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ON A QUESTION RELATED TO SYLVESTER–GALLAI THEOREM

TENGIZ TETUNASHVILI

Abstract. In this note, a certain generalized version of the Sylvester–Gallai theorem is considered. More precisely, for each finite set of points $Z \subset \mathbf{R}^2$ of a special type, the existence of a covering of Z by some closed stripe of specially restricted width, depending on a certain characteristic of Z, is examined. A theorem showing the non-existence, in the general case, of such a covering for $\operatorname{card}(Z) > 4$ is presented. In addition, a theorem is presented stating the existence of such a covering for each set $Z \subset \mathbf{R}^2$ of the same type and consisting of four points.

The celebrated Sylvester–Gallai theorem asserts that the answer to the following question is positive.

Let Z be a finite subset of the Euclidean plane such that Z has at least two elements and, for each pair of distinct points P and Q of Z, there exists a point R of Z such that P, Q and R are collinear and $R \neq P$ and $R \neq Q$. Is there a straight line L in the plane such that Z is a subset of L?

It should be noted that the Sylvester–Gallai theorem has a rich history and various essentially different proofs (see, e.g., [2, 4, 6]). Questions related to this theorem remain objects of interesting mathematical researches. An example of such a question is the well-known problem of finding the number of the so-called ordinary lines for a given finite set of points (see, e.g., [1,3,4]). For $\operatorname{card}(Z) = n$, the complexity of an algorithm of finding ordinary lines was estimated from above in [5]. In short, the problem posed by Sylvester on a finite number of collinear points has various aspects: projective, metric, combinatorial, algorithmic. It makes sense to consider the same problem from the point of view of its stability. In this direction, we begin with formulating the question such that a positive answer to it would imply the Sylvester–Gallai theorem.

Let $L_{a,b}$ denote the straight line passing through the two distinct points a and b in the plane.

Also, let $\rho(c, L)$ be the Euclidean distance from a point c to a straight line L.

As usual, N will denote the set of all natural numbers.

In the plane consider a finite set of points Z with cardinality greater than 2. Let

$$d_Z = \max_{\substack{a \in Z \\ b \in Z \\ a \neq b}} \big(\min_{c \in Z \setminus \{a,b\}} \rho(c, L_{a,b}) \big).$$

Is there a function f acting from $((2, +\infty) \cap N) \times (0, +\infty)$ into $(0, +\infty)$ and satisfying these two conditions:

a) $\lim\limits_{x \to 0+} f(n, x) = 0$ for every fixed $n \in ((2, +\infty) \cap N)$;

b) if r is a positive real number, then for each finite set of points Z of cardinality $n > 2$ in the Euclidean plane such that $d_Z \leq r$ there exists a straight line L_Z in the plane such that $\max \{\rho(c, L_Z) : \ c \in Z\} \leq f(n, r)$?

It can easily be seen that the Sylvester–Gallai theorem asserts the following:

If Z is a finite subset of the Euclidean plane such that Z has at least two elements and for each pair of distinct points P and Q of Z there exists a point R of Z such that P, Q and R are collinear and $R \neq P$ and $R \neq Q$, then there exists a closed stripe H_Z of zero width entirely containing Z.

The above question can be equivalently formulated in terms of closed stripes. More precisely, is there a function f acting from $((2, +\infty) \cap N) \times (0, +\infty)$ into $(0, +\infty)$ and satisfying the following two conditions:

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(i) $\lim_{x\to 0+} f(n,x) = 0$ for every $n \in ((2,+\infty) \cap N)$;

(ii) if r is a positive real number and Z is a finite subset of the plane of cardinality $n > 2$ and $d_Z \le r$, then there exists a closed strip H_Z of width $2f(n,r)$ such that Z is a subset of H_Z?

It is not difficult to show that the positive answer to this question would imply the Sylvester–Gallai theorem. To prove this implication one can use the following statement.

Lemma 1. *Let $\triangle ABC$ be a non-degenerate triangle in the Euclidean plane. Then, for each straight line L in this plane, there exists a vertex of $\triangle ABC$ such that the distance from this vertex to L is not less than a half of the length of the shortest altitude of $\triangle ABC$.*

However, we formulate below a theorem which provides a negative answer to the question and shows the non-existence of such a function.

Theorem 2. *Let $n \ge 5$ be a fixed natural number and α and β be any two fixed positive real numbers.*

There exists a subset Z of the Euclidean plane $\mathbf{R}^2$ such that $\operatorname{card}(Z) = n$ and $d_Z < \alpha$.

Also, for each straight line $L \subset \mathbf{R}^2$, there exists a point O_L of Z such that the Euclidean distance $\rho(O_L, L)$ from O_L to L satisfies the inequality $\rho(O_L, L) > \beta$.

In connection with Theorem 2, we present the following statement.

Theorem 3. *There exists a positive real number μ such that for an arbitrary positive real number r and for each set of points Z of cardinality 4 in the Euclidean plane such that $d_Z \le r$ there exists a straight line L_Z in the plane such that the following inequality:*

$$\max\{\rho(c, L_Z) : \ c \in Z\} \le \mu r$$

holds.

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Department of Mathematics, Georgian Technical University, 77 Kostava Str., Tbilisi 0171, Georgia

I. Vekua Institute of Applied Mathematics, 2 University Str., Tbilisi 0186, Georgia
Email address: tengiztetunashvili@gmail.com; t.tetunashvili@gtu.ge