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A STUDY OF FANTASTIC FILTERS IN BL-ALGEBRAS

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Abstract. In this article, we first introduce the concept of the extension of a non-empty subset X in BL-algebras and by an example we see that this set is not a filter, in general. So, by adding a condition, we prove that the concept of the extension of X becomes a filter and with the help of an example, we show the necessity of this condition. Then we examine and study the concept of the extension of X completely. After that, we check this definition in different algebras such as integral BL-algebras, linear SBL-algebras, implication BL-algebras and MV-algebras. Also, with the help of this new concept, we obtain an equivalent property for fantastic filters, which makes checking MV-algebras faster. In fact, we prove some new properties for fantastic filters in BL-algebras. In addition, a new filter in BL-algebras is introduced and characterized. We state and prove some theorems to determine the relationships between this notion and the other types of filters in BL-algebras.

1. Introduction

Various problems in system identification involve characteristics that are essentially non-probabilistic in nature [17]. In response to this situation, L. A. Zadeh introduced in 1965 fuzzy set theory as an alternative to probability theory. His fundamental idea consists in understanding lattice-valued maps as generalized characteristic functions of some new kind of subsets, the so-called fuzzy sets, of a given universe. For historical reasons we quote the original definition (cf. [8] and [18]). Fuzzy logic grows as a new discipline from the necessity to deal with vague data and imprecise information caused by the indistinguishability of objects in certain experimental environments. As a set of mathematical tools, fuzzy logic is only using $[0,1]$-valued maps and certain binary operations $*$ on the real unit interval $[0,1]$ known also as left-continuous t-norms. It took some time to understand that partially ordered monoids of the form $([0,1],\leq,*)$ as algebras for $[0,1]$-valued interpretations of a certain type of non-classical logic, is the so-called monoidal logic. BL-algebras arise naturally in the analysis of the proof theory of propositional fuzzy logic. Indeed, Basic fuzzy logic (BL for short) and its corresponding BL-algebras were introduced by Hájek (see [9] and references therein) with the purpose of formalizing the many-valued semantics induced by the continuous t-norms on the real unit interval $[0,1]$. BL-algebras are the algebraic structures for Hájek's Basic logic [9]. BL-algebras rise as Lindenbaum algebras from certain logical axioms in a similar manner that Boolean algebras or MV-algebras do from the Classical logic or Lukasiewicz logic, respectively. Filters theory plays an important role in studying these logical algebras. From a logical point of view, various filters correspond to various sets of provable formulas. Hájek introduced the concepts of filters and prime filters in BL-algebras. Turunen studied some properties of the prime filters of BL-algebras in [15]. Haveshki et al. in [10] continued the algebraic analysis of BL-algebras and introduced (positive) implicative and fantastic filters of BL-algebras. After that we defined the notions of normal filters and obstinate filters in [2] and [4], respectively.

This paper aims to analyze the structure of BL-algebras by fantastic filters. In previous research, the concept of the fantastic filters has been used to classify BL-algebras in such a way that the researchers showed that a BL-algebra is MV if and only if the filter $\{1\}$ is fantastic. Since filters play a very important role in examining different structures of BL-algebras, in this article we tried to introduce and study a new filter for studying BL-algebras as much as possible. Our motivation was

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to study filters as possible in BL-algebras, to be able to obtain new relationships between filters as well as different structures of BL-algebras, including MV-algebras, Boolean algebras, etc. Therefore we defined the concept of D_s-filter and proved that every filter of MV-algebras is a D_s-filter. In this paper, we define a new concept and with its help we obtain new properties for fantastic filters that are used for the study of MV-algebras.

The structure of the paper is as follows:

In Section 2, we recall the basic definitions and put in evidence many rules of calculus in BL-algebras which we need in the rest of the paper. In this paper, we introduced the notion of the extension of a nonempty subset of BL-algebras and described it. Also, we obtained a new equivalence property for fantastic filters with the help of the new concept of extension of a set in BL-algebras. Finally, we defined the notion of the D_s-filter of BL-algebras and investigated some of its properties.

2. Preliminaries

Definition 2.1 ([9]). A BL-algebra is an algebra $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ with four binary operations $\wedge, \vee, *, \rightarrow$ and two constants $0, 1$ such that:
(BL_1) $(A, \wedge, \vee, 0, 1)$ is a bounded lattice $L(A)$,
(BL_2) $(A, *, 1)$ is a commutative monoid,
(BL_3) $*$ and $\rightarrow$ form an adjoint pair, i.e, $c \leq a \rightarrow b$ if and only if $a * c \leq b$, for all $a, b, c \in A$,
(BL_4) $a \wedge b = a * (a \rightarrow b)$,
(BL_5) $(a \rightarrow b) \vee (b \rightarrow a) = 1$.

It is easy to prove that if A is a BL-algebra and $x, y, z \in A$, we have the following rules of calculus (for more details see [6, 7, 9, 16]):
(BL_6) $x \leq y$ if and only if $x \rightarrow y = 1$,
(BL_7) $1 \rightarrow x = x$ and $x \leq y \rightarrow x$,
(BL_8) $x \rightarrow (y \rightarrow z) = (x * y) \rightarrow z = y \rightarrow (x \rightarrow z)$,
(BL_9) If $x \leq y$, then $y \rightarrow z \leq x \rightarrow z$, $z \rightarrow x \leq z \rightarrow y$, $x * z \leq y * z$ and $y^- \leq x^-$, where $x^- = x \rightarrow 0$,
(BL_{10}) $y \leq (y \rightarrow x) \rightarrow x$ and $x \vee y = ((x \rightarrow y) \rightarrow y) \wedge ((y \rightarrow x) \rightarrow x)$,
(BL_{11}) $x * y \leq x \wedge y \leq x, y \leq x \vee y$, $x \leq x^{--}$, $x^{---} = x^-$, $x * 0 = 0$ and $x * x^- = 0$,
(BL_{12}) $x \rightarrow y^- = y \rightarrow x^- = x^{--} \rightarrow y^- = (x * y)^-$,
(BL_{13}) $x * (y \vee z) = (x * y) \vee (x * z)$. Hájek [9] defined a filter of a BL-algebra A to be a nonempty subset F of A such that (i) $a, b \in F$ implies $a * b \in F$, and (ii) if $a \in F$, $a \leq b$, then $b \in F$. Turunen [15] defined a deductive system of a BL-algebra A to be a nonempty subset D of A such that (i) $1 \in D$ and (ii) $x \in D$ and $x \rightarrow y \in D$ imply $y \in D$. Note that a subset F of a BL-algebra A is a deductive system of A if and only if F is a filter of A [15].

Let $x \in A$ and F, G be the filters of A. We know that

$$[x) = \{a \in A : a \geq x^n, \text{ for some } n \in N\},$$
$$\langle F \cup [x) \rangle = \{a \in A : \ a \geq f * x^n, \text{ for some } f \in F, \ n \in N\},$$
$$\langle F \cup G \rangle = \{a \in A : \ a \geq f * g, \text{ for some } f \in F, \ g \in G\}.$$

Let F be a filter of a BL-algebra A. F is proper if $F \neq A$. A proper filter F of A is called a prime filter of A if for all $x, y \in A$, $x \vee y \in F$ implies $x \in F$ or $y \in F$. Equivalently, F is a prime filter of A if and only if for all $x, y \in A$, either $x \rightarrow y \in F$ or $y \rightarrow x \in F$. A filter of A is maximal if it is proper and not contained in any other proper filter of A. If M is a maximal filter of A and $x \notin M$, then $\langle M \cup [x) \rangle = A$. A proper filter F of a BL-algebra A is an obstinate filter if $x, y \notin F$ imply $x \rightarrow y \in F$ and $y \rightarrow x \in F$, [4].

Let F be a proper filter of A. The intersection of all maximal filters of A containing F is called the radical of F and it is denoted by $\mathrm{Rad}\,(F)$. We have proved that $\mathrm{Rad}\,(F) = \{a \in A : \ (a^n)^- \rightarrow a \in \ F, \text{ for all } n \in N\}$, for any filter F of A (for details, see [14]). It is clear that $F \subseteq \mathrm{Rad}\,(F)$, for any filter F of A. Throughout this paper, it is assumed that $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ (in short A) is a BL-algebra (unless we write otherwise).

3. Extension of a Set in BL-algebras

In this section, the concept of extension of a set in BL-algebras is introduced and also characterized.

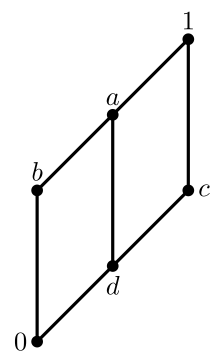
Definition 3.1. For any nonempty subset X of A, define an extension of X as the set $X^e = \{x \in A : x^- \leq a^-, \text{ for some } a \in X\}$. Note that X^e is not a filter, in general, see the following example.

Example. Let $A = \{0, a, b, c, d, 1\}$, where $0 < b < a < 1$ and $0 < d < a, c < 1$. Define $*$ and $\rightarrow$ by the following table.

$*$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	a	b	b	d	0	0
b	b	b	b	0	0	0
c	c	d	0	c	d	0
d	d	0	0	d	0	0
0	0	0	0	0	0	0

$\rightarrow$	1	a	b	c	d	0
1	1	a	b	c	d	0
a	1	1	a	c	c	d
b	1	1	1	c	c	c
c	1	a	b	1	a	b
d	1	1	a	1	1	a
0	1	1	1	1	1	1

The Hasse diagram of this table looks as follows:



Then $(A, \wedge, \vee, *, \rightarrow, 0, 1)$ is a BL-algebra, [11]. Clearly, $X^e = \{a\}^e = \{a, 1\}$ is not a filter of A. Also, $\{a, b\}^e = \{a, b, 1\}$, $\{b\}^e = \{b, 1\}$, $\{c\}^e = \{c, d, 1\}$, $\{d\}^e = \{d, 1\}$ and so, $\{a\}^e$, $\{b\}^e$, $\{c\}^e$, $\{d\}^e$ are not filters.

Remark 3.1. For any nonempty subset X of A, we have

$$x \in X^e \Leftrightarrow x^- \leq a^-, \text{ for some } a \in X;$$
$$\Leftrightarrow x^- * a = 0, \text{ for some } a \in X, \text{by } (BL_3).$$

Therefore

$$X^e = \{x \in A : x^- * a = 0, \text{ for some } a \in X\}.$$

In the following, we add a condition that X^e becomes a filter.

Theorem 3.1. *Let X be a nonempty subset of $A \setminus \{0\}$, which is closed under " $*$ ". Then X^e is a filter of A.*

Proof. Assume that $a \leq b$, for $a, b \in A$ such that $a \in X^e$. Then there exists $x \in X$ such that $a^- * x = 0$. So, $b^- * x = 0$ and therefore $b \in X^e$. Now, let $a, b \in X^e$. Thus there exist $x, y \in X$ such that $a^- * x = b^- * y = 0$. Hence $x \leq a^{--}$ and $y \leq b^{--}$. Then $x * y \leq a^{--} * b^{--} = (a * b)^{--}$ and so, $(a * b)^- * (x * y) = 0$, for $x * y \in X$. Therefore $a * b \in X^e$. Thus X^e is a filter of A. □

Remark 3.2. Note that in Example 3, X is not closed under " $*$ ", and X^e is not a filter of A. Therefore the condition of being closed under " $*$ " is necessary so that X^e becomes a filter of A.

Let X be a nonempty subset of A. The set of double complemented elements X is denoted by $D(X) = \{x \in A : x^{--} \in X\}$ (see [3]). An element a of A is said to be dense if and only if $a^- = 0$. We denote by $D_s(A)$ the set of the dense elements of A.

The following theorem reveals some basic properties of X^e.

Theorem 3.2. *For any two nonempty subsets X and Y of A, we have the following:*

(1) $X \subseteq X^e = D(X^e)$.

(2) $D_s(A) \subseteq X^e = (X^e)^e$, *so* $X \subseteq (X^e)^e$.

(3) *if* $X \subseteq Y$ *then* $X^e \subseteq Y^e$.

(4) $(X \cap Y)^e \subseteq X^e \cup Y^e = (X \cup Y)^e$.

(5) $0 \notin X$ *if and only if* $0 \notin X^e$.

(6) $X^e = \{a \in A : (a \vee x)^- = a^-,$ *for some* $x \in X\} = \{a \in A : (a \wedge x)^- = x^-,$ *for some* $x \in X\} = \{a \in A : x \leq a^{--},$ *for some* $x \in X\}$.

Proof. (1) For any $x \in X$, we have $x^- \leq x^-$. Hence $x \in X^e$. Therefore $X \subseteq X^e$. Now, we have

$$\begin{aligned} D(X^e) &= \{x \in A : x^{--} \in X^e\}; \\ &= \{x \in A : x^{---} \leq a^-, \text{ for some } a \in X\}; \\ &= \{x \in A : x^- \leq a^-, \text{ for some } a \in X\} = X^e. \end{aligned}$$

(2) Let $a \in D_s(A)$. Then $0 = a^- \leq x^-$, for any $x \in X$. And so, $a \in X^e$, i.e., $D_s(A) \subseteq X^e$.

Now, by part (1), we have $X^e \subseteq (X^e)^e$. Again, let $x \in (X^e)^e$. Then, $x^- \leq a^-$, for some $a \in X^e$. Hence $a^- \leq c^-$, for some $c \in X$. Hence $x^- \leq a^- \leq c^-$ and $c \in X$. Thus $x \in X^e$, i.e., $(X^e)^e \subseteq X^e$. Therefore $(X^e)^e = X^e$. Now, by using (1), we get $X \subseteq (X^e)^e$.

(3) Suppose $X \subseteq Y$. Let $x \in X^e$. Then we obtain $x^- \leq a^-$, for some $a \in X \subseteq Y$. Hence it yields $x \in Y^e$. Therefore $X^e \subseteq Y^e$.

(4) We know $X \cap Y \subseteq X, Y$. So, by (3), we get $(X \cap Y)^e \subseteq X^e$ and $(X \cap Y)^e \subseteq Y^e$. Hence $(X \cap Y)^e \subseteq X^e \cup Y^e$.

Now, we know $X, Y \subseteq X \cup Y$. So, from (3), $X^e \cup Y^e \subseteq (X \cup Y)^e$. Now, let $a \in (X \cup Y)^e$. Thus we obtain $a^- \leq x^-$, for some $x \in X \cup Y$. Hence we have three cases: $x \in X$, or $x \in Y$, or $x \in X \cap Y$.

Case (1), if $a^- \leq x^-$, for some $x \in X$. This implies that $a \in X^e$. And so, $a \in X^e \cup Y^e$. Case (2), if $a^- \leq x^-$, for some $x \in Y$. This implies that $a \in Y^e$. And so, $a \in X^e \cup Y^e$. Case (3), if $a^- \leq x^-$, for some $x \in X \cap Y$. This implies that $a \in (X \cap Y)^e$. As $(X \cap Y)^e \subseteq X^e \cup Y^e$, we get $a \in X^e \cup Y^e$. Hence $(X \cup Y)^e \subseteq X^e \cup Y^e$. Therefore $(X \cup Y)^e = X^e \cup Y^e$.

(5) Assume that $0 \notin X$ and $0 \in X^e$. Then there exists $x \in X$ such that $1 = 0^- \leq x^-$. Hence $x^{--} \leq 1^- = 0$, i.e. $x^{--} = 0$. So, from (BL_{11}), we have $x \leq x^{--}$, and thus $x = 0$. That is a contradiction, since $0 \notin X$. Therefore $0 \notin X^e$.

Conversely, the proof is straightforward, by using (1).

(6) We have

$$\begin{aligned} X^e &= \{a \in A : a^- \leq x^-, \text{ for some } x \in X\}; \\ &= \{a \in A : a^- \wedge x^- = a^-, \text{ for some } x \in X\}; \\ &= \{a \in A : (a \vee x)^- = a^-, \text{ for some } x \in X\}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} X^e &= \{a \in A : a^- \leq x^-, \text{ for some } x \in X\}; \\ &= \{a \in A : a^- \vee x^- = x^-, \text{ for some } x \in X\}; \\ &= \{a \in A : (a \wedge x)^- = x^-, \text{ for some } x \in X\}. \end{aligned}$$

By using (BL_3), we have

$$\begin{aligned} X^e &= \{a \in A : a^- \leq x^-, \text{ for some } x \in X\}; \\ &= \{a \in A : a^- * x = 0, \text{ for some } x \in X\}; \\ &= \{a \in A : x \leq a^{--}, \text{ for some } x \in X\}. \end{aligned}$$

□

In the following examples we show that the inverse inclusion of Theorem 3.2(1) may not hold, in general.

Example. Let $A = \{0, a, b, 1\}$, where $0 < a < b < 1$. Define $*$ and $\to$ as follows:

$*$	0	a	b	1
0	0	0	0	0
a	0	0	a	a
b	0	a	b	b
1	0	a	b	1

$\to$	0	a	b	1
0	1	1	1	1
a	a	1	1	1
b	0	a	1	1
1	0	a	b	1

Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra [11]. Take $X = \{1\}$. It is clear that $X^e = \{b, 1\}$. So, $X^e \nsubseteq X$.

A BL-algebra A is called an implication BL-algebra if $(x \to y) \to x = x$, for all $x, y \in A$ such that $y \neq 0$, see [1].

Theorem 3.3. *Let A be an implication BL-algebra, X be a nonempty subset of A and $0 \notin X$. Then $X^e = A \setminus \{0\}$.*

Proof. Let A be an implication BL-algebra. Then by using Theorem 4.6 [1], we get $D_s(A) = A \setminus \{0\}$. By Theorem 3.2(2), we have $D_s(A) \subseteq X^e$. As $0 \notin X$ and by applying Theorem 3.2(5), we conclude that $X^e \subseteq A \setminus \{0\}$. Therefore $D_s(A) = A \setminus \{0\} \subseteq X^e \subseteq A \setminus \{0\}$, i.e., $X^e = A \setminus \{0\}$. □

We recall that a SBL-algebra A is a BL-algebra that satisfies $x^- \wedge x = 0$, for all $x \in A$.

Lemma 3.1. *Let A be a linear SBL-algebra, X be a subset of A and $0 \notin X$. Then $X^e = A \setminus \{0\}$.*

Proof. Let A be a linear SBL-algebra. Then by part (2) of Proposition 3.3 [13], A is a special BL-algebra. So, $D_s(A) = A \setminus \{0\}$, it follows from Proposition 3.1 [13]. By the proof of Theorem 3.3, $X^e = A \setminus \{0\}$. □

A BL-algebra A is called an integral BL-algebra if $x * y = 0$, then $x = 0$ or $y = 0$, for all $x, y \in A$. A proper filter F of a BL-algebra A is called an integral filter if for all $x, y \in A$, $(x * y)^- \in F$ implies $x^- \in F$ or $y^- \in F$ (see [5]).

Lemma 3.2. *Let A be an integral BL-algebra, X be a subset of A and $0 \notin X$. Then*
(i) *$X^e = D_s(A)$.*
(ii) *if X is closed under $*$, X^e is a prime filter.*
(iii) *F^e is a prime filter for any proper filter F of A.*

Proof. (i) Let $x \in X^e$. Then $x^- \leq a^-$, for some $a \in X$. Hence $x^- * a \leq 0$, i.e., $x^- * a = 0$. As A is an integral, we get $x^- = 0$ or $a = 0$. If $a = 0$, then $0 \in X$. That is a contradiction. Hence $x^- = 0$. Then it yeilds $x \in D_s(A)$. Therefore $X^e \subseteq D_s(A)$. By Theorem 3.2(2), we have $D_s(A) \subseteq X^e$. Thus the proof is complete.

(ii) X^e is a proper filter of A. Assume that $a \vee b \in X^e$, for $a, b \in A$. Then there exists $x \in X$ such that $(a \vee b)^- * x = 0$. Thus $(a^- * x) \wedge (b^- * x)(a^- \wedge b^-) * x = 0$. Hence $(a^- * x) * (b^- * x) = 0$. As A is an integral BL-algebra, $a^- * x = 0$ or $b^- * x = 0$. Therefore $a \in X^e$ or $b \in X^e$.

(iii) It is clear by Part (ii). □

Lemma 3.3. *Let F be an integral filter of A. Then $F^e \subseteq D(F)$.*

Proof. Assume that $x \in F^e$. Then there exists $a \in F$ such that $x^- * a = 0$. So, $(x^- * a)^- \in F$ and thus $a^- \in F$ or $x^{--} \in F$. Since $a \in F$, $a^- \notin F$ and therefore $x \in D(F)$. □

In the following lemma, we study the image and inverse image of the extension of a nonempty subset under a BL-homomorphism:

Lemma 3.4. *Let $f : A \longrightarrow B$ be a homomorphism of BL-algebras and $\varnothing \neq X \subseteq A$ and $\varnothing \neq Y \subseteq B$. Then we have:*
(1) *if $f^{-1}(Y) \neq \emptyset$, $(f^{-1}(Y))^e \subseteq f^{-1}(Y^e)$;*

(2) $f(X^e) \subseteq f(X)^e$;
(3) *if* f *is a monomorphism, then* $f(X^e) = f(X)^e$;
(4) *if* f *is a monomorphism and* $Y \subseteq f(A)$, *then* $f^{-1}(Y^e) = (f^{-1}(Y))^e$.

Proof. (1) Let $y \in (f^{-1}(Y))^e$. Then $y^- \leq b^-$, for some $b \in f^{-1}(Y)$. Thus we obtain $f(y^-) \leq f(b^-)$ and $f(b) \in Y$. Hence $f(y)^- \leq f(b)^-$ and $f(b) \in Y$, i.e., $f(y) \in Y^e$. And so, $y \in f^{-1}(Y^e)$. Therefore $(f^{-1}(Y))^e \subseteq f^{-1}(Y^e)$.

(2) Let $b \in f(X^e)$. So, there exists $x \in X^e$ such that $b = f(x)$. Since $x \in X^e$, we get $x^- \leq a^-$, for some $a \in X$. Hence $f(x)^- \leq f(a)^-$ and $f(a) \in f(X)$. Thus it yields $f(x) \in f(X)^e$ and so, $b \in f(X)^e$. Therefore $f(X^e) \subseteq f(X)^e$.

(3) Suppose f is a monomorphism. Let $x \in f(X)^e$. So, there exists $a \in f(X)$ such that $x^- \leq a^-$. As $a \in f(X)$, there exists $b \in X$ such that $a = f(b)$. Hence $x^- \leq f(b)^-$ and so, $f^{-1}(x^-) \leq f^{-1}f(b^-)$. As f is a monomorphism, we get $f^{-1}(x^-) \leq b^-$. Thus $(f^{-1}(x))^- \leq b^-$, for some $b \in X$. It yields $f^{-1}(x) \in X^e$ and so, $x \in f(X^e)$, and hence $f(X)^e \subseteq f(X^e)$. Therefore by part (2), we get $f(X^e) = f(X)^e$.

(4) Let $y \in f^{-1}(Y^e)$. Then $f(y) \in Y^e$, i.e., $f(y)^- \leq b^-$, for some $b \in Y$. Hence $f^{-1}f(y^-) \leq (f^{-1}(b))^-$. Since f is a monomorphism, therefore $y^- \leq (f^{-1}(b))^-$. Now, as $b \in Y$, we get $f^{-1}(b) \in f^{-1}(Y)$. Therefore $y \in (f^{-1}(Y))^e$, i.e., $f^{-1}(Y^e) \subseteq (f^{-1}(Y))^e$. And so, by part (1), $(f^{-1}(Y))^e = f^{-1}(Y^e)$. □

In the next proposition, we study the extension of some sets with special properties.

Proposition 3.1. *Let* X *be a nonempty subset of* A. *Then*
(i) $1 \in X^e$;
(ii) $0 \in X^e$ *if and only if for some* $a \in X$, $a^- = 1$;
(iii) *if* X *is closed under* $\wedge$, *then* X^e *is closed under* $\wedge$;
(iv) *if* X *is closed under* $\vee$, *then* X^e *is closed under* $\vee$;
(v) *if* X *is closed under* $*$, *then* X^e *is closed under* $*$.

Proof. (i), (ii) These parts are easy.

(iii), (iv) Let X be closed under $\wedge$, $\vee$ and $a, b \in X^e$, for $a, b \in A$. Then for some $x, y \in X$, $a^- \leq x^-$ and $b^- \leq y^-$. Hence

$$(a \vee b)^- = a^- \wedge b^- \leq x^- \wedge y^- = (x \vee y)^-;$$
$$(a \wedge b)^- = a^- \vee b^- \leq x^- \vee y^- = (x \wedge y)^-.$$

So, by the hypothesis, $x \wedge y \in X^e$ and $x \vee y \in X^e$.

(v) According to Theorem 3.1, this part is clear. □

4. Extension of a Filter

In this section, we study the extension of filters in BL- algebras with the aim of a more detailed study of BL- algebras.

In Example 3, we show that for any nonempty subset X of A, X^e is not a filter of A, in general. In the following, we prove that for any filter F of A, F^e is a filter.

Theorem 4.1. *For any filter* F *of* A, F^e *is a filter of* A.

Proof. Clearly, $1 \in F^e$. Let $x, y \in F^e$, we have to show that $x * y \in F^e$. As $x, y \in F^e$, there exist $a, b \in F$ such that $x^- \leq a^-$ and $y^- \leq b^-$. Hence $x^{--} \to y^- \leq x^{--} \to b^-$ and so, by (BL_{12}), $(x*y)^- \leq (x^{--}*b)^-$, (I). By $x^- \leq a^-$, we have $a^{--} \leq x^{--}$. Also, by $a \in F$ and $a \leq a^{--}$, we obtain $a^{--} \in F$. And so, $x^{--} \in F$. Hence as $b \in F$, we get $x^{--} * b \in F$. Therefore by (I), $x * y \in F^e$. Now, let $x \in F^e$ and $x \leq y$. Then $y^- \leq x^- \leq a^-$, for some $a \in F$. Hence $y \in F^e$. Therefore F^e is a filter of A. □

Note. In Example 3, $\{a, b\}^e = \{a, b, 1\}$ is a filter and, clearly, $\{a, b\}$ is not a filter of A.

By Theorem 3.2(5), we can obtain the following

Corollary 4.1. *Let F be a filter of A. Then F is a proper filter of A if and only if F^e is a proper filter of A.*

Proof. Let F be a filter of A and $0 \in F$. As $0^- \leq 0^-$, then $0 \in F^e$. Now, let $0 \in F^e$. Then for some $a \in F$, $0^- \leq a^-$ and so, $a^- = 1$. As $a \in F$, then $a^{--} \in F$. Therefore $0 \in F$, since $a^{--} = 0$. □

A BL-algebra A is called an MV-algebra if $x^{--} = x$, for all $x \in A$. The MV-center of a BL-algebra A denoted by $MV(A)$, is defined as

$$MV(A) = \{x \in A : x^{--} = x\}.$$

The following theorem reveals some basic properties of F^e.

Theorem 4.2. *For any two filters F and G of A and $x, a \in A$, we have the following:*
(1) $(F \cap G)^e = F^e \cap G^e$.
(2) $\{1\}^e = D_s(A)$ *and* $A^e = A = \{0\}^e$.
(3) $F^e \cap MV(A) \subseteq F$.
(4) $\{1\}^e \cap MV(A) = \{1\}$, *and so* $D_s(A) \cap MV(A) = \{1\}$.
(5) $x^- \in F^e$ *if and only if* $x^- \in F$.
(6) $x \in F^e$ *implies* $x^{--} \in F$.
(7) $\{a\}^e = \{x \in A : a \leq x^{--}\}$.
(8) $[a)^e = \{x \in A : x^- \leq (a^n)^-, \exists n \in N\} = \{x \in A : a^n \leq x^{--}, \exists n \in N\}$.
(9) $x \in [a)^e$ *if and only if* $x^{--} \in [a)$.
(10) $F = \cup_{a \in F}[a)$, *so,* $F^e = \cup_{a \in F}[a)^e$.
(11) $(D_s(A))^e = D_s(A)$.

Proof. (1) By Theorem 3.2(3), we have $(F \cap G)^e \subseteq F^e \cap G^e$. Conversely, let $x \in F^e \cap G^e$. Then $x^- \leq a^-$ and $x^- \leq b^-$, for some $a \in F$ and $b \in G$. Hence $a^{--} \leq x^{--}$ and $b^{--} \leq x^{--}$, where $a^{--} \in F$ and $b^{--} \in G$. Hence $a^{--} \vee b^{--} \leq x^{--}$. Thus $x^- \leq (a^{--} \vee b^{--})^-$ and $a^{--} \vee b^{--} \in F \cap G$. So, $x \in (F \cap G)^e$, i.e., $F^e \cap G^e \subseteq (F \cap G)^e$. Therefore $(F \cap G)^e = F^e \cap G^e$.

(2) We have

$$\{1\}^e = \{x \in A : x^- \leq 1^- = 0\} = \{x \in A : x^- = 0\} = D_s(A).$$

For any $x \in A$, $x^- \leq 0^- = 1$. Hence $x \in A^e$, so, $A \subseteq A^e$. Therefore $A^e = A$. Now, we know that

$$\{0\}^e = \{a \in A : \ a^- \leq 0^-\} = \{a \in A : \ a^- \leq 1\} = A.$$

(3) Let $x \in F^e \cap MV(A)$. Then $x^- \leq a^-$, for some $a \in F$. And so, $a^{--} \leq x^{--}$, $a^{--} \in F$. Hence $x^{--} \in F$. As $x \in MV(A)$, we get $x^{--} = x$. Therefore $x \in F$, i.e., $F^e \cap MV(A) \subseteq F$.

(4) The proof is clear by parts (2) and (3).

(5) Let $x^- \in F^e$. Then $x^{--} \leq a^-$, for some $a \in F$. So, $a^{--} \leq x^{---} = x^-$ and $a^{--} \in F$. Therefore $x^- \in F$.

Conversely, the proof is clear.

(6) Let $x \in F^e$. Then $x^- \leq a^-$, for some $a \in F$. So, $a^{--} \in F$ and $a^{--} \leq x^{--}$. And thus $x^{--} \in F$.

(7) Applying (BL_3), we have

$$\begin{aligned}\{a\}^e &= \{x \in A : \ x^- \leq a^-\} = \{x \in A : \ x^- * a \leq 0\}, \\ &= \{x \in A : \ a * x^- \leq 0\} = \{x \in A : a \leq x^{--}\}.\end{aligned}$$

(8) Assume that $x \in [a)^e$. Then using (BL_3), we have

$$\begin{aligned} x^- \leq b^-, \ \exists b \in [a) &\Rightarrow x^- \leq b^- \text{ and } b \geq a^n, \ \exists n \in N; \\ &\Rightarrow x^- \leq b^- \leq (a^n)^-, \ \exists n \in N; \\ &\Rightarrow x^- \leq (a^n)^-, \ \exists n \in N; \\ &\Leftrightarrow x^- * a^n \leq 0, \ \exists n \in N; \\ &\Leftrightarrow a^n \leq x^{--}, \ \exists n \in N. \end{aligned}$$

Therefore

$$[a)^e \subseteq \{x \in A : x^- \leq (a^n)^-, \exists n \in N\} = \{x \in A : a^n \leq x^{--}, \exists n \in N\}.$$

Conversely, let $x^- \leq (a^n)^-$, for some $n \in N$. We know $a \in [a)$ and so, by a filter property of $[a)$, we get $a^n \in [a)$, for all $n \in N$. Hence $x \in [a)^e$, i.e.,

$$\{x \in A : x^- \leq (a^n)^-, \exists n \in N\} \subseteq [a)^e.$$

This completes the proof.

(9) Let $x^{--} \in [a)$. Then $x^{--} \geq a^n$, for some $n \in N$. So, $x^- \leq (a^n)^-$, for some $n \in N$. As $a^n \in [a)$, for all $n \in N$, we get $x \in [a)^e$. Conversely is proved in (6), so, the proof is complete.

(10) Let $b \in F$. Then $b \in [b) \subseteq \cup_{a \in F}[a)$. So, $F \subseteq \cup_{a \in F}[a)$. Now, let $x \in \cup_{a \in F}[a)$. So, there exists $b \in F$ such that $x \in [b)$. Thus $x \geq b^n$, for some $n \in N$. As $b \in F$, we conclude that $b^n \in F$, for all $n \in N$. Hence $x \in F$, i.e., $\cup_{a \in F}[a) \subseteq F$. Therefore $F = \cup_{a \in F}[a)$. Now, applying Theorem 3.2(4), we conclude that $F^e = \cup_{a \in F}[a)^e$.

(11) Let $x \in (D_s(A))^e$, for any $x \in A$. Then $x^- \leq a^-$, for some $a \in D_s(A)$. So, $a^- = 0$, and hence $x^- = 0$, i.e., $x \in D_s(A)$. Therefore $(D_s(A))^e \subseteq D_s(A)$. By Theorem 3.2(1), we get $(D_s(A))^e = D_s(A)$. □

If F is a filter of A, then the relation $\sim_F$ defined on A by $(x, y) \in \sim_F$ if and only if $x \to y \in F$ and $y \to x \in F$ is a congruence relation on A. The quotient algebra $A/\sim_F$ denoted by A/F becomes a BL-algebra in a natural way, with the operations induced from those of A. So, the order relation on A/F is given by $x/F \leq y/F$ if and only if $x \to y \in F$. Hence $x/F = 1/F$ if and only if $x \in F$ and $x/F = 0/F$ if and only if $x^- \in F$.

In the following, we characterize fantastic filters.

Theorem 4.3. *Let F be a filter of A. Then the following statements are equivalent:*

(1) $F = F^e$;
(2) $x^{--} \in F$ *implies* $x \in F$, *for all* $x \in A$;
(3) *For* $x, y \in A$, $x^- = y^-$ *and* $x \in F$ *imply that* $y \in F$;
(4) $D(F) = F$;
(5) $D_s(A/F) = \{1/F\}$.
(6) F *is a fantastic filter.*

Proof. (1) ⇒ (2) Let $F = F^e$ and $x^{--} \in F$, for any $x \in A$. Hence $x^{--} \in F^e$. So, there exists $a \in F$ such that $x^- = (x^{--})^- \leq a^-$. Thus $x \in F^e$, and by the hypothesis, we get $x \in F$.

(2) ⇒ (3) Let $x^- = y^-$ and $x \in F$, for any $x, y \in A$. So, $x^{--} \in F$, since $x \leq x^{--}$. By the hypothesis, we get $x^{--} = y^{--}$, so, $y^{--} \in F$. Hence by part (2), $y \in F$.

(3) ⇒ (1) Let $x \in F^e$. Then $x^- \leq a^-$, for some $a \in F$. And so, $a^- = x^- \vee a^- = (x \wedge a)^-$. Hence by part (3) and $a \in F$, we obtain $x \wedge a \in F$. By $x \wedge a \leq x$, we get $x \in F$. And so, $F^e \subseteq F$. Therefore by Theorem 3.2(1), $F = F^e$.

(2) ⇒ (4) Let $x \in D(F)$. Then $x^{--} \in F$. So, by (2), we get $x \in F$. Hence $D(F) \subseteq F$. Now, let $x \in F$. Since $x \leq x^{--}$, we get $x^{--} \in F$. So, $x \in D(F)$. And thus $F \subseteq D(F)$. Therefore $D(F) = F$.

(4) ⇒ (2) Let $D(F) = F$ and $x^{--} \in F$, for any $x \in A$. Then $x \in D(F)$ and by the hypothesis, we get $x \in F$.

(4) ⇒ (5) Let $D(F) = F$ and $x/F \in D_s(A/F)$. Then $(x/F)^- = 0/F$, so, $x^{--} \in F$. Thus $x \in D(F)$. Hence by part (4), we get $x \in F$, i.e., $x/F = 1/F$. Therefore $D_s(A/F) = \{1/F\}$.

(5) ⇒ (4) Let $D_s(A/F) = \{1/F\}$ and $x \in D(F)$. Then $x^{--} \in F$, so, $(x/F)^{--} = 1/F$. Thus $(x/F)^- = 0/F$, i.e., $x/F \in D_s(A/F)$. Hence $x/F = 1/F$ and so, $x \in F$. Therefore $D(F) \subseteq F$. As $F \subseteq D(F)$, we get $F = D(F)$.

(2) ⇔ (6) This part is proved in Lemma 1 [12]. □

Lemma 4.1. *Let F be a proper filter of A such that $F = F^e$ and $x^{--} = 1$, for any $x \in A - \{0\}$. Then $F = A - \{0\}$.*

Proof. Let $x \in A - \{0\}$. Then by the hypothesis, we get $x^{--} = 1 \in F$. So, from Theorem 4.3, $x \in F$. Hence $A \setminus \{0\} \subseteq F$. As F is a proper filter, we conclude that $F \subseteq A - \{0\}$. Thus we obtain $F = A - \{0\}$. □

Remark 4.1. According to (2) $\Leftrightarrow$ (6) of Theorem 4.3 and Theorem 4.4 [10], if F and G are filters of A, $F = F^e$ such that $F \subseteq G$, then $G = G^e$.

Proposition 4.1. *Let F be a filter of A and $[x)^e = [x)$, for some $x \in F$. Then $F^e = F$.*

Proof. Let $[x)^e = [x)$, for some $x \in F$. By Theorem 4.2(10), we have $F = \cup_{a\in F}[a)$. Hence $[x) \subseteq F$. Thus applying Remark 4.1 and $[x) \subseteq F$, we conclude that $F^e = F$. □

Theorem 4.4. *Let F be an integral filter and fantastic filter of A. Then F is an obstinate filter of A.*

Proof. Let $x, y \notin F$, for $x, y \in A$. We will show that $x \to y \in F$ and $y \to x \in F$. We have $(x * x^-)^- = 1 \in F$. As F is an integral filter, we get $x^- \in F$ or $x^{--} \in F$. If $x^{--} \in F$, then $x \in F$. That is a contradiction, so $x^- \in F$. Now, by (BL_9), $0 \leq y$, so, $x \to 0 \leq x \to y$ and since F is a filter and $x^- \in F$, we get $x \to y \in F$. In a similar way, we can prove that $y \to x \in F$. Therefore F is an obstinate filter of A. □

Now, by Theorem 4.4 and Lemma 4.2 [14], we conclude the following

Corollary 4.2. *Let F be an integral filter and fantastic filter of A. Then F is a semi-maximal filter of A.*

Now, by Proposition 4.6 [4] and Theorems 4.4, 4.18 [5], we conclude the following

Theorem 4.5. *Let F be a filter of A. Then the following conditions are equivalent:*
(1) *F is a maximal and positive implicative filter,*
(2) *F is a maximal and implicative filter,*
(3) *F is an obstinate filter,*
(4) *F is an integral and fantastic filter.*

Definition 4.1. A filter F of A is called a D_s-filter of A if $D_s(A) \subseteq F$.

The following example shows that D_s-filter in BL-algebras exists and any filter may not be D_s-filter.

Example. Let $A = \{0, a, b, c, d, 1\}$, where $0 < a < c < 1$ and $0 < b < c, d < 1$. Define $*$ and $\to$ as follows:

$*$	0	a	b	c	d	1
0	0	0	0	0	0	0
a	0	a	0	a	0	a
b	0	0	b	b	b	b
c	0	a	b	c	b	c
d	0	0	b	b	d	d
1	0	a	b	c	d	1

$\to$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	d	1	d	1	d	1
b	a	a	1	1	1	1
c	0	a	d	1	d	1
d	a	a	c	c	1	1
1	0	a	b	c	d	1

Then $(A, \wedge, \vee, *, \to, 0, 1)$ is a BL-algebra [11], and clearly, $D_s(A) = \{c, 1\}$. Hence $F = \{a, c, 1\}$ is a D_s-filter and $G = \{d, 1\}$ is not a D_s-filter of A.

The following theorem and corollary are direct consequences of Definition 4.1.

Theorem 4.6. *Let F and G be two filters of A and F be a D_s-filter of A. If $F \subseteq G$, then G is a D_s-filter of A.*

Corollary 4.3. *Let F and G be two D_s-filters of A. Then we have:*
(1) *$F \cap G$ is a D_s-filter of A.*
(2) *$\langle F \cup G\rangle$ is a D_s-filter of A.*
(3) *$\langle F \cup \{x\}\rangle$ is a D_s-filter of A, for each $x \in A$.*
(4) $\mathrm{Rad}\,(F)$ *is a D_s-filter of A.*

Theorem 4.7. *Every fantastic filter of any BL-algebra is a D_s-filter.*

Proof. Let F be a fantastic filter of A. Then $F^e = F$. Hence by Theorem 3.2(2), $D_s(A) \subseteq F^e$. And so, $D_s(A) \subseteq F$, i.e., F is a D_s-filter of A. □

Corollary 4.4. *Let F be a filter of A. Then F^e is a D_s-filter of A.*

Proof. By Theorem 3.2(2) and Theorem 4.7, the proof is easy. □

Now, by Theorems 4.7 and 4.3, we conclude the following

Corollary 4.5. *We have*
(1) *Every fantastic filter is a D_s-filter.*
(2) *Every positive implicative filter is a D_s-filter.*
(3) *Every obstinate filter is a D_s-filter.*
(3) *Every maximal filter is a D_s-filter.*

Corollary 4.6. *Any filter of Boolean algebra A is a D_s-filter of A.*

Proof. The proof is straightforward from Corollary 4.5(2). □

We consider the set $F^e(A) = \{F \in F(A) : \ F = F^e\}$. By Theorem 4.3, $F^e(A) = \{F \in F(A) : \ F$ is a fantastic filter of $A\}$.

Corollary 4.7. *$D_s(A)$ is the smallest filter of $F^e(A)$.*

Proof. Let $F \in F^e(A)$ and $F \subsetneq D_s(A)$. Then there exists $a \in D_s(A)$, such that $a \notin F$. As $F = F^e$, so, $a \notin F^e$. Hence $a^- \nleq b^-$, for all $b \in F$. Also, as $a \in D_s(A)$, we get $a^- = 0$. Thus $0 = a^- \nleq b^-$, for all $b \in F$, which is a contradiction. Therefore $F = D_s(A)$. □

Theorem 4.8. *Every filter of MV-algebra A is a D_s-filter.*

Proof. Since any filter in MV-algebra is fantastic; by Theorem 4.7, the proof is completed. □

Theorem 4.9. *Let F be a proper filter of A and A/F be an MV-algebra. Then F is a D_s-filter of A.*

Proof. By the hypothesis, F is a fantastic filter and so, by Theorem 4.7, the proof is completed. □

Proposition 4.2. *Let $f : A \longrightarrow B$ be a monomorphism of BL-algebras and F be a D_s-filter of B. Then $f^{-1}(F)$ is a D_s-filter of A.*

Proof. Let F be a D_s-filter of B. Then $D_s(B) \subseteq F$, so, $f^{-1}(D_s(B)) \subseteq f^{-1}(F)$. We have to show that $f^{-1}(D_s(B)) = D_s(A)$. Let $x \in f^{-1}(D_s(B))$, for $x \in A$. Then $f(x) \in D_s(B)$, so, $f(x)^- = 0_B = f(0_A)$. Hence $f^{-1}f(x^-) = f^{-1}f(0_A)$. As f is a monomorphism, we get $x^- = 0_A$. And thus $x \in D_s(A)$, i.e., $f^{-1}(D_s(B)) \subseteq D_s(A)$. Now, let $x \in D_s(A)$. Then $x^- = 0_A$, so, $f(x)^- = 0_B$. Thus $f(x) \in D_s(B)$, i.e., $x \in f^{-1}(D_s(B))$. Therefore $D_s(A) \subseteq f^{-1}(D_s(B))$, and hence $f^{-1}(D_s(B)) = D_s(A)$. Thus as $f^{-1}(D_s(B)) \subseteq f^{-1}(F)$, we get $D_s(A) \subseteq f^{-1}(F)$, i.e., $f^{-1}(F)$ is a D_s-filter of A. □

5. Concluding Remarks and Future Works

BL-algebras have the most important algebraic structure among all the various logical algebras that have been proposed as the semantic systems of non-classical logical systems. Also, they include some important classes of algebras, like the MV. In this article, we tried to take a step towards a more detailed study of BL-algebras by presenting new concepts. In this paper, we introduced the concept of the extension of a nonempty subset X, (X^e), in BL-algebras and we checked this definition in different algebras, such as integral BL-algebras, linear SBL-algebras, implication BL-algebras and MV-algebras. In addition, we have provided the conditions for a filter to be fantastic and obtained some conditions equivalent to them. In fact, we obtained interesting equivalence properties for easier investigation of fantastic filters in BL-algebras and so, we were able to find an easier way to study MV-algebras. Also, we have considered D_s-filters in BL-algebras. The results of this paper will be devoted to studying the local BL-algebras, perfect BL-algebras and SBL-algebras which are different extensions of Basic Logic. And since BL-algebras, MV-algebras and lattice implication algebras are closely related, all results in this paper will contribute much to studying ideals and filters (or, deductive systems) of MV-algebras, lattice implication algebras and related algebraic systems.

Some issues for future work are:

– Study the relationship between X^e and other types of filters in BL-algebras.
– Introducing new topologies on BL-algebras based on X^e.
– Define and study X^e for sets with different properties.
– Introducing a new subclass of BL-algebras.

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References

1. A. Borumand Saeid, N. Mohtashamnia, Implication BL-algebras. *J. Intell. Fuzzy Systems* **26** (2014), no. 3, 1371–1381.
2. A. Borumand Saeid, S. Motamed, Normal filters in BL-algebras. *World Applied Sci. (Special Issue Appl. Math).* **7** (2009), 70–76.
3. A. Borumand Saeid, S. Motamed, Some results in BL-algebras. *MLQ Math. Log. Q.* **55** (2009), no. 6, 649–658.
4. A. Borumand Saeid, S. Motamed, A new filter in BL-algebras. *J. Intell. Fuzzy Systems* **27** (2014), no. 6, 2949–2957.
5. R. A. Borzooei, A. Paad, Integral filters and integral BL-algebras. *Ital. J. Pure Appl. Math.* **30** (2013), 303–316.
6. D. Busneag, D. Piciu, BL-algebra of fractions relative to an $\wedge$-closed system. *An. Ştiinţ. Univ. Ovidius Constanţa Ser. Mat.* **11** (2003), no. 1, 31–40.
7. D. Busneag, D. Piciu, On the lattice of deductive systems of a BL-algebra. *Cent. Eur. J. Math.* **1** (2003), no. 2, 221–237.
8. J. A. Goguen, L-fuzzy sets. *J. Math. Anal. Appl.* **18** (1967), 145–174.
9. P. Hájek, *Metamathematics of Fuzzy Logic.* Trends in Logic-Studia Logica Library, 4. Kluwer Academic Publishers, Dordrecht, 1998.
10. M. Haveshki, A. Borumand Saeid, E. Eslami, Some types of filters in BL algebras. *Soft computing* **10** (2006), 657–664.
11. A. Iorgulescu, *Algebras of Logic as BCK Algebras.* Editura ASE, Bucharest, 2008.
12. M. Kondo, W. A. Dudek, Filter theory of BL algebras. *Soft Computing* **12** (2008): 419–423.
13. N. Mohtashamnia, A. Borumand Saeid, A special type of BL-algebra. *An. Univ. Craiova Ser. Mat. Inform.* **39** (2012), no. 1, 8–20.
14. S. Motamed, L. Torkzadeh, A. Borumand Saeid, N. Mohtashamnia, Radical of filters in BL-algebras. *MLQ Math. Log. Q.* **57** (2011), no. 2, 166–179.
15. E. Turunen, BL-algebras of basic fuzzy logic. *Mathware Soft Comput.* **6** (1999), no. 1, 49–61.
16. E. Turunen, *Mathematics Behind Fuzzy Logic.* Advances in Soft Computing. Physica-Verlag, Heidelberg, 1999.
17. L. A. Zadeh, From circuit theory to system theory. *Proc. IRE* **50** (1962), 856–865.
18. L. A. Zadeh, Fuzzy sets. *Information and Control* **8** (1965), 338–353.

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