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IMPLICATION AND PRODUCTION GRAPHS ON EQ-ALGEBRAS

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Abstract. In this paper, the implication and production graphs of EQ-algebras are investigated. Firstly, a quasi r-prime up set is defined and some related results are provided. Further, implication graphs are constructed and it is shown that these graphs of separated EQ-algebras is a connected graph with diameter at most two and under certain conditions is a star graph. Moreover, the notion of zero divisors by a product operation is defined and by means of zero divisors of an EQ-algebra with bottom element 0, the production graph is introduced and it is proved that the production graph is a connected graph with diameter at most two. Also, some necessary conditions for the production graph to be a star graph are found. Finally, it is proved that the implication graphs and the production graphs coincide in involutive residuated EQ-algebras.

1. Introduction

The EQ-algebras were proposed by Novák and De Baets [13, 14]. One of the motivations was to introduce a special algebra as the correspondence of truth values for the high-order fuzzy type theory (FTT) [12] that generalizes the system of classical type theory [2] in which the sole basic connective is an equality. Analogously, the basic connective in the (FTT) should be a fuzzy equality. Another motivation is from the equational style of proof in logic. It has three connectives: meet $\wedge$, product $\odot$ and fuzzy equality $\sim$. The implication operation $\to$ is the derived of the fuzzy equality $\sim$ and it together with $\odot$ no longer form a strictly adjoint pair, in general. The EQ-algebras are interesting and important for studying and researching; residuated lattices which represent particular cases of the EQ-algebras. In fact, the EQ-algebras generalize non-commutative residuated lattices [7]. From the point of view of logic, the main difference between residuated lattices and the EQ-algebras lies in the way the implication operation is obtained. While in residuated lattices it is obtained from a (strong) conjunction, in the EQ-algebras it is obtained from the equivalence. Consequently, the two kinds of algebras differ at several essential points despite their many similar or identical properties. The graph theory has existed for many years and it has found many applications in engineering and science such as chemical, electrical, civil and mechanical engineering; architecture; management and control; communication; operational research; sparse matrix technology; combinatorial optimization; computer science. Many books have been published on the applied graph theory [5,18,19]. Especially, in the field of universal algebras and graph theory, the graph algebra is a way to get a directed graph of algebraic structure. Algebraic graph theory comprises the study of algebraic objects arising in connection with the graphs. Therefore many authors studied the theory of graphs. For example, in connection with semigroups and rings, I. Beck in [3] introduced the zero-divisor graph associated with the zero-divisor set of a commutative ring, whose vertex set is the set of zero divisors. Two distinct zero divisors x, y are adjacent if and only if $x \cdot y = 0$. In [9], Jun and Lee introduced the notion of associated graph of BCK/BCI-algebras by zero divisors in BCK/BCI-algebras and verified some properties of this graph. In addition, Torkzadeh and Ahmadpanah [16] defined the notions of zero divisors of a non-empty subset of a residuated lattice and a graph associated to a residuated lattice. They proved that this graph is always a connected graph and its diameter is at most two. Aaly Kologani, Borzooei and Kim [1], studied the graph structures on hoop algebras and constructed implicative and productive graphs. Moreover, they proved that these graphs are connected and both complete and tree under certain conditions.

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This paper is organized as follows: In Section 2, the basic definitions, properties and theorems of EQ-algebras and graph theory are reviewed. In Section 3, the notion of a quasi r-prime up set of EQ-algebras is defined and some of their properties are provided. Further, the implication graphs are constructed and it is shown that implication graphs of separated EQ-algebras is a connected graph with diameter at most two and under certain conditions is a star graph. In Section 4, the notion of zero divisors by a product operation is defined and by means of zero divisors of an EQ-algebra with a bottom element 0, the production graph is introduced. Moreover, it is proved that the production graph is a connected graph with diameter at most two. Also, some necessary conditions for the production graph to be a star graph are investigated. Finally, it is proved that the implication graphs and the production graphs coincide in the involutive residuated EQ-algebras.

2. Preliminaries

In this section, we give some fundamental definitions and results. For more details, refer to the references.

Definition 2.1 ([13])**.** An EQ-algebra is an algebra $(E, \wedge, \odot, \sim, 1)$ of type $(2, 2, 2, 0)$ satisfying the following axioms:
$(E1)$ $(E, \wedge, 1)$ is a $\wedge$-semilattice with a top element 1. We set $x \leq y$ if and only if $x \wedge y = x$;
$(E2)$ $(E, \odot, 1)$ is a commutative monoid and $\odot$ is isotone with respect to $\leq$;
$(E3)$ $x \sim x = 1$ (reflexivity axiom);
$(E4)$ $((x \wedge y) \sim z) \odot (s \sim x) \leq z \sim (s \wedge y)$ (substitution axiom);
$(E5)$ $(x \sim y) \odot (s \sim t) \leq (x \sim s) \sim (y \sim t)$ (congruence axiom);
$(E6)$ $(x \wedge y \wedge z) \sim x \leq (x \wedge y) \sim x$ (monotonicity axiom);
$(E7)$ $x \odot y \leq x \sim y$ (boundedness axiom), for all $s, t, x, y, z \in E$.

Let E be an EQ-algebra. Then for all $x, y \in E$, we put

$$x \to y = (x \wedge y) \sim x, \qquad \tilde{x} = x \sim 1.$$

The derived operation $\to$ is called implication. If an EQ-algebra E contains a bottom element 0, then we can define the unary operation $\neg$ on E by $\neg x = x \sim 0$ and for $\emptyset \neq X \subseteq E$, $\overline{X} := \{\neg x \mid x \in X\}$.

Definition 2.2 ([7,13])**.** Let E be an EQ-algebra. Then we say that it is
(i) Separated if $x \sim y = 1$ implies $x = y$, for all $x, y \in E$.
(ii) Spanned if it contains a bottom element 0 and $\tilde{0} = 0$.
(iii) Good, if $\tilde{x} = x$, for all $x \in E$.
(iv) Residuated, if $(x \odot y) \wedge z = x \odot y$ if and only if $x \wedge ((y \wedge z) \sim y) = x$, for all $x, y, z \in E$.
(v) Involutive (IEQ-algebra), if $\neg\neg x = x$, for all $x \in E$.
(vi) IEQ-algebra, if it has a lattice reduct and for all $x, y, z, t \in E$, $((x \vee y) \sim z) \odot (t \sim x) \leq z \sim (y \vee t)$.
Every residuated EQ-algebra or IEQ-algebra is good and every good EQ-algebra is spanned and separated.

Lemma 2.3 ([8,13])**.** *Let E be an* EQ-*algebra. Then for all $x, y, z \in E$, – the following properties hold:*
(i) $x \sim y = y \sim x$, $x \sim y \leq x \to y$. $x \odot y \leq x \wedge y \leq x, y$.
(ii) $(x \to y) \odot (y \to x) \leq x \sim y$.
(iii) $x \sim y \leq (x \wedge z) \sim (y \wedge z)$ *and* $x \sim y \leq (x \sim z) \sim (y \sim z)$.
(iv) $(x \sim y) \odot (y \sim z) \leq x \sim z$ *and* $(x \to y) \odot (y \to z) \leq x \to z$.
(v) *If* $x \leq y$, *then* $x \to y = 1$ *and* $x \sim y = y \to x$.
(vi) *If* $x \leq y$, *then* $z \to x \leq z \to x$ *and* $y \to z \leq x \to z$.
(vii) *If L contains a bottom element* 0, *then* $\neg 0 = 1$, $\neg 1 = 0$ *and* $\neg x = x \to 0$.
(viii) $a \odot (a \to b) \leq \tilde{b}$ (*weak modus ponens*).
(ix) *If E is good, then* $a \odot (a \to b) \leq b$.
(x) *If E is residuated, then* $x \to (y \to z) = (x \odot y) \to z$.
(xi) *If E is separated, then* $x \leq y$ *if and only if* $x \to y = 1$.
(xii) *If E is* IEQ-*algebra, then* $x \to y \leq (x \vee z) \to (y \vee z)$.

Definition 2.4 ([4])**.** Let E be an EQ-algebra and $\emptyset \neq F \subseteq E$. Then (i) F is called a prefilter of E if it satisfies for all $x, y \in E$.

$(F1)$ $1 \in F$.

$(F2)$ If $x \in F$, $x \to y \in F$, then $y \in F$.

A prefilter F is said to be a filter if it satisfies.

$(F3)$ If $x \to y \in F$, then $(x \odot z) \to (y \odot z) \in F$ for all $x, y, z \in E$.

Prefilters and filters coincide in residuated EQ-algebras. Let G be a graph with the vertex set V and edge set E. The edge that connects two distinct vertices x and y is denoted by $e(x, y)$. A graph $\mathrm{G} = (\mathsf{V}, \mathsf{E})$ is called connected if any two distinct vertices x and y of G are linked by a path in G, otherwise the graph is called disconnected. For distinct vertices x and y of G, let $d(x, y)$ be the length of the shortest path from x to y. If there is no path between x and y, then $d(x, y) = \infty$. The diameter of G is

$$\mathrm{diam}(E) = \sup\{d(x, y) \mid x, y \in \mathsf{V}(\Gamma(E))\}.$$

A tree is a connected graph with no cycles. A graph G is called a complete graph if $e(x, y)$ exists for any distinct vertices x and y of G. A graph G is called a star graph in case there is a vertex x in G such that every other vertex in G is an end connected with x and no other vertex by an edge [6].

Note. From now on, in this paper, let E be an EQ-algebra, unless otherwise is stated.

3. Implication Graph of an EQ-algbra

In this section, firstly, we define the sets $r(X)$, $l(X)$ by using implication operation. In the sequel, by means of the set $r(X)$ of an EQ-algebra, the associated implication graph $\Omega(E)$ is introduced and some related results are provided.

Definition 3.1. Let X be a non-empty subset of EQ-algebra E. Then we define $r(X)$, $l(X)$ as follow:

$$r(X) = \{a \in E \mid x \to a = 1, \text{ for every } x \in X\},$$
$$l(X) = \{a \in E \mid a \to x = 1, \text{ for every } x \in X\}.$$

Definition 3.2. Let F be a non-empty subset of E. Then F is called *a quasi r-prime up set* if:

(i) F is a proper up set of E,

(ii) for any $x, y \in E$, if $r(\{x, y\}) \subseteq F$, then $x \in F$ or $y \in F$.

Example 3.3 ([10])**.** Let $E = \{0, a, b, 1\}$ be a chain $0 < a < b < 1$ with the following Cayley tables:

Table 1.

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

Table 2.

$\sim$	0	a	b	1
0	1	a	a	a
a	a	1	b	b
b	a	b	1	1
1	a	b	1	1

Table 3.

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	b	1	1
1	a	b	1	1

Routine calculation shows that $(E, \wedge, \odot, \sim, 1)$ is an EQ-algebra and $r(\{0, b\}) = r(\{a, b\}) = \{1, b\}$, $r(\{0, a\}) = \{1, a, b\}$, $l(\{a, b\}) = \{0, a\}$. Moreover, $F = \{1, a, b\}$ is a quasi r-prime up set of E.

Proposition 3.4. *Let X and Y be two non-empty subsets of E. Then the following statements hold:*
(i) $1 \in r(X)$ *and if E is an* EQ-*algebra with a bottom element* 0*, then* $0 \in l(X)$.
(ii) $X \subseteq r(l(X))$ *and* $X \subseteq l(r(X))$.
(iii) *If* $X \subseteq Y$*, then* $r(Y) \subseteq r(X)$ *and* $l(Y) \subseteq l(X)$.
(iv) $r(X) = r(l(r(X)))$ *and* $l(X) = l(r(l(X)))$.
(v) *If F is a prefilter of E, then* $r(F) \subseteq F$.
(vi) $r(\{a,b\}) = r(\{a\}) \cap r(\{b\})$*, for any* $a, b \in E$.

Proof. (i) Since $X \neq \emptyset$ and by Lemma 2.3(v), $x \to 1 = 1$, for any $x \in X$, we get that $1 \in r(X)$ and since $0 \leq x$, for any $x \in X$, by Lemma 2.3(v), we conclude that $0 \to x = 1$ and so, $0 \in l(X)$.

(ii) Let $x \in X$ and $r(X) = Y$. Then for any $y \in Y$, we have $x \to y = 1$, for any $x \in X$. Hence $x \in l(Y)$ and so, $x \in r(l(X))$. Therefore $X \subseteq r(l(X))$. The proof $X \subseteq l(r(X))$ is similar.

(iii) Let $X \subseteq Y$ and $a \in r(Y)$. Then for any $y \in Y$, $y \to a = 1$ and so, for any $x \in X$, $x \to a = 1$. Thus $a \in r(X)$ and so, $r(Y) \subseteq r(X)$. The proof that $l(Y) \subseteq l(X)$ is similar.

(iv) By (ii), we have $X \subseteq l(r(X))$ and so, by (iii), we get $r(l(r(X))) \subseteq r(X)$. Now, let $Y = r(X)$. Then by (ii), we have $Y \subseteq r(l(Y))$ and so, $r(X) \subseteq r(l(r(X)))$. Hence $r(X) = r(l(r(X)))$ and by a similar way, we have $l(X) = l(r(l(X)))$.

(v) If F is a prefilter of E and $x \in r(F)$, then $f \to x = 1$, for any $f \in F$ and since $1 \in F$, we get $f \to x \in F$ and by $f \in F$, we conclude that $x \in F$. Therefore $r(F) \subseteq F$.

(vi) $t \in r(\{a,b\})$ if and only if $a \to t = 1$ and $b \to t = 1$ if and only if $t \in r(\{a\})$ and $t \in r(\{b\})$ if and only if $t \in r(\{a\}) \cap r(\{b\})$. Therefore $r(\{a,b\}) = r(\{a\}) \cap r(\{b\})$, for any $a, b \in E$. □

Definition 3.5. For any $x \in E$, the set of all elements $y \in E$ such that $r(\{x,y\}) = \{1\}$ is denoted by Z_x, in fact,

$$Z_x = \{y \in E \mid r(\{x,y\}) = \{1\}\}$$

Proposition 3.6. *Let E be a separated* EQ-*algebra. Then for any $x \in E$,* $r(\{x,1\}) = \{1\}$.

Proof. Let E be a separated EQ-algebra and $x \in E$. Then

$$\begin{aligned} r(\{x,1\}) &= \{a \in E \mid x \to a = 1 \,\text{and}\, 1 \to a = 1\} \\ &= \{a \in E \mid x \to a = 1 \,\text{and}\, a = 1\} \\ &= \{1\}. \end{aligned}$$ □

Proposition 3.7. *Let $a, b \in E$ such that $a \leq b$. Then the following statements hold:*
(i) $r(\{b\}) \subseteq r(\{a\})$.
(ii) $r(\{b,x\}) \subseteq r(\{a,x\})$*, for any* $x \in E$.
(iii) $Z_a \subseteq Z_b$.

Proof. (i) Let $a \leq b$ and $x \in r(\{b\})$. Then $b \to x = 1$ and by Lemma 2.3(vi), we have $b \to x \leq a \to x$ and so $a \to x = 1$. Hence, $x \in r(\{a\})$ and so, $r(\{b\}) \subseteq r(\{a\})$.

(ii) Since $a \leq b$, by (i) and Proposition 3.4(vi), for any $x \in E$, we get

$$r(\{b,x\}) = r(\{b\}) \cap r(\{x\}) \subseteq r(\{a\}) \cap r(\{x\}) = r(\{a,x\}).$$

(iii) If $x \in Z_a$, then $r(\{a,x\}) = \{1\}$. Now, if $t \in r(\{b,x\})$, then $b \to t = 1$ and $x \to t = 1$. Since $a \leq b$, by Lemma 2.3(vi), we get $b \to x \leq a \to x$ and so, $a \to x = 1$. Hence $t \in r(\{a,x\})$ and so, $t = 1$. Therefore $r(\{b,x\}) = \{1\}$ and so, $x \in Z_b$. Thus $Z_a \subseteq Z_b$. □

Corollary 3.8. *Let E be a separated* EQ-*algebra and $a, b \in E$ such that $a \to b = 1$. Then:*
(i) $r(\{b\}) \subseteq r(\{a\})$,
(ii) $Z_a \subseteq Z_b$.

Proof. Since E is a separated EQ-algebra and $a \to b = 1$, by Lemma 2.3(xi), we conclude that $a \leq b$. Therefore the assertion holds by Proposition 3.7. □

Theorem 3.9. *Let E be a separated* EQ-*algebra and F be a proper up set of E. Then F is a quasi r-prime up set of E if and only if $r(\{x_1, x_2, \ldots, x_n\}) \subseteq F$ implies that there exists $1 \leq i \leq n$ such that $x_i \in F$.*

Proof. Let E be a separated EQ-algebra and F be a quasi r-prime subset of E. Then we proceed by induction on n. If $n = 2$, then $r(\{x_1, x_2\}) \subseteq F$ implies $x_1 \in F$ or $x_2 \in F$. Now, suppose the statement holds for $n-1$ and $r(\{x_1, x_2, \ldots, x_n\}) \subseteq F$, for $x_1, x_2, \ldots, x_n \in E$. If $y \in r(\{x_1, x_2, \ldots, x_{n-1}\})$, then $x_i \to y = 1$, for any $1 \leq i \leq n-1$ and since E is separated, we get $x_i \leq y$, for any $1 \leq i \leq n-1$. Now, let $a \in r(\{y, x_n\})$. Then $y \to a = 1$ and $x_n \to a = 1$ and so, $y \leq a$ and $x_n \leq a$. By $x_i \leq y$, for any $1 \leq i \leq n-1$, we have $x_i \leq a$, for any $1 \leq i \leq n-1$. Hence $x_i \leq a$, for any $1 \leq i \leq n$ and so, $x_i \to a = 1$, for any $1 \leq i \leq n$. Thus, $a \in r(\{x_1, x_2, \ldots, x_n\})$ and so, $r(\{y, x_n\}) \subseteq r(\{x_1, x_2, \ldots, x_n\})$ and since $r(\{x_1, x_2, \ldots, x_n\}) \subseteq F$, we get $r(\{y, x_n\}) \subseteq F$ which implies $y \in F$ or $x_n \in F$. If $x_n \notin F$, then $y \in F$ and so, $r(\{x_1, x_2, \ldots, x_{n-1}\}) \subseteq F$. Now, by the induction hypothesis, we conclude that $x_i \in F$, for some $1 \leq i \leq n-1$. The converse is obvious. □

The following example shows that the condition separated in Theorem 3.9 is necessary.

Example 3.10 ([17])**.** Let $E = \{0, a, b, c, d, 1\}$, where $0 \leq a \leq b \leq d \leq 1$, $a \leq c \leq d$. The multiplication and fuzzy equality are defined as follows:

Table 4.

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	a	a	a	b
c	0	0	a	0	a	c
d	0	0	a	a	a	d
1	0	a	b	c	d	1

Table 5.

$\sim$	0	a	b	c	d	1
0	1	0	0	0	0	0
a	0	1	d	d	d	d
b	0	d	1	d	d	d
c	0	d	d	1	d	d
d	0	d	d	d	1	1
1	0	d	d	d	1	1

Thus $(E, \wedge, \odot, \sim, 1)$ is an EQ-algebra, but not separated. Now, let $F = \{1, d\}$. Then F is an up set and $r(\{b, c\}) = \{1, d\} \subseteq F$, while $b \notin F$ and $c \notin F$. Therefore F is not a quasi r-prime up set of E.

Proposition 3.11. *For any $x \in E$, $\emptyset \neq Z_x$ is an up set of E and $1 \in Z_x$.*

Proof. Let $a \leq b$ and $a \in Z_x$. Then $r(\{a, x\}) = \{1\}$ and since by Proposition 3.7(ii), $r(\{b, x\}) \subseteq r(\{a, x\})$ and by Proposition 3.4(i), $1 \in r(\{b, x\})$, we get $r(\{b, x\}) = \{1\}$ and so, $b \in Z_x$. Therefore Z_x is an up set of E. Moreover, since Z_x is a non-empty subset of E, there exists $d \in E$ such that $d \in Z_x$ and since $d \leq 1$ and Z_x is an up set of E, we conclude that $1 \in Z_x$. □

Proposition 3.12. *Let E be a residuated* EQ-*algebra, $a, b, x \in E$ and $r(\{a, x\}) = r(\{b, x\}) = \{1\}$. Then $r(\{a \odot b, x\}) = \{1\}$.*

Proof. Let E be a residuated EQ-algebra, $r(\{a, x\}) = r(\{b, x\}) = \{1\}$ and $t \in r(\{a \odot b, x\})$, for $a, b, x \in E$. Then $(a \odot b) \to t = 1$ and $x \to t = 1$ and so, by Lemma 2.3(x), $a \to (b \to t) = (a \odot b) \to t = 1$ and $x \to (b \to t) = b \to (x \to t) = b \to 1 = 1$. Hence $(b \to t) \in r(\{a, x\}) = \{1\}$. Thus $b \to t = 1$ and since $x \to t = 1$, we get $t \in r(\{b, x\}) = \{1\}$. Therefore $t = 1$ and so, $r(\{a \odot b, x\}) = \{1\}$. □

Theorem 3.13. *Let E be a residuated* EQ-*algebra and $x \in E$. Then $\emptyset \neq Z_x$ is a filter of E.*

Proof. By Proposition 3.11, $1 \in Z_x$, for any $x \in E$. If $a, a \to b \in Z_x$, then $r(\{a, x\}) = r(\{a \to b, x\}) = \{1\}$ and since E is a residuated EQ-algebra, by Proposition 3.18, we get $r(\{a \odot (a \to b), x\}) = \{1\}$ and since by Lemma 2.3(ix), $a \odot (a \to b) \leq b$, by Proposition 3.18(ii), we conclude that $r(\{b, x\}) \subseteq r(\{a \odot (a \to b), x\})$ and so, $r(\{b, x\}) = \{1\}$. Thus $b \in Z_x$ and so, Z_x is a prefilter of E and since E is a residuated EQ-algebra, we conclude that Z_x is a filter of E. □

Theorem 3.14. *Let E be a separated* EQ-*algebra, $x \in E$ and Z_x be maximal in $\{Z_a \mid a \in E, Z_a \neq \emptyset\}$. Then Z_x is a quasi r-prime up set of E.*

Proof. By Proposition 3.11, Z_x is an up set. We prove that Z_x is a quasi r-prime up set. Let $a, b \in E$ such that $r(\{a, b\}) \subseteq Z_x$ and $a \notin Z_x$. By Proposition 3.6, $r(\{x, 1\}) = \{1\}$ and so, $1 \in Z_x$ and since Z_x is maximal, we find that Z_x is proper. Moreover, $r(\{a, b, x\}) = \{1\}$, since if $r(\{a, b, x\}) \neq \{1\}$, by Proposition 3.4(vi), we have $r(\{a,b,x\}) = r(\{a,b\}) \cap r(\{x\}) \subseteq Z_x \cap r(\{x\})$. Now, if there exists $1 \neq t \in r(\{a,b,x\})$, then $t \in Z_x \cap r(\{x\})$ and so, $r(\{x,t\}) = \{1\}$ and $x \to t = 1$, which implies $t \in r(\{x,t\})$. Hence $t = 1$, which is a contradiction. Thus $r(\{a,b,x\}) = \{1\}$. By $a \notin Z_x$, we get $r(\{a,x\}) \neq \{1\}$ and so, there exists $1 \neq w \in r(\{a,x\})$. Hence $a \to w = 1$ and $x \to w = 1$ and so, by Proposition 3.8, $Z_x \subseteq Z_w$. Now, if $Z_w = E$, then by $a \in E$, we have $a \in Z_w$ and so, $r(\{a,w\}) = \{1\}$ and since $a \to w = 1$, we conclude that $w \in r(\{a,w\})$ and so, $w = 1$, which is impossible. Hence, since Z_x is maximal, we get $Z_x = Z_w$. In addition, since $a \to w = 1$ and $x \to w = 1$, by Proposition 3.8, $r(\{w\}) \subseteq r(\{a\})$ and $r(\{w\}) \subseteq r(\{x\})$ and so, we have

$$r(\{b,w\}) = r(\{b\}) \cap r(\{w\}) \subseteq r(\{b\}) \cap r(\{a\}) \cap r(\{x\}) = r(\{a,b,x\}) = \{1\}.$$

Hence $r(\{b,w\}) = \{1\}$ and so, $b \in Z_w = Z_x$. Therefore Z_x is a quasi r-prime up set of E. □

Definition 3.15. Let E be an EQ-algebra. Then $\Omega(E)$ is called an implication graph of E if the vertices are just the elements of E and for distinct $x, y \in E$, there is an edge connecting x and y if and only if $r(\{x,y\}) = \{1\}$.

Example 3.16 ([10])**.** Let $E = \{0,a,b,1\}$ be a chain $0 < a < b < 1$ with the following Cayley tables:

Table 6.

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	0	a
b	0	0	0	b
1	0	a	b	1

Table 7.

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	b	b
b	0	b	1	b
1	0	b	b	1

Then $(E, \wedge, \odot, \sim, 1)$ is a separated EQ-algebra and the graph $\Omega(E)$ is given by the following figure:

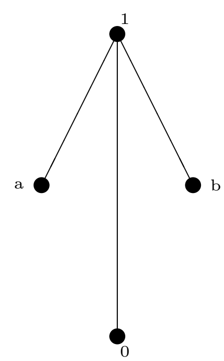

FIGURE 1. Associated implication graph $\Omega(E)$.

Example 3.17 ([11])**.** Let $E = \{0,a,b,c,d,1\}$, where $0 \leq a,b \leq c \leq 1$, $0 \leq b \leq d \leq 1$, but a,b and, respectively c,d are incomparable. The multiplication and a fuzzy equality are defined as follows:

Table 8.

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	0	0	b	b
c	0	a	0	a	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

Table 9.

$\sim$	0	a	b	c	d	1
0	1	d	c	b	a	0
a	d	1	b	c	0	a
b	c	b	1	d	c	b
c	b	c	d	1	b	c
d	a	0	c	b	1	d
1	0	a	b	c	d	1

Table 10.

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	c	c	1	1	1	1
c	b	c	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

$(E, \wedge, \odot, \sim, 1)$ is a good IEQ-algebra. Routine calculation shows that $r(\{1,a\}) = r(\{1,b\}) = r(\{1,c\}) = r(\{1,d\}) = \{1\}$ and $r(\{a,d\}) = r(\{c,d\}) = \{1\}$. Therefore

$$\mathsf{E}(\Omega(E)) = \{e(1,a), e(1,b), e(1,c), e(1,d), e(a,d), e(c,d)\}.$$

$\Omega(E)$ is given by the following figure:

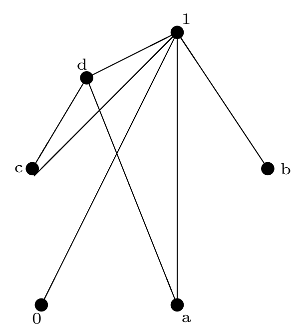

FIGURE 2. Associated implication graph $\Omega(E)$.

Theorem 3.18. *Let E be a separated* EQ-*algebra. Then $\Omega(E)$ is a connected graph with diameter at most two.*

Proof. Let E be a separated EQ-algebra. Then for any $x \in E$,

$$\begin{aligned} r(\{1,x\}) &= \{y \in E \mid x \to y = 1 \text{ and } 1 \to y = 1\} \\ &= \{y \in E \mid x \to y = 1 \text{ and } 1 \sim y = 1\} \\ &= \{y \in E \mid x \to y = 1 \text{ and } y = 1\} = \{1\}. \end{aligned}$$

Hence 1 is connected with all vertices $\Omega(E)$ and so, it is a connected graph. Now, let $x, y \in E$ be two distinct vertices. If $r(\{x,y\}) = \{1\}$, then $d(x,y) = 1$ and if $r(\{x,y\}) \neq \{1\}$, then the path $e(x,1,y)$ exists and so, $d(x,y) = 2$. Hence

$$\operatorname{diam}(\Omega(E)) = \sup\{d(x,y) \mid x, y \in V(\Omega(E))\} \leq 2. \qquad \square$$

Theorem 3.19. *Let E be a finite chain and separated* EQ-*algebra. Then $\Omega(E)$ is a star graph.*

Proof. Suppose E is a finite chain and separated EQ-algebra. By Proposition 3.18, $\Omega(E)$ is a connected graph. Now, we proceed by induction on n, where $|E| = n$, and we prove that $\Omega(E)$ is a star graph. If $n = 3$, then we have $x_1 \leq x_2 \leq 1$ and by Proposition 3.18, we get $r(\{x_1, 1\}) = \{1\}$ and $r(\{x_2, 1\}) = \{1\}$ and since $x_1 \leq x_2$, we get $x_1 \to x_2 = 1$ and so, $r(\{x_1, x_2\}) = \{x_2, 1\}$. Hence, 1 is just connected with all vertices of E, and so, $\Omega(E)$ is a star graph. Now, let $E = \{x_1, \ldots, x_{n-2}, x_{n-1}, 1\}$ such that $x_1 \leq \cdots \leq x_{n-2} \leq x_{n-1} \leq 1$. By the induction hypothesis, we find that $\Omega(D)$ is a star graph such that $D \subseteq E$ and $|D| = n-1$. Consider $D = \{x_1, \ldots, x_{n-2}, 1\}$ such that $x_i \leq x_{n-1}$, for any $1 \leq i \leq n-2$ and so, $x_i \to x_{n-1} = 1$. Hence $r(\{x_i, x_{n-1}\}) = \{x_{n-1}, 1\}$, for any $1 \leq i \leq n-2$ and so x_{n-1} is not connected to any element of $D \backslash \{1\}$ and so, $\Omega(E)$ is a star graph. $\square$

Corollary 3.20. *If E is a finite chain and separated* EQ-*algebra, then $\Omega(E)$ is a tree.*

Proof. It follows from Theorem 3.19. □

Proposition 3.21. *For any $x, y \in E$, $Z_x \neq Z_y$ if and only if $r(\{x,y\}) = \{1\}$, where Z_x, Z_y are the quasi r-prime up sets of E.*

Proof. Let $r(\{x,y\}) = \{1\}$, for $x, y \in E$. Then $y \in Z_x$ and $x \in Z_y$. Now, if $Z_x = Z_y$, then $x \in Z_x$ and $y \in Z_y$ and so, $r(\{x,x\}) = \{1\}$ and $r(\{y,y\}) = \{1\}$. Now, since by $(E3)$, $x \to x = 1$ and $y \to y = 1$, we get $x \in r(\{x,x\})$ and $y \in r(\{y,y\})$, which is impossible. Therefore, $Z_x \neq Z_y$. Conversely, let $Z_x \neq Z_y$ and $r(\{x,y\}) \neq \{1\}$. Then $y \notin Z_x$ and $x \notin Z_y$. Now, if $a \in Z_x$, then $r(\{a,x\}) = \{1\} \subseteq Z_y$ and since Z_y is a quasi r-prime up set of E, we conclude that $a \in Z_y$ or $x \in Z_y$ and since $x \notin Z_y$, we get $a \in Z_y$. Hence $Z_x \subseteq Z_y$ and in a similar way, we conclude that $Z_y \subseteq Z_x$ and so, $Z_x = Z_y$, which is impossible. Therefore $r(\{x,y\}) = \{1\}$. □

4. Production Graph of an EQ-algbra

In this section, firstly, we define the notion of zero divisors by a product operation and provide related results. In what follows, by means of zero divisors of an EQ-algebra with a bottom element 0, the associated production graph $\Gamma(E)$ is introduced.

Note. From now on, in this section, let E be an EQ-algebra with a bottom element 0, unless otherwise is stated.

Definition 4.1. Let X be a non-empty subset of E. Then the set of all zero divisors of X is denoted by Z_X and defined as follows:

$$Z_X = \{a \in E \mid a \odot x = 0, \text{ for any } x \in X\}$$

Example 4.2. Let E be the EQ-algebra given in Example 3.3 and $X = \{a, b\}$. Then $Z_X = \{0, a, b\}$.

Example 4.3. Let E be the EQ-algebra given in Example 3.10, $X = \{a, c\}$ and $Y = \{b, d\}$. Then $Z_X = \{0, a, c\}$ and $Z_Y = \{0, a\}$.

Example 4.4 ([10])**.** Let $E = \{0, a, b, 1\}$ be a chain $0 < a < b < 1$ with the following Cayley tables:

Table 11.

$\odot$	0	a	b	1
0	0	0	0	0
a	0	a	a	a
b	0	a	b	b
1	0	a	b	1

Table. 12

$\sim$	0	a	b	1
0	1	0	0	0
a	0	1	a	a
b	0	a	1	1
1	0	a	1	1

Table. 13

$\to$	0	a	b	1
0	1	1	1	1
a	0	1	1	1
b	0	a	1	1
1	0	a	1	1

Routine calculation shows that $(E, \wedge, \odot, \sim, 1)$ is an EQ-algebra. Now, let $X = \{a\}$, $Y = \{b\}$ and $W = \{a, b\}$. Then $Z_X = Z_Y = Z_W = \{0\}$.

Lemma 4.5. *If E is a residuated* EQ-*algebra, then $a \odot b = 0$ if and only if $a \leq \neg b$, for any $a, b \in E$.*

Proof. Let E be a residuated EQ-algebra and $a, b \in E$. If $a \odot b = 0$, then by Lemma 2.3(x) and (vi), $a \to \neg b = a \to (b \to 0) = (a \odot b) \to 0 = 0 \to 0 = 1$. Hence by Lemma 2.3(xi), $a \leq \neg b$. Conversely, if $a \leq \neg b$, then by $(E2)$ and by Lemma 2.3(vii), $a \odot b \leq b \odot \neg b = b \odot (b \to 0) \leq \tilde{0} = 0$ and so, $a \odot b = 0$. □

Proposition 4.6. *Let X and Y be two non-empty subsets of E. Then the following statements hold:*

(i) $0 \in Z_X$.
(ii) *If $X \subseteq Y$, then $Z_X \subseteq Z_Y$.*
(iii) *If $\overline{Z_X} - \{1\} \neq \emptyset$, then $Z_{\overline{Z_X} - \{1\}} \subseteq Z_X$, where E is residuated.*
(iv) *If $1 \in X$, then $Z_X = \{0\}$.*
(v) *If F is a prefilter of E, then $Z_F = \{0\}$.*
(vi) *$1 \in Z_X$ if and only if $X = \{0\}$ if and only if $Z_X = E$.*
(vii) *If $0 \in X$, then $Z_X = Z_{X - \{0\}}$.*
(viii) *$Z_X \cup Z_Y \subseteq Z_{X \wedge Y}$, where $X \wedge Y = \{x \wedge y \mid x \in X, y \in Y\}$.*
(ix) *Z_X is a down set of E.*

Proof. (i) Since for any $x \in X$, $x \odot 0 = 0$, we get $0 \in Z_X$.

(ii) If $X \subseteq Y$ and $a \in Z_Y$, then for any $y \in Y$, we have $a \odot y = 0$ and so, for $x \in X$, we have $a \odot x = 0$. Hence $a \in Z_X$ and so, $Z_X \subseteq Z_Y$,

(iii) Let $\overline{Z_X} - \{1\} \neq \emptyset$ and $a \in Z_{\overline{Z_X} - \{1\}}$. Then for any $x \in \overline{Z_X} - \{1\}$, $a \odot x = 0$ and since $x \in \overline{Z_X} - \{1\}$, we get $x = \neg t$ such that $t \in Z_X - \{0\}$. Hence $a \odot \neg t = 0$, for any $t \in Z_X - \{0\}$ and since E is a residuated EQ-algebra, by Lemma 4.5, we conclude that $\neg t \leq \neg a$, for any $t \in Z_X - \{0\}$ and since $t \in Z_X - \{0\}$, we get $t \odot b = 0$, for any $b \in X$. Hence by Lemma 4.5, we have $b \leq \neg t$, and since $\neg t \leq \neg a$, we get $b \leq \neg a$ and so, $b \odot a \leq a \odot \neg a = 0$. Thus $a \odot b = 0$, for any $b \in X$ and so, $a \in Z_X$. Therefore $Z_{\overline{Z_X} - \{1\}} \subseteq Z_X$.

(iv) If $1 \in X$ and $a \in Z_X$, then $a \odot 1 = 0$ and so, $a = 0$, Thus $Z_X = \{0\}$.

(v) If F is a prefilter of E, then $1 \in F$ and so, by (iv), $Z_F = \{0\}$.

(vi) Let $1 \in Z_X$. Then for any $x \in X$, $1 \odot x = 0$ and so, $x = 0$. Hence $X = \{0\}$. If $X = \{0\}$, then by $1 \odot 0 = 0$, we get $1 \in Z_X$. Moreover, if $X = \{0\}$, then for any $a \in E$, $a \odot 0 = 0$ and so, $a \in Z_X$. Hence $Z_X = E$. Finally, if $Z_X = E$, then $1 \in Z_X$ and so, $X = \{0\}$.

(vii) Since $X - \{0\} \subseteq X$, by (ii), we get $Z_{X - \{0\}} \subseteq Z_X$. If $0 \in X$ and $a \in Z_X$, then for any $0 \neq x \in X$, $a \odot x = 0$ and so, $a \in Z_{X - \{0\}}$. Hence $Z_{X - \{0\}} = Z_X$.

(viii) Let $a \in Z_X \cup Z_Y$. Then $a \in Z_X$ or $a \in Z_Y$ and so, $a \odot x = 0$ for any $x \in X$ or $a \odot y = 0$ for any $y \in Y$. By Lemma 2.3(i), $x \wedge y \leq x, y$ and by $(E2)$, $a \odot (x \wedge y) \leq a \odot x, a \odot y$. Hence $a \odot (x \wedge y) = 0$ and so, $a \in Z_{X \wedge Y}$. Therefore $Z_X \cup Z_Y \subseteq Z_{X \wedge Y}$.

(ix) Let $a \leq b$ and $b \in Z_X$. Then for any $x \in X$, $b \odot x = 0$ and since by $(E2)$, $a \odot x \leq b \odot x = 0$, we get $a \odot x = 0$, for any $x \in X$. Therefore $a \in Z_X$ and so, Z_X is a down set of E. □

Definition 4.7. For any $x \in E$, the set D_x is called the set of all zero divisors of x and is defined as follows:

$$D_x = \{y \in E \mid Z_{\{x,y\}} = \{0\}\}.$$

By Proposition 4.12 (iv), $D_1 = E$ and $1 \in D_x$, for any $x \in E$.

Example 4.8. Let E be the EQ-algebra given in Example 4.4. Then $D_a = D_b = \{0, a, b, 1\}$.

Example 4.9 ([10])**.** Let $E = \{0, a, b, 1\}$ be a chain $0 < a < b < 1$ with the following Cayley tables:

Table. 14

$\odot$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

Table. 15

$\sim$	0	a	b	1
0	1	a	a	a
a	a	1	b	b
b	a	b	1	1
1	a	b	1	1

Table. 16

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	a	b	1	1
1	a	b	1	1

Then $(E, \wedge, \odot, \sim, 1)$ is an EQ-algebra, but not residuated. Routine calculation shows that $D_a = \{b, 1\}$ and $D_b = \{a, b, 1\}$.

Theorem 4.10. *Let E be a good* EQ-*algbera and $x \in E$. Then D_x is a prefilter of E. Moreover, if E is a residuated* EQ-*algebra, then D_x is a filter of E.*

Proof. Since $1 \in D_x$, we get $D_x \neq \emptyset$. Let $a, a \to b \in D_x$. Then $Z_{\{a,x\}} = \{0\}$ and $Z_{\{a\to b,x\}} = \{0\}$. Now, if $t \in Z_{\{b,x\}} = \{0\}$, then $b \odot t = 0$ and $x \odot t = 0$ and since E is a good EQ-algebra, by Lemma 2.3(ix), we conclude that $a \odot (a \to b) \leq b$ and so, by $(E2)$, $a \odot (a \to b) \odot t \leq b \odot t = 0$. Hence $a \odot (a \to b) \odot t = 0$ and since $x \odot t = 0$, we get $x \odot (a \to b) \odot t = 0$. Thus $(a \to b) \odot t \in Z_{\{a,x\}}$ and so, $(a \to b) \odot t = 0$. Hence $t \in Z_{\{a\to b,x\}}$ and so, $t = 0$. Therefore $Z_{\{b,x\}} = \{0\}$ and so, $b \in D_x$. Hence D_x is a prefilter of E. Finally, if E is a residuated EQ-algebra, then by Definition 2.4, D_x is a filter of E. □

By the following example, we show that the condition good in Theorem 4.10 is necessary.

Example 4.11 ([10])**.** Let $E = \{0, a, b, c, 1\}$ be a chain $0 < a < b < c < 1$ with the following Cayley tables:

Table. 17

$\odot$	0	a	b	c	1
0	0	0	0	0	0
a	0	0	0	0	a
b	0	0	0	0	b
c	0	0	0	0	c
1	0	a	b	c	1

Table. 18

$\sim$	0	a	b	c	1
0	1	0	0	0	0
a	0	1	b	b	b
b	0	b	1	c	c
c	0	b	c	1	1
1	0	b	c	1	1

Table. 19

$\to$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	1	1	1
b	0	b	1	1	1
c	0	b	c	1	1
1	0	b	c	1	1

Then $(E, \wedge, \odot, \sim, 1)$ is an EQ-algebra. Routine calculation shows that $D_a = \{1\}$, $D_b = \{1\}$ and $D_c = \{1\}$. But $\{1\}$ is not a prefilter of E, because $1 \to c = 1 \in \{1\}$, $1 \in \{1\}$ and $c \notin \{1\}$.

Proposition 4.12. *Let E be an* EQ-*algebra. Then for all $a, b, c, d, x \in E$, the following statements hold:*

(i) *If $a, b \in D_x$, then $a \odot b \in D_x$.*
(ii) *If $a \in D_x$ and $a \leq b$, then $b \in D_x$, (up set).*
(iii) *$a \sim b \in D_x$ if and only if $a \to b \in D_x$ and $b \to a \in D_x$.*
(iv) *If $a \sim b \in D_x$ and $b \sim c \in D_x$, then $a \sim c \in D_x$.*
(v) *If $a \to b \in D_x$ and $b \to c \in D_x$, then $a \to c \in D_x$.*

(vi) *If* $a \sim b \in D_x$ *and* $c \sim d \in D_x$*, then* $(a \wedge c) \sim (b \wedge d) \in D_x$.
(vii) *If* $a \sim b \in D_x$ *and* $c \sim d \in D_x$*, then* $(a \sim c) \sim (b \sim d) \in D_x$.
(viii) *If* $a \sim b \in D_x$ *and* $c \sim d \in D_x$*, then* $(a \to c) \sim (b \to d) \in D_x$.
(ix) *If* E *is a* IEQ*-algebra and* $a \sim b \in D_x$, $c \sim d \in D_x$*, then* $(a \vee c) \sim (b \vee d) \in D_x$.
(x) *If* E *is residuated lattice and* $a \sim b \in D_x$*, then* $(a \odot c) \sim (b \odot c) \in D_x$.
(xi) *If* $a \leq b$*, then* $D_a \subseteq D_b$.

Proof. (i) Let $a, b \in D_x$. Then $Z_{\{a,x\}} = \{0\}$ and $Z_{\{b,x\}} = \{0\}$ and if $t \in Z_{\{a \odot b,x\}}$, then $t \odot a \odot b = 0$ and $t \odot x = 0$. Hence $(t \odot a) \odot b = 0$ and since $t \odot x = 0$, we get $(t \odot a) \odot x = 0$. Thus $(t \odot a) \in Z_{\{b,x\}}$ and so, $t \odot a = 0$ and since $t \odot x = 0$, we conclude that $t \in Z_{\{a,x\}}$. Hence $t = 0$ and so, $Z_{\{a \odot b,x\}} = \{0\}$. Therefore $a \odot b \in D_x$.

(ii) Let $a \in D_x$, $a \leq b$ and $t \in Z_{\{b,x\}}$. Then $Z_{\{a,x\}} = \{0\}$ and $t \odot b = 0$, $t \odot x = 0$. Since $a \leq b$, by $(E2)$, we get $a \odot t \leq b \odot t = 0$ and so, $t \odot a = 0$. Hence $t \in Z_{\{a,x\}}$ and so, $t = 0$. Thus $Z_{\{b,x\}} = \{0\}$ and so, $b \in D_x$.

(iii) Let $a \sim b \in D_x$. Then by Lemma 2.3(i), $a \sim b \leq a \to b$ and so, by (ii), we conclude that $a \to b \in D_x$. In a similar way and by Lemma 2.3(i), we get $b \to a \in D_x$. Conversely, let $a \to b \in D_x$ and $b \to a \in D_x$. Then by (i), $(a \to b) \odot (b \to a) \in D_x$ and since by Lemma 2.3(ii), $(a \to b) \odot (b \to a) \leq a \sim b$, by (ii), we get $a \sim b \in D_x$.

(iv) Let $a \sim b \in D_x$ and $b \sim c \in D_x$. Then by (i), $(a \sim b) \odot (b \sim c) \in D_x$ and since by Lemma 2.3(iv), $(a \sim b) \odot (b \sim c) \leq a \sim c$, so, by (ii), we conclude that $a \sim c \in D_x$.

(v) Let $a \to b \in D_x$ and $b \to c \in D_x$. Then by (i), $(a \to b) \odot (b \to c) \in D_x$ and since by Lemma 2.3(iv), $(a \to b) \odot (b \to c) \leq a \to c$, so, by (ii), we conclude that $a \to c \in D_x$.

(vi) Let $a \sim b \in D_x$ and $c \sim d \in D_x$. Then by Lemma 2.3(iii), we have $a \sim b \leq (a \wedge c) \sim (b \wedge c)$ and $c \sim d \leq (c \wedge b) \sim (d \wedge b)$ and so, by (ii), we get $(a \wedge c) \sim (b \wedge c) \in D_x$ and $(c \wedge b) \sim (d \wedge b) \in D_x$. Hence by (iv), we conclude that $(a \wedge c) \sim (b \wedge d) \in D_x$.

(vii) Let $a \sim b \in D_x$ and $c \sim d \in D_x$. Then by Lemma 2.3(iii), we have $a \sim b \leq (a \sim c) \sim (b \sim c)$ and $c \sim d \leq (c \sim b) \sim (d \sim b)$ and so, by (ii), we get $(a \sim c) \sim (b \sim c) \in D_x$ and $(c \sim b) \sim (d \sim b) \in D_x$. Hence by (iv), we conclude that $(a \sim c) \sim (b \sim d) \in D_x$.

(viii) Let $a \sim b \in D_x$ and $c \sim d \in D_x$. Since by $(E3)$, $c \sim c = 1 \in D_x$ and $d \sim d = 1 \in D_x$, by (vi), we conclude that $(a \wedge c) \sim (b \wedge c) \in D_x$ and $(b \wedge c) \sim (b \wedge d) \in D_x$ and so, by (iv), we get $(a \wedge c) \sim (b \wedge d) \in D_x$ and since $a \sim b \in D_x$, by (vi), we conclude that $(a \sim (a \wedge c)) \sim (b \sim (b \wedge d)) \in D_x$. Therefore $(a \to c) \sim (b \to d) \in D_x$.

(ix) Let E be an IEQ-algebra and $a \sim b \in D_x$, $c \sim d \in D_x$. Then by Lemma 2.3(i), we have $a \sim b \leq a \to b$ and $c \sim d \leq c \to d$ and so, by (ii), we conclude that $a \to b \in D_x$ and $c \to d \in D_x$. Now, by Lemma 2.3xii, we have $a \to b \leq (a \vee c) \to (b \vee c)$ and $c \to d \leq (c \vee b) \to (d \vee b)$ and so, by (ii), we get $(a \vee c) \to (b \vee c) \in D_x$ and $(c \vee b) \to (d \vee b) \in D_x$. Hence by (v), we conclude that $(a \vee c) \to (b \vee d) \in D_x$. Moreover, since $a \sim b = b \sim a$ and $c \sim d = d \sim c$, we get $b \sim a \in D_x$, $d \sim c \in D_x$ and in a similar way, we conclude that $(b \vee d) \to (a \vee c) \in D_x$. Therefore by (iii), we have $(a \vee c) \sim (b \vee d) \in D_x$.

(x) Let E be a residuated lattice and $a \sim b \in D_x$. Then by Theorem 4.10, D_x is a filter of E and since by Lemma 2.3(i), $a \sim b \leq a \to b$, we get $a \to b \in D_x$ and so, $(a \odot c) \to (b \odot c) \in D_x$. In a similar way, we conclude that $(b \odot c) \to (a \odot c) \in D_x$. Therefore by (iii), $(a \odot c) \sim (b \odot c) \in D_x$.

(xi) Let $a \leq b$ and $x \in D_a$. Then $Z_{\{a,x\}} = \{0\}$ and if $t \in Z_{\{b,x\}}$, then $t \odot b = 0$ and $t \odot x = 0$. By $(E2)$, $t \odot a \leq t \odot b = 0$, hence $t \odot a = 0$ and so, $t \in Z_{\{a,x\}}$. Thus $t = 0$ and so, $Z_{\{b,x\}} = \{0\}$. Therefore $x \in D_b$ and so, $D_a \subseteq D_b$. □

Definition 4.13. The set of dense elements of EQ-algebra E is denoted by $D_s(E)$ and defined as follows:

$$D_s(E) = \{x \in E \mid \neg x = 0\}.$$

Example 4.14. Let E be the EQ-algebra given in Example 4.11. Then one can see that $D_s(E) = \{a, b, c, 1\}$.

Proposition 4.15. *Let* E *be a spanned* EQ*-algebra. Then* $D_0 \subseteq D_s(E)$.

Proof. Let E be a spanned EQ-algebra and $a \in D_0$. Then $Z_{\{a,0\}} = \{0\}$ and since by Lemma 2.3(viii), $a \odot \neg a \leq \tilde{0} = 0$, we get $a \odot \neg a = 0$ and so, $\neg a \in Z_{\{a,0\}}$. Hence $\neg a = 0$ and so, $a \in D_s(E)$. Therefore $D_0 \subseteq D_s(E)$. □

Theorem 4.16. *Let E be a residuated* EQ-*algebra. Then $D_0 = D_s(E)$.*

Proof. Let E be a residuated EQ-algebra. Then E is a good EQ-algebra and so it is a spanned EQ-algebra. Hence by Proposition 4.15, $D_0 \subseteq D_s(E)$. Now, let $a \in D_s(E)$. Then $\neg a = 0$. If $t \in Z_{\{a,0\}}$, then $a \odot t = 0$ and so, by Lemma 4.5, $t \leq \neg a$ and since $\neg a = 0$, we get $t = 0$ and so, $Z_{\{a,0\}} = \{0\}$. Hence $a \in D_0$ and so, $D_s(E) \subseteq D_0$. Therefore $D_0 = D_s(E)$. □

By the following example we show that the condition spanned in Proposition 4.15 and the condition residuated in Theorem 4.16 are necessary.

Example 4.17. Let E be the EQ-algebra given in Example 4.9. Then E is not a spanned and residuated EQ-algebra and $D_0 \nsubseteq D_s(E)$. Because $D_0 = \{a, 1\}$ and $D_s(E) = \emptyset$ so, $D_0 \nsubseteq D_s(E)$ and $D_0 \neq D_s(E)$.

Definition 4.18. Let E be an EQ-algebra with the bottom element 0. Then $\Gamma(E)$ is called an associated production graph, if the vertices are just the elements of E, and for distinct $a, b \in E$, there is an edge connecting a and b if and only if $Z_{\{a,b\}} = \{0\}$. The edge that connects two vertices a and b is denoted by $e(a, b)$.

Example 4.19. Let E be the EQ-algebra given in Example 4.9. Then $Z_{\{0,a\}} = \{0, a\}$ and $Z_{\{0,b\}} = Z_{\{0,1\}} = Z_{\{a,b\}} = Z_{\{a,1\}} = Z_{\{b,1\}} = \{0\}$. Therefore $\mathsf{E}(\Gamma(E)) = \{e(0, b), e(0, 1), e(a, b), e(a, 1), e(b, 1)\}$ and $\Gamma(E)$ is given by the following figure:

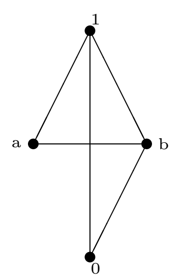

FIGURE 3. Associated production graph $\Gamma(E)$.

Example 4.20 ([17]). Let $E = \{0, a, b, c, d, 1\}$, where $0 \leq a \leq b \leq c \leq d \leq 1$. The multiplication and fuzzy equality are defined as follows:

Table. 20

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	0	0	0	0	a
b	0	0	0	a	a	c
c	0	0	a	a	a	d
d	0	0	a	a	a	d
1	0	a	b	c	d	1

Table. 21

$\sim$	0	a	b	c	d	1
0	1	c	b	a	0	0
a	c	1	b	a	a	a
b	b	b	1	b	b	b
c	a	a	b	1	c	c
d	0	a	b	c	1	d
1	0	a	b	c	d	1

Then $(E, \wedge, \odot, \sim, 1)$ is a good EQ-algebra. Routine calculation shows that
$Z_{\{1,a\}} = Z_{\{1,b\}} = Z_{\{1,c\}} = Z_{\{1,d\}} = \{0\}$. Therefore $\mathsf{E}(\Gamma(E)) = \{e(1, a), e(1, b), e(1, c), e(1, d)\}$ and $\Gamma(E)$ is given the following figure:

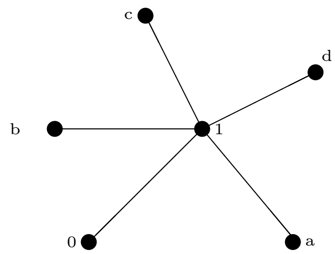

FIGURE 4. Associated production graph $\Gamma(E)$.

Example 4.21. Let E be an EQ-algebra given in Example 3.17. Then routine calculation shows that $Z_{\{1,a\}} = Z_{\{1,b\}} = Z_{\{1,c\}} = Z_{\{1,d\}} = \{0\}$ and $Z_{\{a,d\}} = Z_{\{c,d\}} = \{0\}$. Therefore

$$\mathsf{E}(\Gamma(E)) = \{e(1,a), e(1,b), e(1,c), e(1,d), e(a,d), e(c,d)\}$$

and $\Gamma(E)$ is given by the following figure:

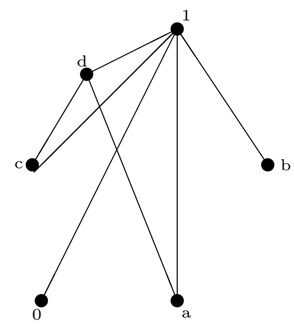

FIGURE 5. Associated production graph $\Gamma(E)$.

Definition 4.22 ([15])**.** An EQ-algebra E with the bottom element 0 is called an *integral* EQ-*algebra.* If $x \odot y = 0$, then $x = 0$ or $y = 0$, for all $x, y \in L$.

Theorem 4.23. *Let E be an integral* EQ*-algebra. Then the associated production graph* $\Gamma(E)$ *is complete.*

Proof. Let E be an integral EQ-algebra and x, y be distinct non-zero elements of E. If $t \in Z_{\{x,y\}}$, then $x \odot t = 0$ and $y \odot t = 0$ and since E is an integral EQ-algebra, we get $t = 0$ and so, $Z_{\{x,y\}} = \{0\}$. If $x = 0$ and $y \neq 0$ and $t \in Z_{\{0,y\}}$, then $y \odot t = 0$ and so, $t = 0$. Hence $Z_{\{0,y\}} = \{0\}$ and so, every two distinct elements of E are connected, Therefore, the associated production graph $\Gamma(E)$ is complete. □

Example 4.24. Let $E = \{0, a, b, c, d, 1\}$, where $0 \leq a, b \leq c \leq d \leq 1$, a, b are incomparable. The multiplication and fuzzy equality are defined as follows:

Table. 22

$\odot$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	a	a	a	a
b	0	a	a	a	a	b
c	0	a	a	a	a	c
d	0	a	a	a	d	d
1	0	a	b	c	d	1

Table. 23

$\sim$	0	a	b	c	d	1
0	1	1	d	d	d	d
a	1	1	d	d	d	d
b	d	d	1	d	d	d
c	d	d	d	1	d	d
d	d	d	d	d	1	d
1	d	d	d	d	d	1

Then $(E, \wedge, \odot, \sim, 1)$ is an integral EQ-algebra. Moreover, $Z_{\{x,y\}} = \{0\}$ for any two distinct elements of E. Therefore $\Gamma(E)$ is a complete graph and it is given by the following figure:

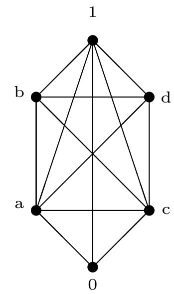

FIGURE 6. Associated production graph $\Gamma(E)$.

Theorem 4.25. *$\Gamma(E)$ is a connected graph with diameter at most two.*

Proof. Since $Z_{\{1,x\}} = \{0\}$, for any $x \in E$, we find that 1 is connected with all elements of $E\backslash\{1\}$. Hence both vertices are connected by a path and so, $\Gamma(E)$ is a connected graph. Now, let x, y be two vertices in $\Gamma(E)$. If $Z_{\{x,y\}} = \{0\}$, then $d(x, y) = 1$ and if $Z_{\{x,y\}} \neq \{0\}$, then the path $e(x, 1, y)$ exists. Hence $d(x, y) = 2$ and so,

$$\operatorname{diam}(\Gamma(E)) = \sup\{d(x, y) \mid x, y \in V(\Gamma(E))\} \leq 2. \qquad \square$$

Theorem 4.26. *Let E be an EQ-algebra. Then:*
(i) *If E is a residuated EQ-algebra and $D_s(E) = E\backslash\{0\}$, then $\Gamma(E)$ is a complete graph.*
(ii) *If E is a spanned EQ-algebra and $\Gamma(E)$ is a complete graph, then $D_s(E) = E\backslash\{0\}$.*

Proof. (i) Let x, y be non-zero elements of E. Then $x, y \in D_s(E)$ and so, $\neg x = \neg y = 0$. Now, if $t \in Z_{\{x,y\}}$, then $x \odot t = 0$ and $y \odot t = 0$ and so, by Lemma 4.5, $t \leq \neg x, \neg y$. Hence $t = 0$ and so, $Z_{\{x,y\}} = \{0\}$. If $x = 0$, $y \neq 0$ and $t \in Z_{\{0,y\}}$, then $y \odot t = 0$ and so, by Lemma 4.5, $t = 0$ and so, $Z_{\{0,y\}} = \{0\}$. Therefore every two distinct elements of E are connected and so, $\Gamma(E)$ is a complete graph.

(ii) Let E be a spanned EQ-algebra, $\Gamma(E)$ be a complete graph and $x \in E\backslash\{0\}$. Then $Z_{\{0,x\}} = \{0\}$ and since by Lemma 2.3(viii), $x \odot \neg x \leq \tilde{0} = 0$, we conclude that $x \odot \neg x = 0$ and so, $\neg x \in Z_{\{0,x\}} = \{0\}$. Hence $\neg x = 0$ and so, $x \in D_s(E)$. Thus $E\backslash\{0\} \subseteq D_s(E)$ and since $\neg 0 = 1$, we get $D_s(E) \subseteq E\backslash\{0\}$. Therefore $D_s(E) = E\backslash\{0\}$. $\square$

Theorem 4.27. *Let E be a residuated EQ-algebra. Then:*
(i) *$D_s(E) = E\backslash\{0\}$ if and only if $\Gamma(E)$ is a complete graph.*
(ii) *If $D_s(E) = E\backslash\{0\}$ and $|E| > 2$, then $\Gamma(E)$ is not a tree.*
(iii) *If $\Gamma(E)$ is a tree, then $D_s(E) = \{1\}$.*

Proof. (i) It follows from Theorem 4.26.

(ii) Let $D_s(E) = E\backslash\{0\}$. Then by (i), $\Gamma(E)$ is a complete graph and since $|E| > 2$, we conclude that there exists a path which is a circle and so, $\Gamma(E)$ is not a tree.

(iii) Since $\neg 1 = 0$, we get $1 \in D_s(E)$. Now, if $1 \neq x \in D_s(E)$, then $\neg x = 0$ and since $Z_{\{1,x\}} = \{0\}$, we conclude that 1 and x are connected. Moreover, since E is a residuated EQ-algebra, we have

$$Z_{\{0,x\}} = \{a \in E \mid a \odot x = 0\} = \{a \in E \mid a \leq \neg x\} = \{a \in E \mid a \leq 0\} = \{0\}.$$

Hence 0 and x are connected and so, we have a path $e(0, x, 1, 0)$ which is a circle, but this is impossible. Therefore $D_s(E) = \{1\}$. $\square$

Theorem 4.28. *Let E be a spanned EQ-algebra that satisfies in the following conditions:*
(i) $|D_s(E)| = 1$.
(ii) *There is $a \in E\backslash\{0\}$ such that $a \leq x$, for any $x \in E\backslash\{0\}$.*
Then $\Gamma(E)$ is a star graph.

Proof. Since $Z_{\{1,x\}} = \{0\}$, we find that 1 is connected with every element of $E\backslash\{1\}$. Now, let $x, y \in E\backslash\{1\}$ such that $x \neq y$. Since $|D_s(E)| = 1$, we conclude that $\neg x \neq 0$ and $\neg y \neq 0$ and so, by (ii), there exists $a \in E$ such that $a \leq \neg x$ and $a \leq \neg y$. Now, by $(E2)$ and Lemma 2.3(viii), we have $a \odot x \leq x \odot \neg x \leq \tilde{0} = 0$ and $a \odot y \leq y \odot \neg y \leq \tilde{0} = 0$. Hence $a \odot x = 0$ and $a \odot y = 0$ and so, $a \in Z_{\{x,y\}}$. Thus $Z_{\{x,y\}} \neq \{0\}$ and so x, y are not connected. Therefore $\Gamma(E)$ is a star graph. □

By the following examples, we show that both conditions listed in Theorem 4.28 are necessary.

Example 4.29. Let E be the EQ-algebra given in Example 4.9. Then $|D_s(E)| = 0$ and $\Gamma(E)$ is not a star graph. Moreover, the graph $\Gamma(E)$ is given by the following figure:

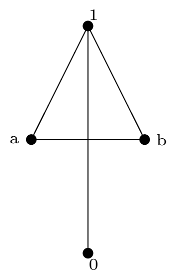

FIGURE 7. Associated production graph $\Gamma(E)$.

Example 4.30. Let E be the EQ-algebra given in Example 4.24. Then E is not a spanned EQ-algebra and $\Gamma(E)$ is not a star graph.

Example 4.31. Let E be the EQ-algebra given in Example 3.17. Then E is a spanned EQ-algebra and by routine calculation, we can see that $|D_s(E)| = 1$ and condition (ii) of Theorem 4.28 does not hold. Thus $\Gamma(E)$ is not a star graph, because $Z_{\{c,d\}} = \{0\}$ and so, $e(c, d)$ exists. Moreover, the graph $\Gamma(E)$ is given by the following figure:

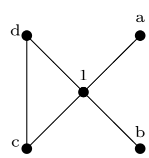

FIGURE 8. Associated production graph $\Gamma(E)$.

In the following, we study the relation between implication graphs and production graphs of residuated and involutive EQ-algebras.

Theorem 4.32. *Let E be a residuated* EQ-*algebra and $x \in E$. Then $Z_x \subseteq D_x$.*

Proof. Let $y \in Z_x$. Then $r(\{x, y\}) = \{1\}$, if $t \in Z_{\{x,y\}}$, then $x \odot t = 0$ and $y \odot t = 0$ and since E is a residuated EQ-algebra, by Lemma 4.5, we get $x \leq \neg t$ and $y \leq \neg t$ and so, $x \to \neg t = 1$ and $y \to \neg t = 1$. Hence $\neg t \in r(\{x, y\})$ and so, $\neg t = 1$ and since E is a separated EQ-algebra, we conclude that $t = 0$ and so, $Z_{\{x,y\}} = \{0\}$. Therefore $y \in D_x$ and so, $Z_x \subseteq D_x$. □

Theorem 4.33. *Let E be an involutive* EQ-*algebra and $x \in E$. Then $D_x \subseteq Z_x$.*

Proof. Let $y \in D_x$. Then $Z_{\{x,y\}} = \{0\}$, if $t \in r(\{x, y\})$, then $x \to t = 1$ and $y \to t = 1$ and so, $x \leq t$ and $y \leq t$. Now, by $(E2)$ and Lemma 2.3(ix), we have $x \odot \neg t \leq t \odot \neg t \leq 0$ and $y \odot \neg t \leq t \odot \neg t \leq 0$. Hence $x \odot \neg t = 0$ and $y \odot \neg t = 0$ and so, $\neg t \in Z_{\{x,y\}}$. Thus $\neg t = 0$ and since E is an involutive EQ-algebra, we get $t = \neg\neg t = \neg 0 = 1$ and so $r(\{x, y\}) = \{1\}$. Therefore $y \in Z_x$ and so, $D_x \subseteq Z_x$. □

Theorem 4.34. *Let E be an involutive residuated* EQ-*algebra. Then the implication graph $\Omega(E)$ and the production graph $\Gamma(E)$ coincide.*

Proof. Let E be an involutive residuated EQ-algebra and $x \in E$. Then by Theorem 4.32 and Theorem 4.33, we conclude that $D_x = Z_x$. Therefore the implication graph $\Omega(E)$ and the production graph $\Gamma(E)$ coincide. □

Example 4.35. Let E be the EQ-algebra in Example 3.17. Then the implication graph $\Omega(E)$ and the production graph $\Gamma(E)$ coincide.

5. Conclusion

The results of this paper are devoted to the study of implication and production graphs of EQ-algebras. A quasi r-prime up set and some of its related properties are investigated. Further, implication graphs are constructed and it is shown that implication graphs of separated EQ-algebras are connected graph with diameter at most two and under certain conditions they are star graphs. Also, several examples of implication graphs are given. Moreover, the notion of zero divisors by a product operation is defined and by means of zero divisors of an EQ-algebra with the bottom element 0, the production graph is introduced and several examples of production graphs are provided. Also, it is proved that the production graph is a connected graph with diameter at most two and under certain conditions, it is a star graph. Finally, when studying the implication graph and the production graph in involutive residuated EQ-algebras, we have proved that they coincide.

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