(C,d:C\to C,\Delta:C\to C\otimes C)$ is a dga. If C is additionally equipped with a multiplication $\mu:C\otimes C\to C$ turning it into a dg-bialgebra, there arises the question: how this structure reflects on the cobar construction ΩC? In [14], it is shown that μ gives rise to the hGa structure on ΩC. And again, the starting operation $E_{1,1}$ is classical: this is Adams's $\smile_1$-product defined for ΩC in [1] by using the *multiplication* of C.

3. Berikashvili's Functor D for hGa

In this section, we present an analogue of the notion of Brown's twisting element (see Section 1) in a homotopy G-algebra replacing in the defining equation $da=a\cdot a$ the dot product by the $\smile_1$ product, i.e., $da=a\smile_1 a$. An appropriate notion of the equivalence and the corresponding functor D will be introduced, as well.

We need a version of the *bigraded* homotopy G-algebra $(C^{*,*},d,\cdot,\{E_{1,k}\})$, that is, a bigraded algebra $(C^{*,*},\cdot)$, $C^{m,n}\cdot C^{p,q}\subset C^{m+p,n+q}$ with the differential (derivation) $d(C^{m,n})\subset C^{m+1,n}$ equipped with a sequence of operations

$$E_{1,k}:C^{m,n}\otimes C^{p_1,q_1}\otimes\cdots\otimes C^{p_k,q_k}\to C^{m+p_1+\cdots+p_k-k,n+q_1+\cdots+q_k},$$

so, the *total complex* (the total degree of $C^{p,q}$ is $p+q$) is the hGa.

Bellow, we introduce two versions of the notion of *twisting element* in a bigraded homotopy G-algebra. Although it is possible to reduce them to each other by changing gradings, we prefer to consider them separately in order to emphasize different areas of their applications. The first one controls Stasheff's A_∞-deformation of graded algebras and the second controls Gerstenhaber's deformation of associative algebras (see the next sections).

3.1. Twisting Elements and Functor D in a Bigraded Homotopy G-algebra (Version 1).

This version was used for classification of minimal A_∞ algebras (see [12]).

A *twisting element* in $C^{*,*}$ we define as

$$m=m_3+m_4+\cdots+m_p+\cdots\ ,\ m_p\in C^{p,2-p}$$

satisfying the condition $dm = E_{1,1}(m;m)$, or changing the notation, $dm = m \smile_1 m$. This condition can be rewritten in terms of the components components as

$$dm_p = \sum_{i=3}^{p-1} m_i \smile_1 m_{p-i+2}. \tag{3.1}$$

Particularly,

$$dm_3 = 0,\ dm_4 = m_3 \smile_1 m_3,\ dm_5 = m_3 \smile_1 m_4 + m_4 \smile_1 m_3, \dots\ .$$

The set of all twisting elements we denote as $Tw(C^{*,*})$.

Consider the set $G = \{g = g_2 + g_3 + \cdots + g_p + \cdots\ ;\ g_p \in C^{p,1-p}\}$ and introduce on G the following operation:

$$\overline{g} * g = \overline{g} + g + \sum_{k=1}^{\infty} E_{1,k}(\overline{g}; g, \dots, g),$$

particularly,

$$\begin{aligned}(\overline{g} * g)_2 =& \overline{g}_2 + g_2;\\ (\overline{g} * g)_3 =& \overline{g}_3 + g_3 + \overline{g}_2 \smile_1 g_2;\\ (\overline{g} * g)_4 =& \overline{g}_4 + g_4 + \overline{g}_2 \smile_1 g_3 + \overline{g}_3 \smile_1 g_2 + E_{1,2}(\overline{g}^2; g_2, g_2).\end{aligned}$$

Using the defining conditions of the hGa (2.1), (2.2), (2.3), it is possible to check that the above operation is associative, has the unit $e = 0 + 0 + \cdots$ and the opposite g^{-1} can be solved inductively from the equation $g * g^{-1} = e$. Thus G is a group.

This G acts on the set $Tw(C^{*,*})$ by the rule $g * m = \overline{m}$, where

$$\overline{m} = m + dg + g \cdot g + E_{1,1}(g; m) + \sum_{k=1}^{\infty} E_{1,k}(\overline{m}; g, \dots, g), \tag{3.2}$$

particularly,

$$\begin{aligned}\overline{m}_3 =& m_3 + dg_2;\\ \overline{m}_4 =& m_4 + dg_3 + g_2 \cdot g_2 + g_2 \smile_1 m_3 + \overline{m_3} \smile_1 g_2;\\ \overline{m}_5 =& m_5 + dg_4 + g_2 \cdot g_3 + g_3 \cdot g_2 + g_2 \smile_1 m_4 + g_3 \smile_1 m_3 +\\ & \overline{m}_3 \smile_1 g_3 + \overline{m}_4 \smile_1 g_2 + E_{1,2}(\overline{m}_3; g_2, g_2).\end{aligned}$$

Despite the fact that in the right-hand side of formula (3.2) participates $\overline{m}$, but of less dimension than in the left-hand side $\overline{m}$, this action is well defined: the components of $\overline{m}$ can be solved from this equation inductively. It is possible to check that the resulting $\overline{m}$ is a twisting element. By $D(C^{*,*})$ we denote the set of orbits $Tw(C^{*,*})/G$.

Perturbation. This action allows us to *perturb* twisting elements in the following sense. Let $g_n \in C^{n,1-n}$ be an arbitrary element, then for $g = 0 + \cdots + 0 + g_n + 0 + \cdots$ the twisting element $\overline{m} = g * m$ has the form

$$\overline{m} = m_3 + \cdots + m_n + (m_{n+1} + dg_n) + \overline{m}_{n+2} + \overline{m}_{n+3} + \cdots\ ,$$

so the components $m_3, \dots, m_n$ remain unchanged and $\overline{m}_{n+1} = m_{n+1} + dg_n$.

The perturbations allow us to consider the following two problems: integrability and rigidity (Gerstenhaber's terminology and the following two theorems are analogous to his results from the theory of deformations (see the next section)).

Integrability. Let us first mention that for a twisting element $m = \sum_{k=3}^{\infty} m_k$, the first component $m_3 \in C^{3,-1}$ is a cycle and any perturbation does not change its homology class $[m_3] \in H^{3,-1}(C^{*,*})$. Thus we have the correct map $\phi : D(C^{*,*}) \to H^{3,-1}(C^{*,*})$.

An *integation* of a homology class $\alpha \in H^{3,-1}(C^{*,*})$ is defined as a twisting element $m = m_3 + m_4 + \cdots$ such that $[m_3] = \alpha$. Thus α is integrable if it belongs to the image of ϕ. The obstructions for integrability lay in the homologies $H^{k,3-k}(C^{*,*}),\ k \geq 5$.

Theorem 3.1. *If $H^{k,3-k}(C^{*,*}) = 0$ for $k \geq 5$, then each $\alpha \in H^{3,-1}(C^{*,*})$ is integrable.*

Proof. Let $m_3 \in C^{3,-1}$ be a cycle from α. Using the defining conditions of hGa, it is easy to see that $m_3 \smile_1 m_3$ is a cycle in $C^{5,-2}$ and, since $H^{5,-2}(C^{*,*}) = 0$, there exists m_4 such that $dm_4 = m_3 \smile_1 m_3$. Again, using the property (2.6), it is possible to see that $m_3 \smile_1 m_4 + m_4 \smile_1 m_3$ is a cycle in $C^{6,-1}$ and, since $H^{6,-1}(C^{*,*}) = 0$, there exists m_5 such that $dm_5 = m_3 \smile_1 m_4 + m_4 \smile_1 m_3$. Continuation of this inductive process completes the proof. □

Rigidity. A twisting element $m = m_3 + m_4 + \cdots + m_p + \cdots$ is called *trivial* if it is equivalent to 0. A bigraded hGa $C^{*,*}$ is *rigid* if each twisting element is trivial, i.e., if $D(C^{*,*}) = \{0\}$. The obstructions to triviality of a twisting element lay in the homologies $H^{k,2-k}(C^{*,*}),\ k \geq 3$.

Theorem 3.2. *If $H^{k,2-k}(C^{*,*}) = 0$ for $k \geq 3$, then each $m = m_3 + m_4 + \cdots$ is trivial, thus the hGa $C^{*,*}$ is rigid.*

Proof. For a twisting element $m = m_3 + m_4 + \cdots + m_p + \cdots$, the first component $m_3 \in C^{3,-1}$ is a cycle and since $H^{3,-1}(C^{*,*}) = 0$, we can choose $g_2 \in C^{2,-1}$ such that $dg_2 = m_3$. Perturbing m by $g = g_2 + 0 + 0 + \cdots$, we kill the first component m_3, i.e., we get the twisting element $\overline{m} \sim m$, which looks as $\overline{m} = 0 + \overline{m}_4 + \overline{m}_5 + \cdots$. Now, owing to (3.1), the component $\overline{m}_4$ becomes a cycle, and its homology class is the second obstruction, which is also zero as $H^{4,-2}(C^{*,*}) = 0$, we can kill $\overline{m}_4$ as well, and so, we obtain $\overline{\overline{m}} = 0 + 0 + \overline{\overline{m}}_5 + \cdots$. Now, $\overline{\overline{m}}_5$ becomes a cycle, this completes the proof. □

Application: Classification of minimal A_∞ algebras. This application is described in [15]. For a graded algebra (H, μ), let us take the hGa from subsection (2.2), the Hochschild cochain complex of (H, μ) which in this case is a *bigraded* hGa. For each $\smile_1$-twisting element, $m = m_3 + m_4 +, \cdots, m_2 = \mu, dm = m \smile_1 m$, the object $(H, \{m_1 = 0, m_2 = \mu, m_3, m_4, \cdots\})$ is a minimal A_∞ algebra extending (A_∞ *deformation of*) (H, μ). Besides, for an equivalent twisting element $m' = g \star m$, the corresponding A_∞ algebras are isomorphic.

So, we obtain the following

Theorem 3.3. *The set of isomorphism classes of all $A(\infty)$ deformations of a graded algebra (H, μ) is bijective to the set of equivalence classes of twisting elements $D(C^{*,*}(H, \mu))$.*

Moreover, from the rigidity Theorem 3.2, we get the following

Theorem 3.4. *If for a graded algebra (H, μ) its Hochschild cohomology modules $HH^{n,2-n}(H, H)$ are trivial for $n \geq 3$, then (H, μ) is intrinsically formal.*

3.2. **Twisting Elements in a Bigraded Homotopy G-algebra (Version 2).** This version will be used for the Gerstenhaber deformations of associative algebras (see bellow). It can be obtained from the previous Version 1 by changing grading: take new bigraded module $\overline{C}^{p,q} = C^{p+q-2,2-q}$. The same operations turn $\overline{C}^{*,*}$ into the bigraded hGa.

A twisting element from version 1, $m = m_3 + m_4 + \cdots$, $m_n \in C^{n,2-n}$, in this case looks as $b = b_1 + b_2 + \cdots + b_n + \cdots \ b_n \in \overline{C}^{2,n}$, where b_k is the analogue of m_{k+2} and satisfies the condition $db = b \smile_1 b$, or equivalently, $db_n = \sum_{i=2}^{n-1} b_i \smile_1 b_{n-i}$.

Particularly, $db_1 = 0$, $db_2 = b_1 \smile_1 b_1$, $db_3 = b_1 \smile_1 b_2 + b_2 \smile_1 b_1, \ldots$. The set of all twisting elements we denote by $Tw(\overline{C}^{*,*})$.

Here, we have the group $G' = \{g = g_1 + g_2 + \cdots + g_p + \cdots\ ;\ g_p \in \overline{C}^{1,p}\}$ with the operation $g' * g = g' + g + \sum_{k=1}^{\infty} E_{1,k}(g'; g, \ldots, g)$.

This group acts on the set $Tw(\overline{C}^{*,*})$ by the rule $g * b = b'$, where

$$b' = b + \delta g + g \smile g + g \smile_1 b + E_{1,1}(b'; g) + \sum_{k=1}^{\infty} E_{1,k}(b'; g, \ldots, g). \tag{3.3}$$

Here, we are going to work with the hGa structure of Hochschil cochain complex of an associative algebra $(A, \cdot)$ described above in Subsection 2.2:

$$(C^*(A, A), \delta, \smile, \{E_{1,k}\}),\ C^n(A, A) = \mathrm{Hom}(A^{\otimes^n}, A).$$

We remark here that for any $f \in C^n(A,A)$, we have

$$E_{1,k}(f; g_1, \dots, g_k) = 0 \quad \text{for} \quad k > n, \tag{3.4}$$

since there will be no space to substitute in f more than n g'-s.

For one more step to work with our Version 2 we need a *bigraded* gHa. For our $(A, \cdot)$ algebra over a field k, let $k[[t]]$ be the algebra of formal power series in variable t and $A[[t]] = A \otimes k[[t]]$ be the algebra of formal power series with coefficients from A.

As the Hochschild complex $C^*(A,A)$ is the hGa, then the tensor product $\overline{C}^{*,*} = C^*(A,A) \otimes k[[t]]$ is a *bigraded* hGa with the following structure:

$$\overline{C}^{p,q} = C^p(A,A) \cdot t^q, \; \delta(f \cdot t^q) = \delta f \cdot t^q,$$
$$f \cdot t^p \smile g \cdot t^q = (f \smile g) \cdot t^{p+q},$$
$$E_{1,k}(f \cdot t^p; g_1 \cdot t^{q_1}, \dots, g_k \cdot t^{q_k}) = E_{1,k}(f; g_1, \dots, g_k) \cdot t^{p+q_1+\cdots+q_k},$$

here we use the notation $f \otimes t^p = f \cdot t^p$.

Now, since $g \in \overline{C}^{1,*}(A,A)$ and $b' \in \overline{C}^{2,*}(A,A))$, we have $E_{1,k}(b'; g, \dots, g) = 0$ for $k \geq 3$ (see 3.4). So, the defining equation of action (3.3) looks as

$$b' = b + \delta g + g \smile g + g \smile_1 b + E_{1,1}(b'; g) + E_{1,2}(b'; g, g). \tag{3.5}$$

Particularly,

$$\begin{cases} b'_1 = & b_1 + dg_1; \\ b'_2 = & b_2 + dg_2 + g_1 \cdot g_1 + g_1 \smile_1 b_1 + b'_1 \smile_1 g_1; \\ b'_3 = & b_3 + dg_3 + g_1 \cdot g_2 + g_2 \cdot g_1 + g_1 \smile_1 b_2 + g_2 \smile_1 b_1 + \\ & b'_1 \smile_1 g_3 + b'_2 \smile_1 g_2 + E_{1,2}(b'_1; g_1, g_1). \end{cases}$$

Note that as in Version 1, the components of b' can be solved from this equation inductively. It is possible to check that the resulting b' is a twisting element. By $D(\overline{C}^{*,*})$ we denote the set of orbits $Tw(\overline{C}^{*,*})/G$.

This group action allows us to *perturb* twisting elements in the following sense. Let $g_n \in \overline{C}^{1,n}$ be an arbitrary element, then for $g = 0 + \cdots + 0 + g_n + 0 + \cdots$, the twisting element $b' = g * b$ looks as

$$b' = b_1 + \cdots + b_n + (b_{n+1} + dg_n) + b'_{n+2} + b'_{n+3} + \cdots, \tag{3.6}$$

so, the components $b_1, \dots, b_n$ remain unchanged and $b'_{n+1} = b_{n+1} + dg_n$.

As in Version 1, the perturbations allow us to consider the following two problems, integrability and rigidity (Gerstenhaber's terminology from deformations of algebras) in this Vertion 2, as well.

Integrability. The first component $b_1 \in \overline{C}^{2,1}$ of a twisting element $b = \sum b_i$ is a cycle and any perturbation does not change its homology class $\alpha = [b_1] \in H^{2,1}(\overline{C}^{*,*})$. Thus we have a correct map $\psi : D'(\overline{C}^{*,*}) \to H^{2,1}(\overline{C}^{*,*})$.

An *integration* of a homology class $\alpha \in H^{2,1}(\overline{C}^{*,*})$ is defined as a twisting element $b = b_1 + b_2 + \cdots$ such that $[b_1] = \alpha$. Thus α is integrable if $\alpha \in Im\ \psi$.

The argument similar to above from Version 1 shows that the obstructions to the integrability lay in homologies $H^{2,n}(\overline{C}^{*,*})$, $n \geq 2$, and we have the following analogue of Theorem 3.1:

Theorem 3.5. *If $H^{3,n}(\overline{C}^{*,*}) = 0$ for $n \geq 2$, then each $\alpha \in H^{2,1}(\overline{C}^{*,*})$ is integrable.*

Proof. Let $b_1 \in \overline{C}^{2,1}$ be a cycle from α. Using the defining conditions of hGa, it is easy to see that $b_1 \smile_1 b_1$ is a cycle in $\overline{C}^{3,2}$, and since $H^{3,2}(\overline{C}^{*,*}) = 0$, there exists b_2 such that $db_2 = b_1 \smile_1 b_1$. Again, using the property (2.6), it is possible to see that $b_1 \smile_1 b_2 + b_2 \smile_1 b_1$ is a cycle in $\overline{C}^{3,3}$, and since $H^{3,3}(\overline{C}^{*,*}) = 0$, there exists b_3 such that $db_3 = b_1 \smile_1 b_2 + b_2 \smile_1 b_1$. Continuation of this inductive process completes the proof. □

Rigidity. A twisting element $b = b_1 + b_2 + \cdots$ is called *trivial* if it is equivalent to 0. A bigraded hGa $\overline{C}^{*,*}$ is *rigid* if each twisting element is trivial, i.e., if $D'(\overline{C}^{*,*}) = \{0\}$. The obstructions to the triviality of a twisting element lay in the homologies $H^{2,n}(\overline{C}^{*,*})$, $n \geq 1$. So, we have the following analogue of Theorem 3.2 also in Version 2:

Theorem 3.6. *If for a bigraded hGa $\overline{C}^{*,*}$ we have $H^{2,n}(\overline{C}^{*,*}) = 0$, $n \geq 1$, then $D'(\overline{C}^{*,*}) = 0$, thus $\overline{C}^{*,*}$ is rigid.*

Proof. For a twisting element $b = b_1 + b_2 + \cdots + b_p + \cdots$, the first component $b_1 \in \overline{C}^{2,1}$ is a cycle and since $H^{2,1}(\overline{C}^{*,*}) = 0$, we can choose $g_1 \in \overline{C}^{1,1}$ such that $dg_1 = b_1$. Perturbing b in the sense of (3.6) by $g = g_1 + 0 + 0 + \cdots$, we kill the first component b_1, i.e., we get the twisting element $\overline{b} \sim b$, which looks as $\overline{b} = 0 + \overline{b}_2 + \overline{b}_3 + \cdots$. Now, owing to (3.1), the component $\overline{b}_2 \in \overline{C}^{2,2}$ becomes a cycle and its homology class is the second obstruction, which also is zero since $H^{2,2}(\overline{C}^{*,*}) = 0$, so killing likewise $\overline{b}_2$, we obtain $\overline{\overline{b}} = 0 + 0 + \overline{\overline{b}}_3 + \overline{\overline{b}}_4 + \cdots$. Now, $\overline{\overline{b}}_3$ becomes a cycle, etc. This completes the proof. □

4. Interpretation of Deformations in Terms of Twisting Elements in hGa

Here, we present the interpretation of Gerstenhaber's deformations of associative algebras in terms of twisting elements of Version 2 type in the hGa of Hochschild cochains.

Deformation of Algebras. Let $(A, \cdot)$ be an algebra over a field k, $k[[t]]$ be the algebra of formal power series in variable t and $A[[t]] = A \otimes k[[t]]$ be the algebra of formal power series with coefficients from A.

Gerstenhaber's deformation of an algebra $(A, \cdot)$ is defined as a sequence of homomorphisms

$$B_i : A \otimes A \to A, \quad i = 0, 1, 2, \dots ; \quad B_0(a \otimes b) = a \cdot b$$

satisfying the *associativity condition*: for an arbitrary $n \geq 0$,

$$\sum_{i=0}^{n} B_i(a \otimes B_{n-i}(b \otimes c)) = \sum_{i=0}^{n} B_i(B_{n-i}(a \otimes b) \otimes c). \tag{4.1}$$

Let us denote $B = \sum_{i \geq 0} B_i$.

Such a sequence determines the *star product*

$$a \star_B b = a \cdot b + B_1(a \otimes b)t + B_2(a \otimes b)t^2 + B_3(a \otimes b)t^3 + \cdots \in A[[t]],$$

which can be naturally extended to a $k[[t]]-$bilinear product

$$\star_B : A[[t]] \otimes A[[t]] \to A[[t]]$$

and condition (4.1) guarantees the associativity of this algebra $(A[[t]], \star_B)$.

Equivalence of deformations. Two deformations $\{B_i\}$ and $\{B'_i\}$ are called *equivalent* if there exists a sequence of homomorphisms $\{G_i : A \to A;\ i = 0, 1, 2, \dots;\ G_0 = id\}$ such that

$$\sum_{r+s=n} G_r(B_s(a \otimes b)) = \sum_{i+j+k=n} B'_i(G_j(a) \otimes G_k(b)). \tag{4.2}$$

The sequence $\{G_i\}$ determines the homomorphism $G = \sum G_i t^i : A \to A[[t]]$. In its turn, this G extends naturally to a $k[[t]]-$linear bijection

$$G : (A[[t]], \star_B) \to (A[[t]], \star_{B'})$$

and condition (4.2) guarantees that this extension is multiplicative, i.e., is an isomorphism of the deformed algebras.

Trivial Deformation. A deformation $\{B_i\}$ is called *trivial* if $\{B_i\}$ is equivalent to $\{B_0, 0, 0, \dots\}$. The algebra A is called *rigid* if each its deformation is trivial.

Interpretation of deformations as twisting elements.

Now, we present the interpretation of deformations and their equivalence in terms of twisting elements of Version 2 type and their equivalence in the hGa of Hochschild cochains.

Each deformation $\{B_i : A^{\otimes^2} \to A,\ i = 0, 1, 2, \dots\}$ can be interpreted as a version 2 type twisting element $b = b_1 + b_2 + \cdots + b_k + \cdots,\ b_k = B_k \cdot t^k \in \overline{C}^{2,k}$: the associativity condition (4.2) can be rewritten for each $n \geq 1$ as

$$\delta B_n \cdot t^n = \sum_{i=1}^{n} B_i \cdot t^i \smile_1 B_{n-i} \cdot t^{n-i},$$

that is,

$$\delta b_n = \sum_{i+j=n} b_i \smile_1 b_j.$$

Suppose now that the two deformations $\{B_i\}$ and $\{B'_i\}$ are equivalent, i.e., there exists $\{G_i\}$ such that condition (4.2) is satisfied. In terms of the Hochschild cochsins, this condition looks like (3.5),

$$b' = b + \delta g + g \smile g + g \smile_1 b + E_{1,1}(b'; g) + E_{1,2}(b'; g, g),$$

where $g = g_1 + \cdots + g_k + \cdots$, $g_k = G_k \cdot t^k \in C^{1,k}$.

So, we find that deformations are equivalent if and only if the corresponding twisting elements b and b' are equivalent. Consequently, the set of equivalence classes of deformations is bijective to $D'(C^{*,*})$.

As we are working on the bigraded variant of hGa generated by the Hochschild cochain complex $\overline{C}^{*,*} = C^*(A, A) \otimes k[[t]]$, it is clear that

$$H^{p,q}(C^{*,*}) = HH^p(A, A) \cdot t^q,$$

where $HH^p(A, A)$ is the Hohchschild cohomology of A. Then from Section 3.2 follow two classical results of Gerstenhaber from [8].

By Theorem 3.5, if $H^{3,n}(\overline{C}^{*,*}) = 0$ for $n \geq 2$, then each $b_1 : A \otimes A \to A$ is integrable, but since $H^{3,n}(\overline{C}^{*,*}) = HH^3(A, A) \cdot t^n$, we get a classical result of Gerstenhaber: the triviality of the Hochschild cohomology $HH^3(A, A)$ guarantees the integrability of each b_1.

By Theorem 3.6, if $H^{2,n}(\overline{C}^{*,*}) = 0$ for $n \geq 1$, then $\overline{C}^{*,*}$ is rigid, i.e., each twisting element b is triwial. And again, since $H^{2,n}(\overline{C}^{*,*}) = HH^2(A, A) \cdot t^n$, we get another classical result of Gerstenhaber: the triviality of the Hochschild cohomology $HH^2(A, A)$ guarantees the triviality of each deformation of A, that is, the rigidity of A.

Remark 4.1. As we see, in determining the equivalence of deformations there take place only the operations $E_{1,1}$ and $E_{1,2}$, the higher operations $E_{1,k}$, $k > 2$ disappear due to (3.4). Therefore considering only the deformation problem, it is impossible to establish a general formula (3.3) for transformation of twisting elements. The operation $E_{1,1}(a, b)$ which we denote here as $a \smile_1 b$, was introduced by Gerstehaber and denoted as the product of circles $a \circ b$, he also mentioned that although this product is not associative, its commutator $[a, b] = a \circ b - b \circ a$ satisfies the Jacobi identity, and the obtained Lie algebra structure is well related to the cup-product in the Hochschild complex, and this induces the so-called Gerstenhaber algebra structure on the Hochschild cohomology. Because of this work with deformations, some authors use the $db = [b, b]$-Maurer–Cartan solution instead of $db = b \smile_1 b$. We hope that application of higher hGa operations $E_{1,k}$ will be useful for the theory of deformations.

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