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IN MEMORY OF NODAR BERIKASHVILI

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Abstract. I write to praise Nodar Berikashvili's pioneering work in developing the study of strong homotopy structures, trying to highlight some of the most important aspects, with a hint at future developments.

1. Introduction

I last wrote in honor of the 90th birthday of Nodar Berikashvili, whom I first encountered through his publications on the homology of fiber spaces (1968, 1976, 1987). It was the fertile ground prepared by Nodar Berikashvili that allowed strong homotopy structures to flourish in Tbilisi well before they became popular in the West. In 1987, I was warmly welcomed by the Georgian delegation to participate at the International Conference on Topology and its Applications, in Baku, in neighboring Azerbajian. The topic was "Cohomological Methods in Physics", which prominently included fiber bundles, hence related to Berikashvili's work on fibrations in terms of twisting cochains or analogous differentials. Victor Gugenheim and Johannes Huebschmann were also at the 1987 meeting in Baku. It would be nice to know how long we had been in contact with the Georgiance before that meeting, although not in person. Tornike writes to me and recalls:

Actually, the first contact was your letter to me sent by mail about my Ainfty article, you told me there that you popularized it at some conference. The next was Dieter Puppe, he visited Tbilisi in 1985, and invited Nodar, me and Samson to Heidelberg, where I met Johannes Huebschmann. That is how all it started.

This marks the birth (or at least the early days) of what is nowadays called homological perturbation theory[1].

2. Some History

Today I will eschew the physical relevance of his work, concentrating instead on his contributions to the everlasting importance of twisting cochains and twisting *functions*.

Let $F \hookrightarrow E \to B$ be a fibre bundle. In 1959, E. H. Brown [5] showed that for a path-connected base B, the homology of the total space E is isomorphic to that of the twisted tensor product

$$C(F) \otimes_t C(B). \tag{2.1}$$

In fact, his main theorem states that there is a chain homotopy equivalence

$$C(F) \otimes_t C(B) \to C(E). \tag{2.2}$$

Here, "twisted" means that the usual tensor product differential is modified to a twisted differential $d_t = d + t$ by adding a *twisting cochain*

$$t \colon C(B) \to C(Aut(F)).$$

Crucially, $(d_t)^2 = 0$ unravels as

$$[d, t] + t^2 = 0,$$

calling to mind the Maurer–Cartan equation. Ed Brown used acyclic models to construct the twisting cochain t.

[1]A proper historical/socialogical study of Berkashvili's school and its interaction with its Western counterparts would provide welcome insight.

Shortly afterwards, Barratt, Gugenheim and Moore [1] constructed twisting *functions* τ in the simplicial setting, that is, for the twisted Cartesian products $F \times_\tau B$, where F and B are simplicial sets and $\tau\colon B_{>0} \to Aut(F)$ is a twisting function with values in the structure group $Aut(F)$, modifying just the 0-th face of the simplicial product $B \times F$:

$$\partial_0^\tau(f,b) = \big(\tau(b) \bullet \partial_0 f,\ \partial_0 b\big)$$

for simplices of positive dimension and $G = Aut(F)$ acting on the left.

Then Szczarba [9] gave an explicit formula for t in terms of τ. Other versions of the proof involved a twisted Eilenberg–Zilber map. Franz has recently pointed out that τ can be expressed in terms of t by solving inductively a certain equation.

3. Berikashvili's Functor $\mathcal{D}$

Today, I will emphasize two aspects of Barikashvili's work: his functor $\mathcal{D}$ and it's specification in terms of twists. (In the West, a similar approach by Victor Gugenheim [6] was developed by using systematically the notions of algebras-coalgebras, modules-comodules, Cotor, etc., in the category of chain complexes). In 1968, Berikashvili introduced the functor $\mathcal{D}$ in terms of "twisting elements" or "twisting cochains" in a differential graded algebra. Around that time, twisting cochains had a presence in topology and differential homological algebra (see [5, 7]). These inspired Berikashvili; in a series of short[2] papers, he developed an approach using his functor $\mathcal{D}$.

This functor $\mathcal{D}$ assigns to a pair (B, K) consisting of a simplicial set B and a simplicial group K a set consisting of equivalence classes of bundles with the base B and structural group K. It helps to think of $\mathcal{D}(B, K)$ as a moduli space of such bundles.

In [2, 3][3] in 1998, he explores the structure of the homology spect ral sequence of a fibration at the level of twisted structures in 1998, he explores the structure of the homology spectral sequence of a fibration at the level of twisted structures involving the chains of the base and of the fibre. The first paper is the abstract version for a very general class of spectral sequences in the category of differential graded algebras. The second paper consists of applications of his functor $\mathcal{D}$ to fibre spaces.

This includes a new look at various chain models, in particular, especially the calculation of differentials in the spectral sequence in the spirit of the work of Shih Wei–Shu [8].

These two papers represent a significant synthesis of and insight into the works of Shih and other mathematicians, especially Hirsch, Ed Brown and Dold.

Later, Berikashvili extended his functor $\mathcal{D}$ to more general settings, including simplicial (see Huebschmanm's paper in this volume for a broad overview).

In 2006 [4], he returned to consider further structure including the behavior of Steenrod's cochain operations $\smile_i$ in the Serre spectral sequence in terms of the singular complex functor and twisting cochains, especially in the context of local coefficient systems. On the other hand, he did not work with the higher homotopies underlying the $\smile_i$ operations, as in Steenrod. The latter played a key role in the theory of "operads" as developed by Peter May.

In addition to Berikashvili's own work, he founded a school of homotopical algebra which lives on and includes a focus on similar higher homotopies. Unfortunately, I have been unable to visit Tblisi myself, but two of my former students, Tom Lada and Ron Umble, have represented me there several times.

4. Recent and Future Developments

There is a more recent functor $\mathfrak{RH}$, related, but in a rather distant context, in algebraic and differential geometry. Block and Smith developed a generalization of the classical Riemann–Hilbert correspondence between covering spaces of a closed, compact, connected manifold M and representations of its fundamental group $\pi_1(M)$.

[2] a Soviet math tradition - due to limitation on paper?

[3] Earlier as a preprint in 1998. Caution with citations: Berikashvili wrote and published in German, Russian and his native Georgian.

Their generalization is a correspondence (in fact, an equivalence of quasi-categories)

$$\mathfrak{RH}\colon \mathfrak{F}(M) \Longleftrightarrow sSet(M, \mathcal{C}_\infty)\colon \mathfrak{HR} \tag{4.1}$$

between certain dg-categories, in particular, between the de Rham dg-algebra of differential forms on a compact manifold M and the dg-category of *infinity-local systems* on M. The latter are the homotopy coherent representations of the (smooth) singular simplicial set $Sing_\bullet(M)$. The relevance of the topological version $Sing(X)$ and corresponding infinity-local systems (without the name) appears already in Berikshvili [4].

A crucial part of the Block and Smith program is a *twisting cochain*.

The Block–Smith correspondence calls out for the generalization of the form

$$\mathfrak{RH}\colon \mathfrak{F}(K) \Leftrightarrow sSet(K, \mathcal{C}_\infty)\colon \mathfrak{HR} \tag{4.2}$$

in which K is a simplicial set and $\mathfrak{F}(K)$ is a dg-category of fibrations over K while $sSet(K, \mathcal{C}_\infty)$ is a dg-category of infinity-local systems on K.

For almost any notion of fibration with homotopy lifting, the functor

$$\mathfrak{RH}\colon \mathfrak{F}(K) \Rightarrow sSet(K, \mathcal{C}_\infty)$$

is straightforward and indeed classical. The functor

$$\mathfrak{F}(K) \Leftarrow sSet(K, \mathcal{C}_\infty)\colon \mathfrak{HR}$$

is more subtle, involving higher homotopies. Block and Smith worked in a more 'geometric' (in fact, algebraic geometric) context in which appropriate 'machinery' exists.

Perhaps a more complicated higher homotopy generalization of Berikashvili's functor $\mathcal{D}$ would help.

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