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COMPLEX COBORDISM MODULO SPHERICAL COBORDISM

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Dedicated to the memory of Academician Nodar Berikashvili

Abstract. We propose a commutative, complex-oriented cohomology theory $MU_S^*(-)$, complex cobordism modulo c_1-spherical cobordism with the coefficient ring, identical to a quotient ring MU^*/S. This exploits the Baas-Sullivan theory of cobordism with singularities and all the Mironov obstructions to the commutativity vanish.

1. Introduction

In [5], we observe that the ideal in a complex cobordism ring MU^* generated by the polynomial generators $S = (x_1, x_k, k \geq 3)$ of a c_1-spherical cobordism ring W^*, viewed as elements in MU^* by a forgetful map, is prime.

Using the Baas–Sullivan theory of cobordism with singularities and proving that all the Mironov obstructions to the commutativity vanish, we define a commutative, complex-oriented cohomology theory $MU_S^*(-)$, complex cobordism modulo c_1-spherical cobordism with the coefficient ring, identical to the quotient ring MU^*/S.

Then any subsequence $\Sigma \subseteq S$ is also regular in MU^* and therefore provides a multiplicative complex-oriented cohomology theory $MU_\Sigma^*(-)$.

In particular, we prove that the generators of $W^*[1/2]$ can be specified as follows:

i) For $\Sigma = (x_k, k \geq 3)$, the corresponding cohomology is identical to the Abel cohomology, constructed previously in [11];

ii) Another example corresponding to $\Sigma = (x_k, k \geq 5)$ gives $MU^*[1/2]/\Sigma$, the coefficient ring of the universal Buchstaber formal group law, i.e., it rationally is identical to the Krichever–Hoehn complex elliptic genus [13, 14].

2. Statements

The theory of complex cobordism $MU^*(-)$ and special unitary cobordism $MSU^*(-)$ play an important role in the cobordism theory. The ring MSU^* is torsion free after localized away from 2,

$$MSU^*[1/2] = \mathbb{Z}[1/2][y_2, y_3, \dots], \quad |y_i| = 2i.$$

With the SU-structure forgetful inclusion in mind

$$MSU^*[1/2] \subset MU^*[1/2] = \mathbb{Z}[1/2][a_1, a_2, a_3, \dots], \quad |a_i| = 2i,$$

the generators y_i can be treated as the elements in $MU^*[1/2]$.

In particular, y_i is a SU-manifold if and only if all Chern numbers of y_i having factor c_1, are zero. Thus we have to check the main Chern number $s_i(y_i)$ for Novikov's criteria [17], in order for the set of polynomial generators to belong to $MSU^*[1/2]$.

In [12], Conner and Floyd introduced c_1-spherical cobordism groups $W^{2n} \subset MU^{2n}$. A complex cobordism class belongs to W^{2n} if and only if every Chern number involving c_1^2 vanishes.

W^* is not a subring of MU^*. However, with respect to some complex orientation and some $*$ multiplication, the ring W^* is polynomial on generators in every positive even degree except 4, [15, 18]:

$$W^* = Z[x_1, x_k : k \geq 3], \quad x_1 = CP_1, \;\; x_k \in W_{2k}. \tag{2.1}$$

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The polynomial generators x_k can be specified by the condition for the main Chern numbers $s_k(x_k) = d(k)d(k-1)$ for $k \geq 3$, where

$$d(k) = \begin{cases} p, & \text{if } k+1 = p^s \text{ for some prime} p, \\ 1, & \text{otherwise.} \end{cases}$$

By [18], the cohomology theory $W^*(-)$ is complex-oriented and (by Proposition 3.12) the ring $W^*[1/2]$ is generated by the coefficients of the corresponding formal group law F_W. In particular,

$$W^*[1/2] \text{ is a free } MSU^*[1/2] \text{ module generated by 1 and } [CP_1].$$

Our main observation is that the sequence of polynomial generators of W^*, $(x_k, k \geq 3)$ in (2.1) viewed as the elements in MU^* by a forgetful map, generates the same ideal in MU^* as the coefficients $\alpha_{ij}, i, j \geq 2$ of the universal formal group law.

With that in mind, we prove one of our main results

Theorem 2.1. *Let $S = (x_1, x_k, k \geq 3)$ be a sequence of polynomial generators of c_1-spherical cobordism W^*. Then:*
i) *S is regular in MU^*.*
ii) *any subsequence Σ of S is regular in MU^*.*

The use of the Baas–Sullivan theory of cobordism with singularities defines a multiplicative cohomology theory $MU^*_\Sigma(-)$, with the scalar ring $MU^*/(\Sigma)$. All obstructions to multiplicativity vanish bay [16]. Clearly, $h^*_\Sigma(-)$ is complex-oriented.

One interesting example of Theorem 2.1 is the Abel cohomology h^*_{Ab} constructed in [11].

Theorem 2.2. *For $\Sigma = (x_i, i \geq 3)$, the cohomology theory $MU^*_\Sigma(-)[1/2]$ is identical to the Abel cohomology $h^*_{Ab}[1/2]$.*

The Abel cohomology is commutative by [11]. For any Σ, the question of commutativity is not trivial. The obstruction to commutativity is studied in [16]. In some cases, all obstructions vanish only for dimensional reasons.

In particular, using $\Sigma = S$ in Theorem 2.1, we define the complex cobordism modulo c_1-spherical cobordism.

Theorem 2.3. *There is a commutative complex-oriented cohomology theory $MU^*_S(-)$ with the coefficient ring MU^*/S.*

Another example of Theorem 2.1 is related to the Buchstaber formal group law F_B [2–4, 6]. After tensored with rational numbers, the corresponding classifying map f_B is identical [7, 8] to the Krichever–Hoehn complex elliptic genus [13, 14]. The scalar ring Λ_B of F_B is calculated in [10]. In particular, it has only 2-torsion and the quotient

$$\Lambda_{\mathcal{B}} = \Lambda_B/\text{torsion}$$

is an integral domain. We will consider this example and prove the following

Theorem 2.4. *Let $\Sigma = (x_i, i \geq 5)$, where x_i as in Theorem 2.1. Let Λ_B be the scalar ring of the universal Buchstaber formal group law. Then the cohomology theory $MU^*_\Sigma(-)[1/2]$ is multiplicative, commutative, complex-oriented and has the coefficient ring, identical to $MU^*[1/2]/\Sigma := \Lambda_B[1/2]$.*

On the c_1-spherical cobordism W^*, we discuss a genus

$$\phi_W : W^* \to \mathbb{Z}[x_1, x_3, x_4].$$

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References

1. N. A. Baas, On bordism theory of manifolds with singularities. *Math. Scand.* **33** (1973), 279–302.
2. M. Bakuradze, The formal groups of Bukhshtaber, Krichever and Nadiradze coincide. (Russian) *translated from Uspekhi Mat. Nauk* **68** (2013), no. 3(411), 189–190; *Russian Math. Surveys* **68** (2013), no. 3, 571–573.
3. M. Bakuradze, On the Buchstaber formal group law and some related genera. *Proc. Steklov Math. Inst.* **286** (2014), 7–21.
4. M. Bakuradze, Computing the Krichever genus. *J. Homotopy Relat. Struct.* **9** (2014), no. 1, 85–93.
5. M. Bakuradze, Complex cobordism modulo c_1-spherical cobordism and related genera. *Proc. Steklov Math. Inst.* (to appear).
6. M. Bakuradze, Cohomological realization of the Buchstaber formal group law. (Russian) *translated from Uspekhi Mat. Nauk* **77** (2022), no. 5, 189–190; *Russian Math. Surveys* **77** (2022), no. 5, 949–951.
7. M. Bakuradze, Polynomial generators of $\mathbf{MSU}^*[1/2]$ related to classifying maps of certain formal group laws. *Homology Homotopy Appl.* **26** (2024), no. 1, 1–14.
8. M. Bakuradze, V. V. Vershinin, On addition theorems related to elliptic integrals. (Russian) *translated from Tr. Mat. Inst. Steklova* **305** (2019); Algebraicheskaya Topologiya Kombinatorika i Matematicheskaya Fizika, 29–39 *Proc. Steklov Inst. Math.* **305** (2019), no. 1, 22–32.
9. V. M. Buchstaber, T. E. Panov, *Toric Topology.* Mathematical Surveys and Monographs, 204. American Mathematical Society, Providence, RI, 2015.
10. V. M. Buchstaber, A. V. Ustinov, Coefficient rings of formal groups. (Russian) *translated from Mat. Sb.* **206** (2015), no. 11, 19–60; *Sb. Math.* **206** (2015), no. 11-12, 1524–1563.
11. Ph. Busato, Realization of Abel's universal formal group law. *Math. Z.* **239** (2002), no. 3, 527–561.
12. P. E. Conner, E. E. Floyd, Torsion in SU-bordism. *Mem. Amer. Math. Soc.* **60** (1966), 74 pp.
13. G. Höhn, Komplexe elliptische Geschlechter und S^1-äquivariante Kobordismustheorie. (preprint) 2004, `https://arxiv.org/abs/math/0405232`.
14. I. M. Krichever, Generalized elliptic genera and Baker-Akhiezer functions. (Russian) *translated from Mat. Zametki* **47** (1990), no. 2, 34–45, 158; *Math. Notes* **47** (1990), no. 1-2, 132–142.
15. I. Yu. Limonchenko, T. E. Panov, G. S. Chernykh, *SU*-bordism: structure results and geometric representatives. (Russian) *translated from Uspekhi Mat. Nauk* **74** (2019), no. 3(447), 95–166; *Russian Math. Surveys* **74** (2019), no. 3, 461–524.
16. O. K. Mironov, Multiplications in cobordism theories with singularities, and Steenrod–tom Dieck operations. *Math. USSR Izv.* **13** (1979), no. 1, pp 89.
17. S. P. Novikov, Methods of algebraic topology from the point of view of cobordism theory. (Russian) *Izv. Akad. Nauk SSSR Ser. Mat.* **31** (1967), 855–951.
18. T. E. Panov, G. Chernykh, *SU*-linear operations in complex cobordism and the c_1-spherical bordism theory. (Russian) *translated from Izv. Ross. Akad. Nauk Ser. Mat.* **87** (2023), no. 4, 133–165; *Izv. Math.* **87** (2023), no. 4, 768–797.

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