Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 2, 259–274

CONTROLLED CONTINUOUS $K-g$–FUSION FRAME IN HILBERT C^*–MODULES

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Abstract. Frame theory has been a great revolution for recent years. This theory has several properties applicable in many fields of mathematics and engineering and plays a significant role in signal and image processing, which lead to many applications in informatics, medicine and in the theory of probability. In this paper, we introduce the concept of controlled continuous g-fusion frame and controlled continuous $K-g$-fusion frame in Hilbert C^*-modules. Then we investigate some of their properties. Also, we discuss the perturbation problem for a controlled continuous $K-g$-fusion frame.

1. Introduction and Preliminaries

In 1952, the concept of frame in Hilbert spaces has been introduced by Duffin and Schaeffer [6] to study some deep problems in nonharmonic Fourier series by abstracting the fundamental notion of Gabor [9] for signal processing. Frames have been used in image processing, data compression and sampling theory.

In 2000, Frank–Larson [8] introduced the concept of frames in Hilbet C^*-modules as a generalization of frames in Hilbert spaces. The basic idea was to consider modules over C^*-algebras of linear spaces and to allow the inner product to take values in the C^*-algebras [13].

Many generalizations of the concept of frame have been defined in Hilbert C^*-modules [10,12,16–20].

Controlled frames have been introduced to improve the numerical efficiency of iterative algorithms for inverting the frame operator [3].

The paper is organized as follows, we continue this introductory section by recalling briefly the definitions and basic properties of C^*-algebra and Hilbert C^*-modules. In Section 2, we introduce the concept of $(C, C^{'})$-controlled continuous g-fusion frame, the $(C, C^{'})$-controlled continuous g-fusion frame operator and establish some results. In Section 3, we introduce the concept of $(C, C^{'})$-controlled continuous $K-g$-fusion frame and give some properties. Finally, in Section 4, we discuss the perturbation problem for $(C, C^{'})$-controlled continuous $K-g$-fusion frame.

In the following, we briefly recall the definitions and basic properties of C^*-algebra and Hilbert $\mathscr{A}$-modules. Our reference for C^*-algebras is [4,5]. For a C^*-algebra $\mathscr{A}$, if $a \in \mathscr{A}$ is positive, we write $a \geq 0$, and $\mathscr{A}^+$ denotes the set of positive elements of $\mathscr{A}$.

Definition 1.1 ([4]). If $\mathscr{A}$ is a Banach algebra, an involution is a map $a \to a^*$ of $\mathscr{A}$ into itself such that for all a and b in $\mathscr{A}$ and all scalars α the following conditions:

(1) $(a^*)^* = a$.
(2) $(ab)^* = b^*a^*$.
(3) $(\alpha a + b)^* = \bar{\alpha} a^* + b^*$

hold.

Definition 1.2 ([4]). A C^*-algebra $\mathscr{A}$ is a Banach algebra with involution such that
$$\|a^*a\| = \|a\|^2$$
for every a in $\mathscr{A}$.

2020 *Mathematics Subject Classification.* 42C15, 41A58.

Key words and phrases. Controlled continuous frame; Controlled continuous g-fusion frame; Controlled continuous $K-g$-fusion frame; C^*-algebra; Hilbert C^*-modules.

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Example. $\mathscr{B} = B(\mathscr{H})$, the algebra of bounded operators on a Hilbert space, is a C^*-algebra, where for each operator A, A^* is the adjoint of A.

Definition 1.3 ([11])**.** Let $\mathscr{A}$ be a unital C^*-algebra and U be a left $\mathscr{A}$-module such that the linear structures of $\mathscr{A}$ and U are compatible. U is a pre-Hilbert $\mathscr{A}$-module if U is equipped with an $\mathscr{A}$-valued inner product $\langle .,. \rangle : U \times U \to \mathscr{A}$ such that is sesquilinear, positive definite and respects the module action. In other words,

(i) $\langle x,x\rangle \geq 0$ for all $x \in U$ and $\langle x,x\rangle = 0$ if and only if $x = 0$,
(ii) $\langle ax+y,z\rangle = a\langle x,z\rangle + \langle y,z\rangle$ for all $a \in \mathscr{A}$ and $x,y,z \in U$,
(iii) $\langle x,y\rangle = \langle y,x\rangle^*$ for all $x,y \in U$.

For $x \in U$, we define $\|x\| = \|\langle x,x\rangle\|^{\frac{1}{2}}$. If U is complete with $\|.\|$, it is called a Hilbert $\mathscr{A}$-module or a Hilbert C^*-module over $\mathscr{A}$. For every a in a C^*-algebra $\mathscr{A}$, we have $|a| = (a^*a)^{\frac{1}{2}}$ and the $\mathscr{A}$-valued norm on U is defined by $|x| = \langle x,x\rangle^{\frac{1}{2}}$ for $x \in U$.

Throughout this paper, U is considered to be a Hilbert C^*-modules over a C^*-algebra, we denote that I_U is the identity operator on U, $\{H_w\}_{w\in\Omega}$ is a sequence of Hilbert C^*-submodules of U and $\{V_w\}_{w\in\Omega}$ is a sequence of Hilbert C^*-modules.

We denote by $End^*_{\mathscr{A}}(U,V_w)$ a set of all adjointable operators. In particular, $End^*_{\mathscr{A}}(U)$ denotes the set of all adjointable operators on U. $\mathscr{R}(T)$ for the range of T and $GL^+(U)$ denotes the set of all bounded positive linear operators which have bounded inverse.

The following lemmas will be used to prove our mains results

Lemma 1.1 ([15])**.** *Let U be a Hilbert $\mathscr{A}$-module. If $T \in End^*_{\mathscr{A}}(U)$, then*

$$\langle Tx,Tx\rangle \leq \|T\|^2\langle x,x\rangle, \ \ \forall x \in U.$$

Lemma 1.2 ([2])**.** *Let U and H be two Hilbert $\mathscr{A}$-modules and $T \in End^*_{\mathscr{A}}(U,H)$. Then the following statements:*

(i) *T is surjective,*
(ii) *T^* is bounded below with respect to the norm, i.e., there is $m > 0$ such that $\|T^*x\| \geq m\|x\|$ for all $x \in H$,*
(iii) *T^* is bounded below with respect to the inner product, i.e., there is $m' > 0$ such that $\langle T^*x,T^*x\rangle \geq m'\langle x,x\rangle$ for all $x \in H$*

are equivalent.

Lemma 1.3 ([1])**.** *Let U and H be two Hilbert $\mathscr{A}$-modules and $T \in End^*_{\mathscr{A}}(U,H)$. Then:*

(i) *If T is injective and T has closed range, then the adjointable map T^*T is invertible and*
$$\|(T^*T)^{-1}\|^{-1} \leq T^*T \leq \|T\|^2.$$
(ii) *If T is surjective, then the adjointable map TT^* is invertible and*
$$\|(TT^*)^{-1}\|^{-1} \leq TT^* \leq \|T\|^2.$$

Lemma 1.4 ([2])**.** *Let U be a Hilbert $\mathscr{A}$-module over a C^*-algebra $\mathscr{A}$, and $T \in End^*_{\mathscr{A}}(U)$ such that $T^* = T$. The following statements:*

(i) *T is surjective,*
(ii) *There are $m,M > 0$ such that $m\|x\| \leq \|Tx\| \leq M\|x\|$, for all $x \in U$,*
(iii) *There are $m',M' > 0$ such that $m'\langle x,x\rangle \leq \langle Tx,Tx\rangle \leq M'\langle x,x\rangle$, for all $x \in U$*

are equivalent.

Lemma 1.5 ([7])**.** *Let E,H and L be Hilbert $\mathscr{A}$-modules, $T \in End^*_{\mathscr{A}}(E,L)$ and $T' \in End^*_{\mathscr{A}}(H,L)$. Then the following two statements:*

(1) *$T'(T')^* \leq \lambda TT^*$ for some $\lambda > 0$,*
(2) *There exists $\mu > 0$ such that $\|(T')^*z\| \leq \mu\|T^*z\|$, for all $z \in L$*

are equivalent.

Lemma 1.6 ([1])**.** *If $\phi : \mathscr{A} \to \mathscr{B}$ is an $*$-homomorphism between C^*-algebras, then ϕ is increasing, that is, if $a \leq b$, then $\phi(a) \leq \phi(b)$.*

2. Controlled Continuous g-fusion Frame in Hilbert C^*-modules

Let X be a Banach space, (Ω,μ) a measure space and a measurable function $f:\Omega\to X$. The integral of the Banach-valued function f has been defined by Bochner and others. Most of the properties of this integral are similar to those of the integral of real-valued functions. Since every C^*-algebra and Hilbert C^*-module is a Banach space, thus we can use this integral and its properties.

Definition 2.1. Let $\{H_w\}_{w\in\Omega}$ be a sequence of closed submodules orthogonally complemented in U, P_{H_w} be the orthogonal projection from U to H_w, $\Lambda_w\in End^*_{\mathscr{A}}(U,V_w)$, $w\in\Omega$ and $\{v_w\}_{w\in\Omega}$ be a family of weights in $\mathscr{A}$, i.e., each v_w is a positive invertible element from the center of the C^*-algebra $\mathscr{A}$. We say $\Lambda=\{H_w,\Lambda_w,v_w\}_{w\in\Omega}$ is a continuous g-fusion frame for U if

(1) for each $x\in U$, $\{P_{H_w}x\}_{w\in\Omega}$ is measurable;
(2) for each $x\in U$, the function $\tilde{\Lambda}:\Omega\to V_w$ defined by $\tilde{\Lambda}(w)=\Lambda_w x$ is measurable;
(3) there exist $0<A\le B<\infty$ such that

$$A\langle x,x\rangle\le\int_\Omega v_w^2\langle\Lambda_wP_{H_w}x,\Lambda_wP_{H_w}x\rangle d\mu(w)\le B\langle x,x\rangle,\qquad\forall x\in U.\tag{2.1}$$

We call A and B the lower and upper continuous g-fusion frame bounds, respectively. If $A=B$, we call Λ the tight continuous g-fusion frame. Moreover, if $A=B=1$, Λ is called the Parseval continuous g-fusion frame.

Definition 2.2. Let $C,C^{'}\in GL^+(U)$ and $\{H_w\}_{w\in\Omega}$ be a sequence of closed submodules orthogonally complemented in U, P_{H_w} be the orthogonal projection from U to H_w, $\Lambda_w\in End^*_{\mathscr{A}}(U,V_w)$, $\forall w\in\Omega$ and $\{v_w\}_{w\in\Omega}$ be a family of weights in $\mathscr{A}$, i.e., each v_w is a positive invertible element from the center of the C^*-algebra $\mathscr{A}$. We say $\Lambda=\{H_w,\Lambda_w,v_w\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous g-fusion frame for U if

(1) for each $x\in U$, $\{P_{H_w}x\}_{w\in\Omega}$ is measurable,
(2) for each $x\in U$, the function $\tilde{\Lambda}:\Omega\to V_w$ defined by $\tilde{\Lambda}(w)=\Lambda_w x$ is measurable,
(3) there exist $0<A\le B<\infty$ such that

$$A\langle x,x\rangle\le\int_\Omega v_w^2\langle\Lambda_wP_{H_w}Cx,\Lambda_wP_{H_w}C^{'}x\rangle d\mu(w)\le B\langle x,x\rangle,\quad\forall x\in U.\tag{2.2}$$

We call A and B the lower and upper $(C,C^{'})$-controlled continuous g -fusion frame bounds, respectively. If only the right-hand inequality of (2.2) is satisfied, we call Λ the $(C,C^{'})$-controlled continuous g-fusion Bessel sequence. If $A=B$, we call Λ the tight $(C,C^{'})$-controlled continuous g-fusion frame. Moreover, if $A=B=1$, Λ is called the Parseval $(C,C^{'})$-controlled continuous g-fusion frame.

Proposition 2.1. *If $\{v_w\Lambda_wP_{H_w}\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous g-frame for U, then $\Lambda=\{H_w,\Lambda_w,v_w\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous g-fusion frame for U.*

Proof. Since $\{v_w\Lambda_wP_{H_w}\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous g- frame for U, we have

$$A\langle x,x\rangle\le\int_\Omega\langle v_w\Lambda_wP_{H_w}Cx,v_w\Lambda_wP_{H_w}C^{'}x\rangle d\mu(w)\le B\langle x,x\rangle$$

for each $x\in U$. Then

$$A\langle x,x\rangle\le\int_\Omega v_w^2\langle\Lambda_wP_{H_w}Cx,\Lambda_wP_{H_w}C^{'}x\rangle d\mu(w)\le B\langle x,x\rangle.$$

Hence Λ is a $(C,C^{'})$-controlled continuous g-fusion frame for U. □

Suppose that $\Lambda=\{H_w,\Lambda_w,v_w\}_{w\in\Omega}$ be a $(C,C^{'})$-controlled continuous g-fusion Bessel sequence for U. The bounded linear operator $T_{(C,C^{'})}:\oplus_{w\in\Omega}V_w\to U$ is defined by

$$T_{(C,C^{'})}(\{x_w\}_{w\in\Omega})=\int_\Omega v_w(CC^{'})^{\frac{1}{2}}P_{H_w}\Lambda_w^*x_wd\mu(w),\quad\forall\{x_w\}_{w\in\Omega}\in\oplus_{w\in\Omega}V_w.\tag{2.3}$$

$T_{(C,C')}$ is called the synthesis operator for the $(C,C^{'})$-controlled continuous g-fusion frame Λ.

The adjoint operator $T^*_{(C,C')} : U \to \oplus_{w\in\Omega} V_w$ given by

$$T^*_{(C,C')}(y) = \big\{v_w \Lambda_w P_{H_w}(C^{'}C)^{\frac{1}{2}} y\big\}_{w\in\Omega}, \tag{2.4}$$

is called the analysis operator for the $(C,C^{'})$-controlled continuous g-fusion frame Λ.

When C and $C^{'}$ commute with each other and commute with the operator $P_{H_w}\Lambda^*_w\Lambda_w P_{H_w}$, for each $w \in \Omega$, then the $(C,C^{'})$-controlled continuous g-fusion frame operator $S_{(C,C')} : U \to U$ is defined as

$$S_{(C,C')}(x) = T_{(C,C')}T^*_{(C,C')}(x) = \int_\Omega v_w^2 C^{'} P_{H_w}\Lambda^*_w\Lambda_w P_{H_w} C x d\mu(w), \quad \forall x \in U. \tag{2.5}$$

From now we assume that C and $C^{'}$ commute with each other and commute with the operator $P_{H_w}\Lambda^*_w\Lambda_w P_{H_w}$, for each $w \in \Omega$.

Lemma 2.1. *Let Λ be a $(C,C^{'})$-controlled continuous g-fusion frame for U. Then the $(C,C^{'})$-controlled continuous g-fusion frame operator $S_{(C,C')}$ is positive, self-adjoint and invertible.*

Proof. For each $f \in H$, we have $S_{(C,C')}(x) = \int_\Omega v_w^2 C^{'} P_{H_w}\Lambda^*_w\Lambda_w P_{H_w} C x d\mu(w)$and

$$\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} C x, \Lambda_w P_{W_w} C^{'} x\rangle d\mu(w) = \Big\langle \int_\Omega v_w^2 C^{'} P_{H_w}\Lambda^*_w\Lambda_w P_{H_w} C x d\mu(w), x\Big\rangle = \langle S_{(C,C')}(x), x\rangle.$$

Since Λ is a $(C,C^{'})$-controlled continuous g-fusion frame for U, it follows that

$$A\langle x, x\rangle \leq \langle S_{(C,C')}(x), x\rangle \leq B\langle x, x\rangle, \quad \forall x \in U. \tag{2.6}$$

So, $S_{(C,C')}$ is a positive. Also, it is clearly bounded and linear. On the other hand, for each x, $y \in U$

$$\begin{aligned}\langle S_{(C,C')}(x), y\rangle &= \Big\langle \int_\Omega v_w^2 C^{'} P_{H_w}\Lambda^*_w\Lambda_w P_{H_w} C x d\mu(w), y\Big\rangle \\ &= \Big\langle x, \int_\Omega v_w^2 C P_{H_w}\Lambda^*_w\Lambda_w P_{H_w} C^{'} y d\mu(w)\Big\rangle \\ &= \langle x, S_{(C',C)}(y)\rangle.\end{aligned}$$

That implies $S^*_{(C,C')} = S_{(C',C)}$. Also, as C and $C^{'}$ commute with each other and commute with the operator $P_{H_w}\Lambda^*_w\Lambda_w P_{H_w}$, for each $w \in \Omega$, we have $S_{(C,C')} = S_{(C',C)}$. So, the $(C,C^{'})$-controlled continuous g-fusion frame operator $S_{(C,C')}$ is self-adjoint and we have

$$AI_H \leq S_{(C,C')} \leq BI_H. \tag{2.7}$$

Therefore the $(C,C^{'})$-controlled continuous g-fusion frame operator $S_{(C,C')}$ is invertible. □

Theorem 2.1. *If $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous g-fusion frame for U with frame bounds A and B, then $T_{(C,C')}$ is surjective with $\|T_{(C,C')}\| \leq \sqrt{B}$ and $T^*_{(C,C')}$ is injective, closed.*

Proof. For each $x \in U$, we have

$$A\langle x, x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} C x, \Lambda_w P_{H_w} C^{'} x\rangle d\mu(w) \leq B\langle x, x\rangle.$$

Then

$$A\langle x, x\rangle \leq \langle T^*_{(C,C')}x, T^*_{(C,C')}x\rangle \leq B\langle x, x\rangle. \tag{2.8}$$

Hence

$$\sqrt{A}\|x\| \leq \|T^*_{(C,C')}x\|. \tag{2.9}$$

So, $T^*_{(C,C')}$ is injective, we now show that the $\mathscr{R}(T^*_\Lambda)$ is closed.

Let $\{T^*_{(C,C')}(x_n)\}_{n\in\mathbb{N}} \in \mathscr{R}(T^*_{(C,C')})$ such that $\lim_n T^*_{(C,C')}(x_n) = y$.

Let n, $m \in \mathbb{N}$, from (2.8), we have

$$\langle x_n - x_m, x_n - x_m\rangle \leq A^{-1}\langle T^*_{(C,C')}(x_n - x_m), T^*_{(C,C')}(x_n - x_m)\rangle.$$

Then

$$\|\langle x_n - x_m, x_n - x_m\rangle\| \leq A^{-1}\|T^*_{(C,C')}(x_n - x_m)\|^2.$$

Since $\{T^*_{(C,C')}(x_n)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\oplus_{w\in\Omega}V_w$, so, $\|\langle x_n - x_m, x_n - x_m\rangle\| \to 0$. Therefore the sequence $\{x_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in U and there exists $x \in U$ such that $\lim_n x_n = x$. Again, by (2.8), we have

$$\|T^*_{(C,C')}x_n - T^*_{(C,C')}x\|^2 \leq B\|\langle x_n - x, x_n - x\rangle\|,$$

thus $\|T^*_{(C,C')}(x_n) - T^*_{(C,C')}(x)\| \to 0$ implies that $T^*_{(C,C')}(x) = y$, hence $\mathscr{R}(T^*_{(C,C')})$ is closed, finally $T_{(C,C')}$ is surjective. □

We estabilish an equivalent definition of $(C, C^{'})$-controlled continuous g-fusion frame.

Theorem 2.2. *Let C, $C^{'} \in GL^+(U)$ and $\{H_w\}_{w\in\Omega}$ be a sequence of closed submodules orthogonally complemented in U, P_{H_w} be the orthogonal projection from U to H_w, $\Lambda_w \in End^*_{\mathscr{A}}(U, V_w)$, $\forall w \in \Omega$ and $\{v_w\}_{w\in\Omega}$ be a family of weights in $\mathscr{A}$, then $\Lambda = \{W_j, \Lambda_j, v_j\}_{j\in J}$ is a $(C, C^{'})$-controlled g-fusion frame for U if and only if there exist two constants $0 < A \leq B < \infty$ such that*

$$A\|x\|^2 \leq \left\|\int_\Omega v_w^2\langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'}x\rangle d\mu(w)\right\| \leq B\|x\|^2, \qquad \forall x \in U. \tag{2.10}$$

Proof. If Λ is a $(C, C^{'})$-controlled g-fusion frame for U, then we have inequality (2.10). Conversely, assume that (2.10) holds. From (2.4), the $(C, C^{'})$-controlled g-fusion frame operator $S_{(C,C')}$ is positive, self-adjoint and invertible. Then for all $x \in U$, we have

$$\langle (S_{(C,C')})^{\frac{1}{2}}x, (S_{(C,C')})^{\frac{1}{2}}x\rangle = \langle S_{(C,C')}x, x\rangle = \int_\Omega v_w^2\langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'}x\rangle d\mu(w). \tag{2.11}$$

From (2.11) and (2.10), for each $x \in U$, we have

$$\sqrt{A}\|x\| \leq \|S^{\frac{1}{2}}_{(C,C')}x\| \leq \sqrt{B}\|x\|, \quad \forall x \in U.$$

So, by Lemma 1.4, we conclude that Λ is a $(C, C^{'})$-controlled continuous g-fusion frame for U. □

Theorem 2.3. *If the operator $T_{(C,C')} : \oplus_{w\in\Omega}V_w \to U$ defined by $T(\{x_w\}_{w\in\Omega}) = \int_\Omega v_w(CC^{'})^{\frac{1}{2}} \times P_{H_w}\Lambda^*_w x_w d\mu(w)$ is well-definite and surjective, then $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a $(C, C^{'})$-controlled continuous g-fusion frame for U.*

Proof. For each $x \in U$, we have

$$\begin{aligned}
\left\|\int_\Omega v_w^2\langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'}x\rangle d\mu(w)\right\| &= \left\|\int_\Omega v_w^2\langle \Lambda_w P_{H_w} (CC^{'})^{\frac{1}{2}}x, \Lambda_w P_{H_w} (CC^{'})^{\frac{1}{2}}x\rangle d\mu(w)\right\| \\
&= \left\|\langle x, \int_\Omega v_w^2 (CC^{'})^{\frac{1}{2}} P_{H_w}\Lambda^*_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\rangle d\mu(w)\right\| \\
&= \|\langle x, T_{(C,C')}(\{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}\}_{w\in\Omega})\rangle\| \\
&\leq \|x\|\,\|T_{(C,C')}\|\,\|\{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}\}_{w\in\Omega}| \quad \leq \|x\|\,\|T_{(C,C')}\| \\
&\quad \times \left\|\int_\Omega v_w^2\langle \Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x, \Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\rangle d\mu(w)\right\|^{\frac{1}{2}}
\end{aligned}$$

$$= \|x\|\, \|T_{(C,C')}\| \Big\| \int_{\Omega} v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'} x\rangle d\mu(w) \Big\|^{\frac{1}{2}},$$

hence

$$\Big\| \int_{\Omega} v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'} x\rangle d\mu(w) \Big\| \leq \|T_{(C,C')}\|^2 \|x\|^2. \tag{2.12}$$

Since $T_{(C,C')}$ is surjective, by Lemma 1.2, there exists $\nu > 0$ such that

$$\nu\|x\| \leq \|T^*_{(C,C')}x\|, \qquad \forall x \in U,$$

so, $T^*_{(C,C')}$ is injective and this implies that $T^*_{(C,C')} : U \to \mathscr{R}(T^*_{(C,C')})$ is invertible. Therefore $(T^*_{/\mathscr{R}(T^*_{(C,C')})})^{-1} T^*_{(C,C')} x = x$, for each $x \in U$, then

$$\|x\|^2 \leq \|(T^*_{/\mathscr{R}(T^*_{(C,C')})})^{-1}\|^2 \|T^*_{(C,C')}(x)\|^2, \qquad \forall x \in U,$$

hence

$$\|(T^*_{/\mathscr{R}(T^*_{(C,C')})})^{-1}\|^{-2} \|x\|^2 \leq \|T^*_{(C,C')}(x)\|^2, \qquad \forall x \in U.$$

So,

$$\|(T^*_{/\mathscr{R}(T^*_{(C,C')})})^{-1}\|^{-2} \|x\|^2 \leq \Big\| \int_{\Omega} v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'} x\rangle d\mu(w) \Big\|. \tag{2.13}$$

From inequalities (2.12) and (2.13), we conclude that $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a $(C, C^{'})$-controlled continuous g-fusion frame for U. □

Theorem 2.4. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a $(C, C^{'})$-controlled continuous g-fusion frame for U and $\Gamma = \{H_w, \Gamma_w, v_w\}_{w\in\Omega}$ be a $(C, C^{'})$-controlled continuous g-fusion Bessel sequence for U, suppose that C and $C^{'}$ commute with $P_{H_w}\Gamma_w^*\Lambda_w P_{H_w}$ for each $w \in \Omega$. If the operator $Q : U \to U$ defined by $Q(x) = \int_\Omega v_w^2 C P_{H_w}\Gamma_w^*\Lambda_w P_{H_w} C^{'} x d\mu(w)$ is surjective, then Γ is a $(C, C^{'})$-controlled continuous g-fusion frame for U.*

Proof. Let T_Λ and T_Γ be the synthesis operators of Λ and Γ, respectively.

For each $x \in U$, we have

$$\begin{aligned} Q(x) &= \int_{\Omega} v_w^2 C P_{H_w}\Gamma_w^*\Lambda_w P_{H_w} C^{'} x d\mu(w) \\ &= \int_{\Omega} v_w^2 (CC^{'})^{\frac{1}{2}} P_{H_w}\Gamma_w^*\Lambda_w P_{H_w} (CC^{'})^{\frac{1}{2}} x d\mu(w) \\ &= T_\Gamma\big(\{v_w \Lambda_w P_{H_w} (CC^{'})^{\frac{1}{2}} x\}_{w\in\Omega}\big) \\ &= T_\Gamma T_\Lambda^*(x). \end{aligned}$$

Since Q is surjective, for each $y \in U$, there exists $x \in U$ such that $y = Q(x)$, hence $y = T_\Gamma T_\Lambda^*(x)$, because $T_\Lambda^* x \in \oplus_{w\in\Omega} V_w$, then T_Γ is surjective and therefore by Theorem 2.3, Γ is a $(C, C^{'})$-controlled continuous g-fusion frame for U. □

Theorem 2.5. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a $(C, C^{'})$-controlled continuous g-fusion frame for U with frame bounds A and B. If $\theta \in End^*_{\mathscr{A}}(U)$ is injective, has closed range, $\theta P_{H_w} C = P_{H_w} C\theta$ and $\theta P_{H_w} C^{'} = P_{H_w} C^{'}\theta$ for each $w \in \Omega$, then $\{H_w, \Lambda_w\theta, v_w\}_{w\in\Omega}$ is a $(C, C^{'})$-controlled continuous g-fusion frame for U.*

Proof. Let Λ be a $(C, C^{'})$-controlled continuous g-fusion frame for U with frame bounds A and B, then for each $x \in U$,

$$A\langle x, x\rangle \leq \int_{\Omega} v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'} x\rangle d\mu(w) \leq B\langle x, x\rangle,$$

and for each $x \in U$, we have

$$\int\limits_{\Omega} v_w^2 \langle \Lambda_w \theta P_{H_w} C x, \Lambda_w \theta P_{H_w} C^{'} x \rangle d\mu(w) = \int\limits_{\Omega} v_w^2 \langle \Lambda_w P_{H_w} C \theta x, \Lambda_w P_{H_w} C^{'} \theta x \rangle d\mu(w) \tag{2.14}$$

From equality (2.14), for each $x \in U$,

$$\begin{aligned} \int\limits_{\Omega} v_w^2 \langle \Lambda_w \theta P_{H_w} C x, \Lambda_w \theta P_{H_w} C^{'} x \rangle d\mu(w) &\leq B \langle \theta x, \theta x \rangle \\ &\leq B \|\theta\|^2 \langle x, x \rangle, \end{aligned}$$

and

$$A \langle \theta x, \theta x \rangle \leq \int\limits_{\Omega} v_w^2 \langle \Lambda_w \theta P_{H_w} C x, \Lambda_w \theta P_{H_w} C^{'} x \rangle d\mu(w).$$

Since θ is injective, has closed range, by Lemma 1.3,

$$\|(\theta^* \theta)^{-1}\|^{-1} \langle x, x \rangle \leq \langle \theta \theta^* x, x \rangle$$

So,

$$A \|(\theta^* \theta)^{-1}\|^{-1} \langle x, x \rangle \leq \int\limits_{\Omega} v_w^2 \langle \Lambda_w \theta P_{H_w} C x, \Lambda_w \theta P_{H_w} C^{'} x \rangle d\mu(w).$$

Then $\{H_w, \Lambda_w \theta, v_w\}_{w \in \Omega}$ is a $(C, C^{'})$-controlled continuous g-fusion frame for U. □

In the next theorem, we take $V_w \subseteq U$ for all $w \in \Omega$.

Theorem 2.6. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w \in \Omega}$ be a $(C, C^{'})$-controlled continuous g-fusion frame for U with frame bounds A and B. If $\theta \in End_{\mathscr{A}}^*(U, V_w)$ is injective, has closed range, suppose that $\theta \Lambda_w P_{H_w} C = \Lambda_w P_{H_w} C \theta$ and $\theta \Lambda_w P_{H_w} C^{'} = \Lambda_w P_{H_w} C^{'} \theta$ for all $w \in \Omega$, then $\{H_w, \theta \Lambda_w, v_w\}_{w \in \Omega}$ is a $(C, C^{'})$-controlled continuous g-fusion frame for U.*

Proof. For each $x \in U$, we have

$$A \langle x, x \rangle \leq \int\limits_{\Omega} v_w^2 \langle \Lambda_w P_{H_w} C x, \Lambda_w P_{H_w} C^{'} x \rangle d\mu(w) \leq B \langle x, x \rangle$$

and

$$\int\limits_{\Omega} v_w^2 \langle \theta \Lambda_w P_{H_w} C x, \theta \Lambda_w P_{H_w} C^{'} x \rangle d\mu(w) = \int\limits_{\Omega} v_w^2 \langle \Lambda_w P_{H_w} C \theta x, \Lambda_w P_{H_w} C^{'} \theta x \rangle d\mu(w). \tag{2.15}$$

From equality (2.15) follows

$$\begin{aligned} \int\limits_{\Omega} v_w^2 \langle \theta \Lambda_w P_{H_w} C x, \theta \Lambda_w P_{H_w} C^{'} x \rangle d\mu(w) &\leq B \langle \theta x, \theta x \rangle \\ &\leq B \|\theta\|^2 \langle x, x \rangle. \end{aligned}$$

Also, for each $x \in U$,

$$A \langle \theta x, \theta x \rangle \leq \int\limits_{\Omega} v_w^2 \langle \theta \Lambda_w P_{H_w} C x, \theta \Lambda_w P_{H_w} C^{'} x \rangle d\mu(w),$$

Since θ is injective, has closed range, therefore

$$A \|(\theta^* \theta)^{-1}\|^{-1} \langle x, x \rangle \leq \langle \theta^* \theta x, x \rangle,$$

so,

$$A \|(\theta^* \theta)^{-1}\|^{-1} \langle x, x \rangle \leq \int\limits_{\Omega} v_w^2 \langle \theta \Lambda_w P_{H_w} C x, \theta \Lambda_w P_{H_w} C^{'} x \rangle d\mu(w).$$

Thus $\{H_w, \theta \Lambda_w, v_w\}_{w \in \Omega}$ is a $(C, C^{'})$-controlled continuous g-fusion frame for U. □

Theorem 2.7. *Let for every $w \in \Omega$, $\Lambda_w \in End_{\mathscr{A}}^{*}(U, V_w)$ and $\{y_{w,v}\}_{v\in\Omega_w}$ be a (C,C)-controlled continuous frame for V_w with frame bounds C_w and D_w such there exist C and D such that $C \leq C_w$ and $D_w \leq D$, suppose that C commutes with $P_{H_w}\Lambda_w^*$ for all $w \in \Omega$, and the following conditions are equivalent:*

1. *$\{v_w C P_{H_w}\Lambda_w^* y_{w,v}\}_{v\in\Omega_w}$ is a (C,C)-controlled continuous frame for U.*
2. *$\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a (C,C)-controlled continuous g-fusion frame for U.*

Proof. Let $\{y_{w,v}\}_{v\in\Omega_w}$ be a (C,C)-controlled continuous frame for V_w, then for each $x \in U$, we have

$$\begin{aligned} C_w\langle v_w\Lambda_w P_{H_w}Cx, v_w\Lambda_w P_{H_w}Cx\rangle &\leq \int_{\Omega_w} \langle v_w\Lambda_w P_{H_w}Cx, y_{w,v}\rangle\langle Cy_{w,v}, v_w\Lambda_w P_{H_w}Cx\rangle d\mu(v) \\ &\leq D_w\langle v_w\Lambda_w P_{H_w}Cx, v_w\Lambda_w P_{H_w}Cx\rangle, \end{aligned}$$

then

$$\begin{aligned} C_w\langle v_w\Lambda_w P_{H_w}Cx, v_w\Lambda_w P_{H_w}Cx\rangle &\leq \int_{\Omega_w} \langle x, v_w C P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle v_w C P_{H_w}\Lambda_w^* C y_{w,v}, x\rangle d\mu(v) \\ &\leq D_w\langle v_w\Lambda_w P_{H_w}Cx, v_w\Lambda_w P_{H_w}Cx\rangle, \end{aligned}$$

so,

$$\begin{aligned} C\langle v_w\Lambda_w P_{H_w}Cx, v_w\Lambda_w P_{H_w}Cx\rangle &\leq \int_{\Omega_w} \langle x, v_w C P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle C v_w C P_{H_w}\Lambda_w^* y_{w,v}, x\rangle d\mu(v) \\ &\leq D\langle v_w\Lambda_w P_{H_w}Cx, v_w\Lambda_w P_{H_w}Cx\rangle, \end{aligned}$$

hence

$$\begin{aligned} C\int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}Cx\rangle d\mu(w) &\leq \int_{\Omega}\int_{\Omega_w} \langle x, v_w C P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle C v_w C P_{H_w}\Lambda_w^* y_{w,v}, x\rangle d\mu(v) d\mu(w) \\ &\leq D\int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}Cx\rangle d\mu(w). \end{aligned} \tag{2.16}$$

Suppose that $\{v_w C P_{H_w}\Lambda_w^* y_{w,v}\}_{v\in\Omega_w}$ is a (C,C)-controlled continuous frame for U with frame bounds C' and D', then for each $x \in U$,

$$C'\langle x, x\rangle \leq \int_{\Omega}\int_{\Omega_w} \langle x, v_w C P_{H_w}\Lambda_w^* y_{w,v}\rangle\langle C v_w C P_{H_w}\Lambda_w^* y_{w,v}, x\rangle d\mu(v) d\mu(w) \leq D'\langle x, x\rangle, \tag{2.17}$$

by (2.16) and (2.17), we have

$$C\int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}Cx\rangle d\mu(w) \leq D'\langle x, x\rangle$$

and

$$C'\langle x, x\rangle \leq D\int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}Cx\rangle d\mu(w).$$

Therefore

$$D^{-1}C'\langle x, x\rangle \leq \int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}Cx\rangle d\mu(w) \leq C^{-1}D'\langle x, x\rangle, \quad \forall x \in U.$$

So, $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a (C,C)-controlled continuous g-fusion frame for U. Conversely, assume that $\{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a (C,C)-controlled continuous g-fusion frame for U with frame bounds C' and D', then for each $x \in U$,

$$C'\langle x, x\rangle \leq \int_{\Omega} v_w^2\langle\Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}Cx\rangle d\mu(w) \leq D'\langle x, x\rangle,$$

and by (2.16), we have

$$CC^{'}\langle x,x\rangle \leq \int_{\Omega}\int_{\Omega_w} \langle x, v_w CP_{H_w}\Lambda_w^* y_{w,v}\rangle\langle Cv_wCP_{H_w}\Lambda_w^* y_{w,v}, x\rangle d\mu(v)d\mu(w) \leq DD^{'}\langle x,x\rangle.$$

Thus we can conclude that $\{v_wCP_{H_w}\Lambda_w^* y_{w,v}\}_{v\in\Omega_w}$ is a (C,C)-controlled continuous frame for U. □

Under which conditions a $(C,C^{'})$-controlled continuous g-fusion frame for U with U a C^*-module over a unital C^*-algebras A is also a $(C,C^{'})$-controlled continuous g-fusion frame for U with U a C^*-module over a unital C^*-algebras B? the following theorem answers this question. In the next theorem, we take $V_w \subset U$, $\forall w \in \Omega$.

Theorem 2.8. *Let $(U,\mathscr{A},\langle .,.\rangle_{\mathscr{A}})$ and $(U,\mathscr{B},\langle .,.\rangle_{\mathscr{B}})$ be two Hilbert C^*-modules and let $\phi:\mathscr{A}\to\mathscr{B}$ be a $*$-homomorphisme and θ be a map on H such that $\langle\theta x,\theta y\rangle_{\mathscr{B}} = \phi(\langle x,y\rangle_{\mathscr{A}})$ for all x, $y\in U$. Suppose that $\Lambda=\{H_w,\Lambda_w,v_w\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous g-fusion frame for $(U,\mathscr{A},\langle .,.\rangle_{\mathscr{A}})$ with the frame operator $S_{\mathscr{A}}$ and lower and upper bounds A and B, respectively. If θ is surjective such that $\theta\Lambda_wP_{H_w}=\Lambda_wP_{H_w}\theta$ for each $w\in\Omega$ and $\theta C=C\theta$ and $\theta C^{'}=C^{'}\theta$, then $\{H_w,\Lambda_w,\phi(v_w)\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous g-fusion frame for $(U,\mathscr{B},\langle .,.\rangle_{\mathscr{B}})$ with frame operator $S_{\mathscr{B}}$ and lower and upper bounds A and B, respectively, and $\langle S_{\mathscr{B}}\theta x,\theta y\rangle_{\mathscr{B}}=\phi(\langle S_{\mathscr{A}}x,y\rangle_{\mathscr{A}})$.*

Proof. Since θ is surjective, for every $y\in U$, there exists $x\in U$ such that $\theta x=y$. Using the definition of a $(C,C^{'})$-controlled continuous g-fusion frame for $(U,\mathscr{A},\langle .,.\rangle_{\mathscr{A}})$, we have

$$A\langle x,x\rangle_{\mathscr{A}} \leq \int_{\Omega} v_w^2\langle\Lambda_wP_{H_w}Cx,\Lambda_wP_{H_w}C^{'}x\rangle_{\mathscr{A}}d\mu(w) \leq B\langle x,x\rangle_{\mathscr{A}}.$$

Then

$$\phi\big(A\langle x,x\rangle_{\mathscr{A}}\big) \leq \phi\bigg(\int_{\Omega} v_w^2\langle\Lambda_wP_{H_w}Cx,\Lambda_wP_{H_w}C^{'}x\rangle_{\mathscr{A}}d\mu(w)\bigg) \leq \phi\big(B\langle x,x\rangle_{\mathscr{A}}\big).$$

Frome the definition of the $*$-homomorphism, we have

$$A\phi\big(\langle x,x\rangle_{\mathscr{A}}\big) \leq \int_{\Omega}\phi(v_w^2)\phi\big(\langle\Lambda_wP_{H_w}Cx,\Lambda_wP_{H_w}C^{'}x\rangle_{\mathscr{A}}d\mu(w)\big) \leq B\phi\big(\langle x,x\rangle_{\mathscr{A}}\big).$$

Using the relation between θ and ϕ, we get

$$A\langle\theta x,\theta x\rangle_{\mathscr{B}} \leq \int_{\Omega}\phi(v_w)^2\langle\theta\Lambda_wP_{H_w}Cx,\theta\Lambda_wP_{H_w}C^{'}x\rangle_{\mathscr{B}}d\mu(w) \leq B\langle\theta x,\theta x\rangle_{\mathscr{B}}.$$

Since $\theta\Lambda_jP_{W_j}=\Lambda_jP_{W_j}\theta$ for each $j\in J$ and $\theta C=C\theta$ and $\theta C^{'}=C^{'}\theta$, we have

$$A\langle\theta x,\theta x\rangle_{\mathscr{B}} \leq \int_{\Omega}\phi(v_w)^2\langle\Lambda_wP_{H_w}C\theta x,\Lambda_wP_{H_w}C^{'}\theta x\rangle_{\mathscr{B}}d\mu(w) \leq B\langle\theta x,\theta x\rangle_{\mathscr{B}}.$$

Therefore

$$A\langle y,y\rangle_{\mathscr{B}} \leq \int_{\Omega}\phi(v_w)^2\langle\Lambda_wP_{H_w}Cy,\Lambda_wP_{H_w}C^{'}y\rangle_{\mathscr{B}} \leq B\langle y,y\rangle_{\mathscr{B}}, \quad \forall y\in U.$$

This implies that $\{H_w,\Lambda_w,\phi(v_w)\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous g-fusion frame for $(U,\mathscr{B},\langle .,.\rangle_{\mathscr{B}})$.

And for each $x\in U$,

$$\begin{aligned}\phi\big(\langle S_{\mathscr{A}}x,y\rangle_{\mathscr{A}}\big) &= \phi\bigg(\Big\langle\int_{\Omega} v_w^2C^{'}P_{H_w}\Lambda_w^*\Lambda_wP_{H_w}Cxd\mu(w),y\Big\rangle_{\mathscr{A}}\bigg)\\ &= \phi\bigg(\int_{\Omega} v_w^2\langle\Lambda_wP_{H_w}Cx,\Lambda_wP_{H_w}C^{'}y\rangle_{\mathscr{A}}d\mu(w)\bigg)\end{aligned}$$

$$= \int_\Omega \phi(v_w)^2 \langle \Lambda_w P_{H_w} C\theta x, \Lambda_w P_{H_w} C^{'} \theta y\rangle_{\mathscr{B}} d\mu(w)$$

$$= \Big\langle \int_\Omega \phi(v_w)^2 C^{'} P_{H_w} \Lambda_w^* \Lambda_w P_{H_w} C\theta x d\mu(w), \theta y\Big\rangle_{\mathscr{B}}$$

$$= \langle S_{\mathscr{B}} \theta x, \theta y\rangle_{\mathscr{B}}.$$

□

3. Controlled Continuous $K-g$-fusion Frames in Hilbert C^*-modules

We begin this section with the following

Lemma 3.1 ([14]). *Let $\{H_w\}_{w\in\Omega}$ be a sequence of orthogonally complemented closed submodules of U and $T \in End_{\mathscr{A}}^*(U)$ invertible, if $T^*TH_w \subseteq H_w$ for each $w \in \Omega$, then $\{TH_w\}_{w\in\Omega}$ is a sequence of orthogonally complemented closed submodules and $P_{H_w}T^* = P_{H_w}T^*P_{TH_w}$.*

Definition 3.1. Let $K \in End_{\mathscr{A}}^*(U)$ and let $\{H_w\}_{w\in\Omega}$ be a sequence of closed submodules orthogonally complemented in U, P_{H_w} be the orthogonal projection from U to H_w, $\Lambda_w \in End_{\mathscr{A}}^*(U, V_w)$, for each $w \in \Omega$ and $\{v_w\}_{w\in\Omega}$ be a family of weights in $\mathscr{A}$, i.e., each v_w is a positive invertible element from the center of $\mathscr{A}$; we say that $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a continuous $K-g$-fusion frame for U if

(1) for each $x \in U$, $\{P_{H_w}x\}_{w\in\Omega}$ is measurable;
(2) for each $x \in U$, the function $\tilde{\Lambda} : \Omega \to V_w$ defined by $\tilde{\Lambda}(w) = \Lambda_w x$ is measurable;
(3) there exist $0 < A \le B < \infty$ such that

$$A\langle K^*x, K^*x\rangle \le \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} x, \Lambda_w P_{H_w} x\rangle d\mu(w) \le B\langle x, x\rangle, \quad \forall x \in U. \tag{3.1}$$

We call A and B lower and upper frame bounds of a continuous $K-g$-fusion frame, respectively.

Definition 3.2. Let $C,C^{'} \in GL^+(U)$ and $K \in End_{\mathscr{A}}^*(U)$, $\{H_w\}_{w\in\Omega}$ be a sequence of closed submodules orthogonally complemented of U, $\{v_w\}_{w\in\Omega}$ be a family of weights, i.e., each v_w is a positive invertible element from the center of $\mathscr{A}$ and $\Lambda_w \in End_A^*(U, V_w)$ for each $w \in \Omega$. We say $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a $(C, C^{'})$-controlled continuous $K-g$-fusion frame for U if there exist $0 < A \le B < \infty$ such that

$$A\langle K^*x, K^*x\rangle \le \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'} x\rangle d\mu(w) \le B\langle x, x\rangle, \quad \forall x \in U. \tag{3.2}$$

The constants A and B are called a lower and an upper bounds of a $(C, C^{'})$-controlled continuous $K-g$-fusion frame, respectively. If the left-hand inequality of (3.2) is an equality, we say that Λ is a tight $(C, C^{'})$-controlled continuous $K-g$-fusion frame.

Remark 3.1. If Λ is a $(C, C^{'})$-controlled continuous $K-g$-fusion frame for U with bounds A and B, we have

$$AKK^* \le S_{(C,C^{'})} \le BI_H. \tag{3.3}$$

From inequality (3.3) and equality (2.5), we have

Lemma 3.2. *Let $K \in End_{\mathscr{A}}^*(U)$ and Λ be a $(C, C^{'})$-controlled continuous g-fusion Bessel sequence for U. Then Λ is a $(C, C^{'})$-controlled continuous $K-g$-fusion frame for U if and only if there exists a constant $A > 0$ such that $AKK^* \le S_{(C,C^{'})}$, where $S_{(C,C^{'})}$ is the frame operator for Λ.*

Theorem 3.1. *Let $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ and $\Gamma = \{W_w, \Gamma_w, u_w\}_{w\in\Omega}$ be two $(C, C^{'})$-controlled continuous g-fusion Bessel sequences for U with bounds B_1 and B_2, respectively. Suppose that T_Λ and T_Γ are their synthesis operators such that $T_\Gamma T_\Lambda^* = K^*$ for some $K \in End_{\mathscr{A}}^*(U)$. Then both Λ and Γ are $(C, C^{'})$-controlled continuous K and $K^* - g$-fusion frames for U, respectively.*

Proof. For each $x \in U$, we have

$$\begin{aligned}\langle K^*x, K^*x\rangle = \langle T_\Gamma T_\Lambda^* x, T_\Gamma T_\Lambda^* x\rangle &\leq \|T_\Gamma\|^2 \langle T_\Lambda^*, T_\Lambda^* f\rangle \\ &\leq B_2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'} x\rangle d\mu(w).\end{aligned}$$

Hence

$$B_2^{-1}\langle K^*x, K^*x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'} x\rangle d\mu(w).$$

This means that Λ is a $(C, C^{'})$-controlled continuous $K-g$-fusion frame for U. Similarly, Γ is a $(C, C^{'})$-controlled continuous K^*-g-fusion frame for H with the lower bound B_1^{-1}. □

Theorem 3.2. *Let $Q \in End_{\mathscr{A}}^*(U)$ be an invertible operator on U and $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a $(C, C^{'})$-controlled continuous $K-g$-fusion frame for U for some $K \in End_{\mathscr{A}}^*(U)$. Suppose that $Q^*QH_w \subset H_w$, $\forall w \in \Omega$ and C, $C^{'}$ commute with Q. Then $\Gamma = \{QH_w, \Lambda_w P_{H_w} Q^*, v_w\}_{w\in\Omega}$ is a $(C, C^{'})$-controlled continuous QKQ^*-g-fusion frame for U.*

Proof. Since Λ is a $(C, C^{'})$-controlled continuous $K-g$-fusion frame for U, $\exists\ A, B > 0$ such that

$$A\langle K^*x, K^*x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'} x\rangle d\mu(w) \leq B\langle x, x\rangle, \quad \forall x \in U.$$

Also, Q is an invertible linear operator on U, so, for any $w \in \Omega$, QH_w is closed in U. Now, for each $x \in U$, using Lemma 3.1, we obtain

$$\begin{aligned}\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Q^* P_{QH_w} Cx, \Lambda_w P_{H_w} Q^* P_{QH_w} C^{'} x\rangle d\mu(w) &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Q^* Cx, \Lambda_w P_{H_w} Q^* C^{'} x\rangle d\mu(w) \\ &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} CQ^* x, \Lambda_w P_{H_w} C^{'} Q^* x\rangle d\mu(w) \\ &\leq B\langle Q^* f, Q^* f\rangle \leq B\|Q\|^2 \langle x, x\rangle.\end{aligned}$$

On the other hand, for each $f \in H$,

$$\begin{aligned}A\langle (QKQ^*)^* x, (QKQ^*)^* x\rangle &= A\langle QK^*Q^*x, QK^*Q^*x\rangle \\ &\leq A\|Q\|^2 \langle K^*Q^*x, K^*Q^*x\rangle \\ &\leq \|Q\|^2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} C(Q^*x), \Lambda_w P_{H_w} C^{'}(Q^*x)\rangle d\mu(w) \\ &= \|Q\|^2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Q^* Cx, \Lambda_w P_{H_w} Q^* C^{'} x\rangle d\mu(w) \\ &= \|Q\|^2 \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Q^* P_{QH_w} Cx, \Lambda_w P_{H_w} Q^* P_{QH_w} C^{'} x\rangle d\mu(w).\end{aligned}$$

Then

$$\frac{A}{\|Q\|^2}\langle (QKQ^*)^* x, (QKQ^*)^* x\rangle \leq \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Q^* P_{QH_w} Cx, \Lambda_w P_{H_w} Q^* P_{QH_w} C^{'} x\rangle d\mu(w).$$

Therefore Γ is a $(C, C^{'})$-controlled continuous QKQ^*-g-fusion frame for U. □

Theorem 3.3. *Let $Q \in End_{\mathscr{A}}^*(U)$ be an invertible operator on U and $\Gamma = \{QH_w, \Lambda_w P_{H_w} Q^*, v_w\}_{w\in\Omega}$ be a $(C, C^{'})$-controlled continuous $K-g$-fusion frame for U for some $K \in End_{\mathscr{A}}^*(U)$. Suppose that $Q^*QH_w \subset H_w$, $\forall w \in \Omega$ and C, $C^{'}$ commute with Q. Then $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a $(C, C^{'})$-controlled continuous $Q^{-1}KQ-g$-fusion frame for U.*

Proof. Since $\Gamma = \{QH_w, \Lambda_w P_{H_w}, v_w\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous $K-g$-fusion frame for U, there exists two constants A, $B>0$ such that

$$A\langle K^*x, K^*x\rangle \leq \int_\Omega v_w^2\langle \Lambda_w P_{H_w}Q^*P_{QH_w}Cx, \Lambda_w P_{H_w}Q^*P_{QH_w}C^{'}x\rangle d\mu(w) \leq B\langle x,x\rangle, \quad \forall x\in U.$$

Letting $x\in U$, we have

$$\begin{aligned}
A\langle (Q^{-1}KQ)^*x, (Q^{-1}KQ)^*x\rangle &= A\langle Q^*K^*(Q^{-1})^*x, Q^*K^*(Q^{-1})^*x\rangle \\
&\leq A\|Q\|^2\langle K^*(Q^{-1})^*x, K^*(Q^{-1})^*x\rangle \\
&\leq \|Q\|^2\int_\Omega v_w^2\langle \Lambda_w P_{H_w}Q^*P_{QH_w}C(Q^{-1})^*x, \\
&\qquad\qquad \Lambda_w P_{H_w}Q^*P_{QH_w}C^{'}(Q^{-1})^*x\rangle d\mu(w) \\
&\leq \|Q\|^2\int_\Omega v_w^2\langle \Lambda_w P_{H_w}Q^*C(Q^{-1})^*x, \Lambda_w P_{H_w}Q^*C^{'}(Q^{-1})^*x\rangle d\mu(w) \\
&= \|Q\|^2\int_\Omega v_w^2\langle \Lambda_w P_{H_w}Q^*(Q^{-1})^*Cx, \Lambda_w P_{H_w}Q^*(Q^{-1})^*C^{'}x\rangle d\mu(w) \\
&= \|Q\|^2\int_\Omega v_w^2\langle \Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}C^{'}x\rangle d\mu(w).
\end{aligned}$$

Then, for each $f\in H$, we have

$$\frac{A}{\|Q\|^2}\langle (Q^{-1}KQ)^*x, (Q^{-1}KQ)^*x\rangle \leq \int_\Omega v_w^2\langle \Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}C^{'}x\rangle d\mu(w).$$

Also, for each $x\in U$, we have

$$\begin{aligned}
\int_\Omega v_w^2\langle \Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}C^{'}x\rangle d\mu(w) &= \int_\Omega v_w^2\langle \Lambda_w P_{H_w}CQ^*(Q^{-1})^*x, \Lambda_w P_{H_w}C^{'}Q^*(Q^{-1})^*x\rangle d\mu(w) \\
&= \int_\Omega v_w^2\langle \Lambda_w P_{H_w}Q^*C(Q^{-1})^*x, \Lambda_w P_{H_w}Q^*C^{'}(Q^{-1})^*x\rangle d\mu(w) \\
&= \int_\Omega v_w^2\langle \Lambda_w P_{H_w}Q^*P_{QH_w}C(Q^{-1})^*x, \\
&\qquad\qquad \Lambda_w P_{H_w}Q^*P_{QH_w}C^{'}(Q^{-1})^*x d\mu(w)\rangle \\
&\leq B\langle (Q^{-1})^*x, (Q^{-1})^*x\rangle \\
&\leq B\|Q^{-1}\|^2\langle x,x\rangle.
\end{aligned}$$

Thus Λ is a $(C,C^{'})$-controlled continuous $Q^{-1}KQ-g$-fusion frame for U. □

Theorem 3.4. *Let $K\in End^*_{\mathscr{A}}(U)$ be an invertible operator on U and $\Lambda = \{H_w, \Lambda_w, v_w\}_{w\in\Omega}$ be a $(C,C^{'})$-controlled continuous g-fusion frame for U with frame bounds A, B and let $S_{(C,C^{'})}$ be the associated $(C,C^{'})$-controlled continuous g-fusion frame operator. Suppose that for all $w\in\Omega$, $T^*TH_w\subset H_w$, where $T=KS^{-1}_{(C,C^{'})}$ and C, $C^{'}$ commute with T. Then $\{KS^{-1}_{(C,C^{'})}H_w, \Lambda_w P_{H_w}S^{-1}_{(C,C^{'})}K^*, v_w\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous $K-g$-fusion frame for U with the corresponding $(C,C^{'})$-controlled continuous g-fusion frame operator $KS^{-1}_{(C,C^{'})}K^*$.*

Proof. Now, $T=KS^{-1}_{(C,C^{'})}$ is invertible on U and $T^* = (KS^{-1}_{(C,C^{'})})^* = S^{-1}_{(C,C^{'})}K^*$. For each $x\in U$, we have

$$\langle K^*x, K^*x\rangle = \langle S_{(C,C^{'})}S^{-1}_{(C,C^{'})}K^*x, S_{(C,C^{'})}S^{-1}_{(C,C^{'})}K^*x\rangle$$

$$\leq \|S_{(C,C')}\|^2 \langle S^{-1}_{(C,C')} K^* x, S^{-1}_{(C,C')} K^* x\rangle$$
$$\leq B^2 \langle S^{-1}_{(C,C')} K^* x, S^{-1}_{(C,C')} K^* x\rangle.$$

Now, for each $x \in U$, we get

$$\begin{aligned}
\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* P_{TH_w} C(x), \Lambda_w P_{H_w} T^* P_{TH_w} C^{'}(x)\rangle d\mu(w) &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* C(x), \\
&\qquad \Lambda_w P_{H_w} T^* C^{'}(x)\rangle d\mu(w) \\
&= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} C T^*(x), \\
&\qquad \Lambda_w P_{H_w} C^{'} T^*(x)\rangle d\mu(w) \\
&\leq B\langle T^* x, T^* x\rangle \\
&\leq B\|T\|^2 \langle x, x\rangle \\
&\leq B\|S^{-1}_{(C,C')}\|^2 \|K\|^2 \langle x, x\rangle \\
&\leq \frac{B}{A^2} \|K\|^2 \langle x, x\rangle.
\end{aligned}$$

On the other hand, for each $x \in U$, we have

$$\begin{aligned}
\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* P_{TH_w} C(x), \Lambda_w P_{H_w} T^* P_{TH_w} C^{'}(x)\rangle d\mu(w) &= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} T^* C(x), \\
&\qquad \Lambda_w P_{H_w} T^* C^{'}(x)\rangle d\mu(w) \\
&= \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} C T^*(x), \\
&\qquad \Lambda_w P_{H_w} C^{'} T^*(x)\rangle d\mu(w) \\
&\geq A\langle T^* x, T^* x\rangle \\
&= A\langle S^{-1}_{(C,C')} K^* x, S^{-1}_{(C,C')} K^* x\rangle \\
&\geq \frac{A}{B^2} \langle K^* x, K^* x\rangle.
\end{aligned}$$

Thus $\{KS^{-1}_{(C,C')} H_w, \Lambda_w P_{H_w} S^{-1}_{(C,C')} K^*, v_w\}_{w\in\Omega}$ is a $(C, C^{'})$-controlled continuous $K-g$-fusion frame for U.

For each $x \in U$, we have

$$\begin{aligned}
\int_\Omega v_w^2 C^{'} P_{TH_w} (\Lambda_w P_{H_w} T^*)^* (\Lambda_w P_{H_w} T^*) P_{TH_w} C x d\mu(w) &= \int_\Omega v_w^2 C^{'} P_{TH_w} T P_{H_w} \Lambda_w^* \\
&\qquad (\Lambda_w P_{H_w} T^*) P_{TH_w} C x d\mu(w) \\
&= \int_\Omega v_w^2 C^{'} (P_{H_w} T^* P_{TH_w})^* \Lambda_w^* \Lambda_w \\
&\qquad (P_{H_w} T^* P_{THw}) C x d\mu(w) \\
&= \int_\Omega v_w^2 C^{'} T P_{H_w} \Lambda_w^* \Lambda_w P_{H_w} T^* C x d\mu(w) \\
&= \int_\Omega v_w^2 T C^{'} P_{H_w} \Lambda_w^* \Lambda_w P_{H_w} C T^* x d\mu(w)
\end{aligned}$$

$$= T\Big(\int_\Omega v_w^2 C^{'} P_{H_w}\Lambda_w^*\Lambda_w P_{H_w} C T^* x d\mu(w)\Big)$$
$$= TS_{(C,C^{'})}T^*(x) = KS_{(C,C^{'})}^{-1}K^*(x).$$

This implies that $KS_{(C,C^{'})}^{-1}K^*$ is the associated $(C,C^{'})$-controlled continuous g-fusion frame operator. □

In the next theorem we give an equivalent definition of a $(C,C^{'})$-controlled $K-g$-fusion frame.

Theorem 3.5. *Let $K \in End_{\mathscr{A}}^*(U)$. Then $\Lambda = \{W_w, \Lambda_w, v_w\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous $K-g$-fusion frame for U if and only if there exist the constants A, $B>0$ such that*

$$A\|K^*x\|^2 \leq \|\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'}x\rangle d\mu(w)\| \leq B\|x\|^2, \quad \forall x \in U. \tag{3.4}$$

Proof. Evidently, every $(C,C^{'})$-controlled $K-g$-fusion frame for H satisfies (3.4). For the converse, we suppose that (3.4) holds. For any $\{x_w\}_{w\in\Omega} \in \oplus_{w\in\Omega} V_w$,

$$\begin{aligned}
\Big\|\int_\Omega v_w (CC^{'})^{\frac{1}{2}} P_{H_w}\Lambda_w^* x_w d\mu(w)\Big\| &= \sup_{\|y\|=1}\Big\|\langle \int_\Omega v_w (CC^{'})^{\frac{1}{2}} P_{W_j}\Lambda_j^* x_w, y\rangle d\mu(w)\Big\| \\
&= \sup_{\|y\|=1}\Big\|\int_\Omega \langle v_w (CC^{'})^{\frac{1}{2}} P_{H_w}\Lambda_w^* x_w, y\rangle d\mu(w)\Big\| \\
&= \sup_{\|y\|=1}\Big\|\int_\Omega \langle x_w, v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}} y\rangle d\mu(w)\Big\| \\
&\leq \sup_{\|y\|=1}\Big\|\int_\Omega \langle x_w, x_w\rangle d\mu(w)\Big\|^{\frac{1}{2}} \\
&\qquad \Big\|\int_\Omega v_w^2 \langle \Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}} y, \Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}} y\rangle d\mu(w)\Big\|^{\frac{1}{2}} \\
&= \sup_{\|y\|=1}\Big\|\int_\Omega \langle x_w, x_w\rangle d\mu(w)\Big\|^{\frac{1}{2}}\Big\|\int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Cy, \Lambda_w P_{H_w} C^{'} y\rangle d\mu(w)\Big\|^{\frac{1}{2}} \\
&\leq \sup_{\|y\|=1}\Big\|\int_\Omega \langle x_w, x_w\rangle d\mu(w)\Big\|^{\frac{1}{2}}\sqrt{B}\|y\| = \sqrt{B}\|\{x_w\}_{w\in\Omega}\|.
\end{aligned}$$

Thus the $\int_\Omega v_w (CC^{'})^{\frac{1}{2}} P_{H_w}\Lambda_w^* x_w d\mu(w)$ converges in U.

Since

$$\langle Tx, \{x_w\}_{w\in\Omega}\rangle = \int_\Omega \langle v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x, x_w\rangle d\mu(w) = \big\langle x, \int_\Omega v_w (CC^{'})^{\frac{1}{2}} P_{H_w}\Lambda_w^* x_w d\mu(w)\big\rangle,$$

T is adjointable. Now, for each $x \in U$, we have

$$\langle Tx, Tx\rangle = \int_\Omega v_w^2 \langle \Lambda_w P_{H_w} Cx, \Lambda_w P_{H_w} C^{'}x\rangle d\mu(w) \leq \|T\|^2\langle x, x\rangle.$$

On the other hand, the left-hand inequality of (3.4) gives

$$\|K^*x\|^2 \leq \frac{1}{A}\|Tx\|^2, \quad \forall x \in U.$$

Then Lemma 1.5 implies that there exists a constant $\mu > 0$ such that

$$KK^* \leq \mu T^*T,$$

and hence

$$\frac{1}{\mu}\langle K^*x,K^*x\rangle \leq \langle Tx,Tx\rangle = \int_{\Omega} v_w^2\langle \Lambda_w P_{H_w}Cx, \Lambda_w P_{H_w}C^{'}x\rangle d\mu(w), \qquad \forall x\in U.$$

We conclude that Λ is a $(C,C^{'})$-controlled continuous $K-g$-fusion frame for U. □

4. Perturbation of Controlled $K-g$-fusion Frame in Hilbert C^*-modules

Theorem 4.1. *Let $\Lambda=\{H_w,\Lambda_w,v_w\}_{w\in\Omega}$ be a $(C,C^{'})$-controlled continuous $K-g$-fusion frame for U with frame bounds A, B and $\Gamma_w\in End_{\mathscr{A}}^*(U,V_w)$. Suppose that C and $C^{'}$ commute with $P_{W_w}\Gamma_w^*\Gamma_w P_{W_w}$ for all $w\in\Omega$ such that for each $x\in U$,*

$$\begin{aligned}\|\{(v_w\Lambda_w P_{H_w}-u_w\Gamma_w P_{W_w})(CC^{'})^{\frac{1}{2}}x\}_{w\in\Omega}\| \leq& \lambda_1\|\{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\}_{w\in\Omega}\| \\ &+\lambda_2\|\{u_w\Gamma_w P_{W_w}(CC^{'})^{\frac{1}{2}}x\}_{w\in\Omega}\|+\epsilon\|K^*x\|.\end{aligned}$$

where $0<\lambda_1,\lambda_2<1$ and $\epsilon>0$ such that $\epsilon<(1-\lambda_1)\sqrt{A}$.
Then $\{W_w,\Gamma_w,u_w\}_{w\in\Omega}$ is a $(C,C^{'})$-controlled continuous $K-g$-fusion frame for U.

Proof. For each $x\in U$, we have

$$\begin{aligned}\Big\|\int_{\Omega} u_w^2\langle \Gamma_w P_{W_w}Cx,\Gamma_w P_{W_w}C^{'}x\rangle d\mu(w)\Big\|^{\frac{1}{2}} &= \|\{u_w\Gamma_w P_{W_w}(CC^{'})^{\frac{1}{2}}x\}_w\| \\ &= \|\{u_w\Gamma_w P_{W_w}(CC^{'})^{\frac{1}{2}}x\}_w+\{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\}_w- \\ &\qquad \{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\}_w\| \\ &\leq \|\{(u_w\Gamma_w P_{W_w}-v_w\Lambda_w P_{H_w})(CC^{'})^{\frac{1}{2}}x\}_w\| \\ &\qquad +\|\{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\}_w\| \\ &\leq (\lambda_1+1)\|\{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\}_w\| \\ &\qquad +\lambda_2\|\{u_w\Gamma_w P_{W_w}(CC^{'})^{\frac{1}{2}}x\}_w\|+\epsilon\|K^*x\|.\end{aligned}$$

So,

$$(1-\lambda_2)\|\{u_w\Gamma_w P_{W_w}(CC^{'})^{\frac{1}{2}}x\}_w\| \leq (\lambda_1+1)\sqrt{B}\|x\|+\epsilon\|K^*x\|.$$

Then

$$\begin{aligned}\|\{u_w\Gamma_w P_{W_w}(CC^{'})^{\frac{1}{2}}x\}_w\| &\leq \frac{(\lambda_1+1)\sqrt{B}\|x\|+\epsilon\|K^*x\|}{1-\lambda_2} \\ &\leq \Big(\frac{(\lambda_1+1)\sqrt{B}+\epsilon\|K\|}{1-\lambda_2}\Big)\|x\|.\end{aligned}$$

Hence

$$\Big\|\int_{\Omega} u_w^2\langle \Gamma_w P_{W_w}Cx,\Gamma_w P_{W_w}C^{'}x\rangle d\mu(w)\Big\| \leq \Big(\frac{(\lambda_1+1)\sqrt{B}+\epsilon\|K\|}{1-\lambda_2}\Big)^2\|x\|^2.$$

On the other hand, for each $x\in U$,

$$\begin{aligned}\Big\|\int_{\Omega} u_w^2\langle \Gamma_w P_{W_w}Cx,\Gamma_w P_{W_w}C^{'}x\rangle d\mu(w)\Big\|^{\frac{1}{2}} &= \|\{u_w\Gamma_w P_{W_w}(CC^{'})^{\frac{1}{2}}x\}_w\| \\ &= \|\{(u_w\Gamma_w P_{W_w}-v_w\Lambda_w P_{H_w})(CC^{'})^{\frac{1}{2}}x\}_w \\ &\qquad +\{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\}_w\| \\ &\geq \|\{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\}_w\| \\ &\qquad -\|\{(u_w\Gamma_w P_{W_w}-v_w\Lambda_w P_{H_w})(CC^{'})^{\frac{1}{2}}x\}_w\|\end{aligned}$$

$$\geq (1-\lambda_1)\|\{v_w\Lambda_w P_{H_w}(CC^{'})^{\frac{1}{2}}x\}_w\| - \lambda_2\|\{u_w\Gamma_w P_{W_w}(CC^{'})^{\frac{1}{2}}x\}_w\| - \epsilon\|K^*x\|.$$

Thus

$$\Big\|\int_\Omega u_w^2\langle \Gamma_w P_{W_w}Cx, \Gamma_w P_{W_w}C^{'}x\rangle d\mu(w)\Big\| \geq \Big(\frac{(1-\lambda_1)\sqrt{A}-\epsilon}{1+\lambda_2}\Big)^2\|K^*x\|^2. \qquad \square$$

Acknowledgement

It is our great pleasure to thank the referee for his careful reading of the paper and for several helpful suggestions.

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(Received 26.12.2021)

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