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A SUBGROUP OF B_4 THAT CONTAINS THE KERNEL OF BURAU REPRESENTATION

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Dedicated to the memory of Academician Nodar Berikashvili

Abstract. It is known that there are braids α and β in the braid group B_4, such that the group $\langle\alpha,\beta\rangle$ is a fee subgroup [7], which contains the kernel K of the Burau map $\rho_4 : B_4 \to GL\left(3,\mathbb{Z}[t,t^{-1}]\right)$ [3,6]. In this paper we will prove that K is a subgroup of $G=\langle\tau,\Delta\rangle$, where τ and Δ are fourth and square roots of the generator θ of the center Z of the group B_4. Consequently, we will write elements of K in terms of $\tau^i,\ i=1,2,3$ and Δ. Moreover, we will show that the quotient group G/Z is isomorphic to the free product $Z_4 * Z_2$.

1. Introduction

Let $\rho_4 : B_4 \to GL\big(3,\mathbb{Z}\big[t,t^{-1}\big]\big)$ be the reduced Burau representation of the braid group B_4. Consider the matrices $A=\rho_4(\alpha)$ and $B=\rho_4(\beta)$, where $\alpha=\sigma_1\sigma_2\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}$ and $\beta=\sigma_3\sigma_1^{-1}$ ($\sigma_i,\ i=1,2,3$ are standard generators of B_4). It is known that the group $\langle\alpha,\beta\rangle$ generated by α and β is a free group, which contains the kernel of the Burau map $\rho_4 : B_4 \to GL\left(3,\mathbb{Z}[t,t^{-1}]\right)$ [3,5–7]. Note that in [5, Theorem 3.19] it is shown that the kernel of the Burau map ρ_4 is a subgroup of the free group $\langle X,Y\rangle$ of rank 2, where $X=\sigma_3\sigma_1^{-1}$ and $Y=\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}$. On the other hand, since $\beta=X$ and $\alpha=Y^{-1}\cdot X$, we have $\langle X,Y\rangle=\langle\alpha,\beta\rangle$.

In the paper [1] it is shown that there exists an order four matrix T, which satisfies the following equality:

$$A=TBT^{-1},\quad A^{-1}=T^{-1}BT,\quad B^{-1}=T^2BT^2.$$

Using these relations, we have shown that the braid $\tau=\sigma_1\sigma_2\sigma_3\in B_4$ has the properties

$$\alpha=\tau^{-1}\beta\tau,\quad \alpha^{-1}=\tau\beta\tau^{-1},\quad \beta^{-1}=\tau^2\beta\tau^{-2}.$$

On the other hand, $\beta=\Delta^{-1}\tau^2$, where $\Delta=\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$. Consequently, we will obtain that an element of K has the form

$$\omega=\theta^m\tau^2\Delta\tau^{i_1}\Delta\tau^{i_2}\dots\tau^{i_k}\Delta\tau^2,\quad m\in\mathbb{Z},\ \ i_i\in\{1,2,3\}.$$

where θ is the generator of the center Z of the group B_4. Since $\rho_4(\theta)=t^4I$, if $\bar{T}=\rho_4(\tau)$ and $D=\rho_4(\delta)$, then the Burau representation for $n=4$ is faithful if and only if the product of the matrices of the form

$$t^{4m}\bar{T}^2D\bar{T}^{i_1}D\bar{T}^{i_2}\dots\bar{T}^{i_k}D\bar{T}^2,\quad m\in\mathbb{Z},\ \ i_i\in\{1,2,3\}$$

is not the identity matrix.

2. Subgroup Generated by τ and Δ

Let σ_1, σ_2 and σ_3 be the standard generators of the braid group B_4. Consider the corresponding Burau matrices:

$$\rho_4\left(\sigma_1\right)=\begin{bmatrix}-t & t & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix},\quad \rho_4\left(\sigma_2\right)=\begin{bmatrix}1 & 0 & 0\\ 1 & -t & t\\ 0 & 0 & 1\end{bmatrix},\quad \rho_4\left(\sigma_3\right)=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 1 & -t\end{bmatrix}.$$

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Note that there is a minor difference from the standard matrices considered in [5]. Moreover, following [2] we use the left multiplication of matrices.

In the paper [1, Lemma 2.1] is shown that for $\alpha = \sigma_1\sigma_2\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}$ and $\beta = \sigma_3\sigma_1^{-1}$ and corresponding matrices

$$A = \rho_4(\alpha) = \begin{bmatrix} 0 & 0 & -t^{-1} \\ 0 & -t & -t^{-1}+t \\ -1 & 0 & -t^{-1}+1 \end{bmatrix},$$

$$B = \rho_4(\beta) = \begin{bmatrix} -t^{-1} & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -t \end{bmatrix},$$

there is an order four matrix

$$T = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

such that

$$A = TBT^{-1}, \quad A^{-1} = T^{-1}BT, \quad B^{-1} = T^2BT^2. \tag{2.1}$$

Note that, since we use **the left multiplication of matrices**, using (2.1) we obtain the following:

Lemma 2.1. *The braid $\tau = \sigma_1\sigma_2\sigma_3$ (see Figure 1) is the element of B_4, such that*

$$\bar{T} = \rho_4(\tau) = \begin{bmatrix} -t & t & 0 \\ -t & 0 & t \\ -t & 0 & 0 \end{bmatrix} \tag{2.2}$$

and the following condition is fulfilled:

$$\alpha = \tau^{-1}\beta\tau, \quad \alpha^{-1} = \tau\beta\tau^{-1}, \quad \beta^{-1} = \tau^2\beta\tau^{-2}. \tag{2.3}$$

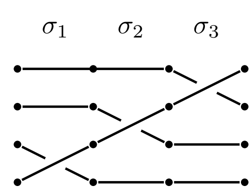

FIGURE 1. $\tau = \sigma_1\sigma_2\sigma_3$.

Proof. The equation (2.2) can be obtained by the direct calculation. For (2.3) we will use the following relations:

$$\sigma_{i+1}^{-1}\sigma_i^{-1}\sigma_{i+1} = \sigma_i\sigma_{i+1}^{-1}\sigma_i^{-1}, \quad \sigma_{i+1}\sigma_i^{-1}\sigma_{i+1}^{-1} = \sigma_i^{-1}\sigma_{i+1}^{-1}\sigma_{i+1}$$
$$\sigma_{i+1}^{-1}\sigma_i\sigma_{i+1} = \sigma_i\sigma_{i+1}\sigma_i^{-1}, \quad \sigma_{i+1}\sigma_i\sigma_{i+1}^{-1} = \sigma_i^{-1}\sigma_{i+1}\sigma_{i+1}.$$

Case 1. $\alpha = \tau^{-1}\beta\tau$ (see Figure 2 and Figure 3):

$$\tau^{-1}\beta\tau = (\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1})(\sigma_3\sigma_1^{-1})(\sigma_1\sigma_2\sigma_3) = \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2\sigma_3$$
$$= \sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2\sigma_3\sigma_2 = \sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}\sigma_3\sigma_2 = \sigma_1\sigma_3^{-1}\sigma_2^{-1}\sigma_3\sigma_1^{-1}\sigma_2$$
$$= \sigma_1\sigma_2\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_2 = \sigma_1\sigma_2\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1} = \alpha.$$

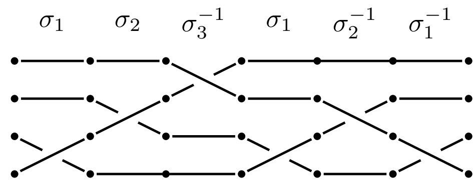

FIGURE 2. $\alpha = \sigma_1\sigma_2\sigma_3^{-1}\sigma_1\sigma_2^{-1}\sigma_1^{-1}$.

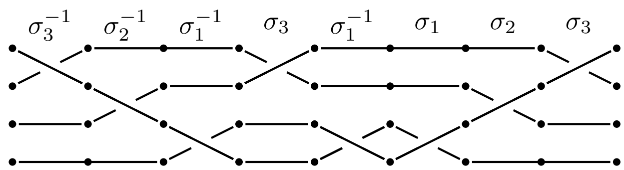

FIGURE 3. $\tau^{-1}\beta\tau$.

Case 2. $\alpha^{-1} = \tau\beta\tau^{-1}$ (see Figure 4 and Figure 5):

$$\tau\beta\tau^{-1} = (\sigma_1\sigma_2\sigma_3)(\sigma_3\sigma_1^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1})$$
$$= \sigma_1\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_1^{-1} = \alpha^{-1}.$$

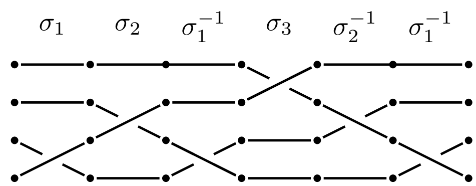

FIGURE 4. $\alpha^{-1} = \sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_1^{-1}$.

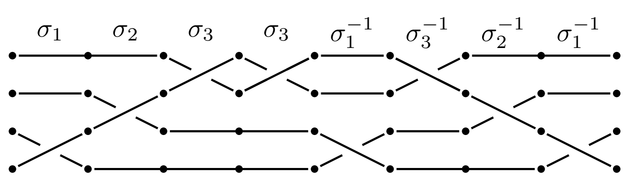

FIGURE 5. $\tau\beta\tau^{-1}$.

Case 3. $\beta^{-1} = \tau^2\beta\tau^{-2}$ (see Figure 6 and Figure 7):

$$\tau^2\beta\tau^{-2} = (\sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3)(\sigma_3\sigma_1^{-1})(\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1})$$
$$= \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_3\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1}$$
$$= \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1^{-1}\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}$$
$$= \sigma_1\sigma_2\sigma_3\sigma_2^{-1}\sigma_1\sigma_2\sigma_2^{-1}\sigma_3^{-1}\sigma_2\sigma_2^{-1}\sigma_1^{-1}\sigma_2^{-1}$$
$$= \sigma_1\sigma_2\sigma_3\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1} = \sigma_1\sigma_2\sigma_3\sigma_3^{-1}\sigma_2^{-1}\sigma_3^{-1} = \sigma_1\sigma_3^{-1} = \beta^{-1}.$$

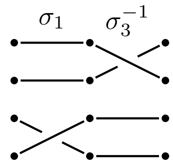

FIGURE 6. $\beta^{-1} = \sigma_1\sigma_3^{-1}$.

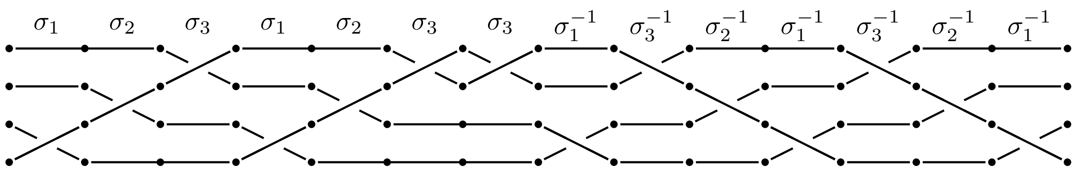

FIGURE 7. $\tau^2\beta\tau^{-2}$.

The lemma is proved. □

Lemma 2.2. *The braid* $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$ *(see Figure 8) is the element of* B_4*, such that*

$$\beta = \Delta^{-1}\tau^2, \quad \alpha = \tau^{-1}\Delta^{-1}\tau^3, \quad \alpha^{-1} = \tau\Delta^{-1}\tau, \quad \beta^{-1} = \tau^2\Delta^{-1}. \tag{2.4}$$

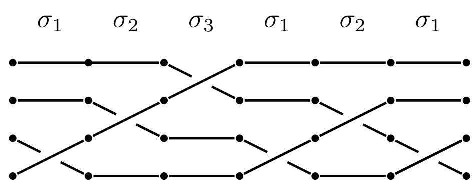

FIGURE 8. $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$.

Proof. Let us show that $\beta = \Delta^{-1}\tau^2$ (see figure 9 and figure 10):

$$\Delta^{-1}\tau^2 = (\sigma_1^{-1}\sigma_2^{-1}\sigma_1^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1})(\sigma_1\sigma_2\sigma_3)(\sigma_1\sigma_2\sigma_3) = \sigma_1^{-1}\sigma_3 = \sigma_3\sigma_1^{-1} = \beta$$

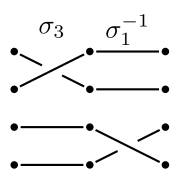

FIGURE 9. $\beta = \sigma_3\sigma_1^{-1}$.

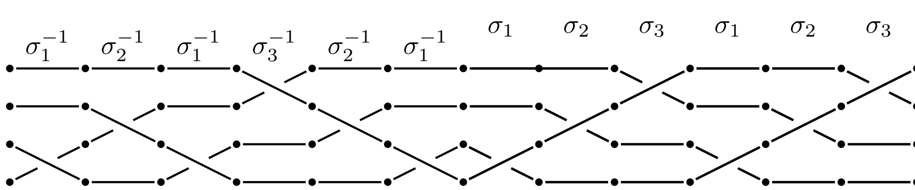

FIGURE 10. $\Delta^{-1}\tau^2$.

On the other hand, Lemma 2.1 and the equality $\beta = \Delta^{-1}\tau^2$ imply remaining part of (2.4). □

Theorem 2.1. *The kernel K of the Burau representation*

$$\rho_4 : B_4 \to GL\big(3, \mathbb{Z}\big[t, t^{-1}\big]\big)$$

is contained in the subgroup G of B_4 generated by the elements τ and Δ. Moreover, if the kernel K is not-trivial, then there exists a non-identity element $\omega \in K$ that can be written in the form

$$\omega = \theta^m \tau^2 \Delta \tau^{i_1} \Delta \tau^{i_2} \dots \tau^{i_k} \Delta \tau^2, \quad m \in \mathbb{Z}, \;\; i_i \in \{1, 2, 3\}. \tag{2.5}$$

Proof. By the corollary 3.2 [3] any kernel element can be written as a word in the Bokut–Vesnin (Gorin–Lin) generators α, β, α^{-1} and β^{-1}. Let ω be a non-identity kernel element, written as an irreducible non-empty word in letters α, β, α^{-1} and β^{-1}. We can assume that ω has the suffix β and prefix β^{-1}. If not we can conjugate it by some power of β. In this case, by substitution $\beta = \Delta^{-1}\tau^2$, $\alpha = \tau^{-1}\Delta^{-1}\tau^3$, $\alpha^{-1} = \tau\Delta^{-1}\tau$, $\beta^{-1} = \tau^2\Delta^{-1}$ and using the property that $\tau^{4m} = \Delta^{2m} = \theta^m$ commutes all elements of B_4, we can reduce ω in the form (2.5). □

Corollary 2.1. *The Burau representation for $n = 4$ is faithful if and only if the product of the matrices of the form*

$$t^{4m}\bar{T}^2 D \bar{T}^{i_1} D \bar{T}^{i_2} \dots \bar{T}^{i_k} D \bar{T}^2, \quad m \in \mathbb{Z}, \;\; i_i \in \{1, 2, 3\}$$

is not the identity matrix, where $\bar{T} = \rho_4(\tau)$ and $D = \rho_4(\Delta)$.

Corollary 2.2. *Let G be the subgroup of B_4 generated by τ and Δ and Z be the center, then a representation of G/Z is given by:*

$$\langle \tau, \Delta | \tau^4 = \Delta^2 = 1 \rangle,$$

where $\tau = \sigma_1\sigma_2\sigma_3$ and $\Delta = \sigma_1\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1$.

References

1. A. Beridze, S. Bigelow, P. Traczyk, On the Burau Representation of B_4 modulo p. *TMJ Special issue* **7**, 2021, 57–62.
2. A. Beridze, P. Traczyk, Burau representation for $n = 4$. *J. Knot Theory Ramifications* **27** (2018), no. 3, 1840002, 6 pp.
3. A. Beridze, P. Traczyk, Forks, noodles and the Burau representation for $n = 4$. *Trans. A. Razmadze Math. Inst.* **172** (2018), no. 3, part A, 337–353.
4. S. Bigelow, The Burau representation is not faithful for $n = 5$. *Geom. Topol.* **3** (1999), 397–404.
5. J. S. Birman, *Braids, Links, and Mapping Class Groups.* Annals of Mathematics Studies, no. 82. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1974.
6. L. Bokut, A. Vesnin, Gröbner-Shirshov bases for some braid groups. *J. Symbolic Comput.* **41** (2006), no. 3-4, 357–371.
7. E. A. Gorin, V. Ya. Lin, Algebraic equations with continuous coefficients and some problems of the algebraic theory of braids. *Math. USSR Sb.* **7** (1969), no. 4, 569-596.
8. D. D. Long, M. Paton, The Burau representation is not faithful for $n \geq 6$. *Topology* **32** (1993), no. 2, 439–447.
9. J. A. Moody, The Burau representation of the braid group B_n is unfaithful for large n. *Bull. Amer. Math. Soc. (N.S.)* **25** (1991), no. 2, 379–384.
10. S. Moran, Matrix representation for the braid group B_4. *Arch. Math. (Basel)* **34** (1980), no. 6, 496–501.

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