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# THE INFLUENCE OF BOUNDARY CONDITIONS OF RIGID FASTENING ON THE DYNAMICAL THERMOSTABILITY OF SHELLS OF REVOLUTION, WITH AN ELASTIC FILLER

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**Abstract.** The aim of the present paper is to investigate the influence of boundary conditions of rigid fastening on the boundaries of regions of dynamical instability of closed shells of revolution, close by their forms to cylindrical ones, with an elastic filler. It is assumed that a shell is under the action of external pressure which varies in time and temperature. We consider the shells of average length whose shape of generatrix of the middle surface is a parabolic function. We consider the shells of positive and negative Gaussian curvature. The formulas are obtained for finding eigenfrequencies and boundaries of regions of dynamical instability depending on the boundary conditions, Gaussian curvature, initial stressed state, temperature and amplitude of a shell deviation from cylindrical form. The focus is on finding the moot dangerous area of dynamical instability and on the lowest eigenfrequencies of shells under consideration.

We consider the problem of influence of boundary conditions of a rigid fastening, temperature, external pressure (depending on time) and elastic filler on the boundaries of regions of dynamical instability of closed shells of revolution which by their form are close to cylindrical ones. The shell is assumed to be thin and elastic. The filler is light for which the influence of tangential stresses on the contact surface and inertia forces may be neglected. Temperature is uniformly distributed in the shell body. An elastic filler is simulated by Winkler's base, its heat expansion does not taken into account. We investigate the shells of average length whose shape of the midsurface generatrix is a parabolic function. The shells of positive and negative Gaussian curvature are also considered. Formulas and universal curves of dependence of the least frequency, shape of wave formation and boundaries of regions of dynamical instability on Gaussian curvature, type of boundary conditions, temperature, rigidity of elastic filler, as well as on the amplitude of shell deviation from a cylinder, are obtained. The focus is on finding the most dangerous area of dynamical instability and on the least eigenfrequencies which are practically most important.

A shell is considered whose middle surface is formed by the rotation of a square parabola around the $z$-axis of a rectangular system of coordinates $z$, $y$, $z$ with the origin in the middle of the segment of the axis of rotation. It is assumed that radius R of the midsurface cross-section is determined by the equality $R = r + \delta_0\big[1 - \xi^2\big(\frac{r}{l}\big)^2\big]$, where $r$ is the radius of the edge cross-section, $\delta_0$ is maximal deviation from a cylindrical form (the shell is convex for $\delta_0 > 0$ and concave for $\delta_0 < 0$); $L = 2l$ is the shell length, $\xi = \frac{z}{r}$.

We consider the shells of middle length [8] and assume that

$$\Big(\frac{\delta_0}{r}\Big)^2, \Big(\frac{\delta_0}{l}\Big)^2 \ll 1. \tag{1.1}$$

For the shells of average length, the forms of oscillations corresponding to the lowest frequencies have weak variability in the longitudinal direction compared to the circumferential, therefore the relation

$$\frac{\partial^2 f}{\partial \xi^2} \ll \frac{\partial^2 f}{\partial \varphi^2} \qquad (f = u, v, w) \tag{1.2}$$

holds true; here, $u$, $v$, $w$ are, respectively, meridional, circumferential, radial components of displacement characterizing a mode of oscillation. According to V. V. Novozhilov's statement [4], as the

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basic equations of oscillations, one can take those corresponding to V. Z. Vlasov's semi-momentless theory [7]. As a result, a simplified system of equations (due to the adopted assumption, temperature terms are equal to zero [5]) takes the form

$$\frac{\partial^2 u}{\partial\varphi^2} = -[1 + 2(2+\nu)\delta]\frac{\partial w}{\partial\xi},$$

$$\frac{\partial^2 v}{\partial\varphi^2} = (1 + 2\nu\delta)\frac{\partial w}{\partial\varphi},$$

$$\varepsilon\frac{\partial^8 w}{\partial\varphi^8} + \frac{\partial^4 w}{\partial\xi^4} + 4\delta\frac{\partial^4 w}{\partial\xi^2\partial\varphi^2} + 4\delta^2\frac{\partial^4 w}{\partial\varphi^4} - t_1^0\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4}$$
$$-t_2^0\frac{\partial^6 w}{\partial\varphi^6} - 2s\frac{\partial^6 w}{\partial\xi\partial\varphi^5} + \gamma_0\frac{\partial^4 w}{\partial\varphi^4} + \frac{\rho r^2}{E}\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial\varphi^4}\Big) = 0, \tag{1.3}$$

$$\varepsilon = \frac{h^2}{12}r^2(1-\nu^2), \quad \delta = \frac{\delta_0 r}{l^2}, \quad t_i = \frac{T_i^0}{Eh}\ \ (i=1,2), \quad s^0 = \frac{S^0}{Eh}, \quad \gamma_0 = \frac{\beta_0 r^2}{Er},$$

where $E$ and $\nu$ are, respectively, elasticity modulus and Poisson coefficient; $T_1^0$ and $T_2^0$ are meridional and circumferential stresses of the initial state; $S^0$ is shearing force of the initial state; $\rho$ is the material density of the shell; $\beta_0$ is a "bed" coefficient of an elastic filler (characterizing elastic rigidity of the filler); $\varphi$ is angular coordinate and $t$ is time.

**1.** Let us consider first the case, where $q = q_0$. Assuming the initial state is momentless, we can calculate internal stresses. When the shell edges are rigidly fixed, meridional displacements are absent. Based on the corresponding solution and taking into account filler's reaction and temperature and also inequality (1.1), we get the following approximate expressions:

$$T_1^0 = -q_0 r\Big\{\nu + \frac{\delta_0}{r}\Big[\frac{1+\nu}{3} + 2(1-2\nu^2)\Big(\frac{r}{l}\Big)^2 - (1-\nu)\xi^2\Big(\frac{r}{l}\Big)^2\Big]\Big\} - \frac{\alpha TEh}{1-\nu},$$
$$T_2^0 = -q_0 r\Big[1 - 2\nu\frac{\delta_0}{r}\Big(\frac{r}{l}\Big)^2\Big] + w_0\beta_0 r, \quad S^0 = 0, \tag{1.4}$$

where $w_0$ and $\beta_0$ are deflection and "bed" coefficient of the filer in the initial state; $\alpha$ is coefficient of linear extension; $T$ is temperature; $q_0$ is external pressure ($q_0 > 0$).

Bearing in mind relations (1.1) and (1.2), we obtain

$$\frac{\delta_0}{r}\Big[\frac{1+\nu}{3} + 2(1-2\nu^2)\Big(\frac{r}{l}\Big)^2 - (1-\nu)\xi^2\Big(\frac{r}{l}\Big)^2\Big]\frac{\partial^2 w}{\partial\xi^2} \ll \frac{\partial^2 w}{\partial\varphi^2}, \quad \nu\frac{\partial^2 w}{\partial\xi^2} \ll \frac{\partial^2 w}{\partial\varphi^2}.$$

Therefore relations (1.4) after substitution into equation (1.3) can be simpified and hence they take the form

$$T_1^0 = -\frac{\alpha TEh}{(1-\nu)}, \quad T_2^0 = -q_0 r\Big[1 - 2\nu\frac{\delta_0}{r}\Big(\frac{r}{l}\Big)^2\Big] + w_0\beta_0 r, \quad T_1^0 = \sigma_i^0 h\ \ (i=1,2). \tag{1.4$'$}$$

Taking into account the fact that in the initial state the shell deformation in circumferential direction $\varepsilon_\varphi^0$ is defined by the equality

$$\varepsilon_\varphi^0 = \frac{\sigma_2^0 - \nu\sigma_1^0}{E} + \alpha T, \qquad \varepsilon_\varphi^0 = -\frac{w_0}{r},$$

we get

$$w_0 = \big(-\sigma_2^0 + \nu\sigma_1^0\big)\frac{r}{E} - \alpha T_r. \tag{1.5}$$

Substituting (1.5) into (1.4′), we obtain

$$\frac{T_1^0}{Eh} = \frac{\sigma_1^0}{E} = -\frac{\alpha T}{1-\nu},$$
$$\frac{T_2^0}{Eh} = \frac{\sigma_2^0}{E} = -\frac{q_0 r}{Eh}\Big[1 - 2\nu\frac{\sigma_0}{r}\Big(\frac{r}{l}\Big)^2\Big] + \frac{\beta_0 r}{Eh}\Big[\big(-\sigma_2^0 + \nu\sigma_1^0\big)\frac{r}{E} - \alpha\Gamma r\Big]. \tag{1.6}$$

Introduce the notation

$$\overline{q} = \frac{qr}{Eh}, \quad \delta = \frac{\delta_0}{r}\Big(\frac{r}{l}\Big)^2, \quad \gamma_0 = \frac{\beta_0 r^2}{Eh}, \quad g = 1 + \gamma_0. \tag{1.6$'$}$$

Then (1.6) takes the form

$$\frac{\sigma_1^0}{E} = -\frac{\alpha T}{1-\nu}, \quad \frac{\sigma_2^0}{E} = -\overline{q}(1-2\nu\delta) + \Big[\Big(-\frac{\sigma_2^0}{E} + \nu\frac{\sigma_1^0}{E}\Big)\gamma_0 - \alpha T\gamma_0\Big], \tag{1.7}$$

whence it follows that

$$\frac{\sigma_2^0}{E}(1+\gamma_0) = -\overline{q}(1-2\nu\sigma) + \nu\gamma_0\frac{\sigma_1^0}{E} - \alpha T\gamma_0. \tag{1.8}$$

Substituting into (1.8) the first expression of (1.7), we have

$$\frac{\sigma_2^0}{E} = -\Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big]g^{-1}.$$

Consequently,

$$-\frac{\sigma_1^0}{E} = \frac{\alpha T}{1-\nu}, \quad -\frac{\sigma_2^0}{E} = \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big]g^{-1}. \tag{1.9}$$

As a result, the third equation of system (1.3) takes the form

$$\varepsilon\frac{\partial^8 w}{\partial\varphi^8} + \frac{\partial^4 w}{\partial\xi^4} + 4\delta\frac{\partial^4 w}{\partial\xi^2\partial\varphi^4} + 4\delta^2\frac{\partial^4 w}{\partial\varphi^4} + \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big]g^{-1}\frac{\partial^6 w}{\partial\varphi^6}$$
$$+\frac{\alpha T}{1-\nu}\frac{\partial^6 w}{\partial\xi^2\partial\varphi^4} + \gamma_0\frac{\partial^4 w}{\partial\varphi^4} + \frac{\rho r^2}{E}\frac{\partial^2}{\partial t^2}\Big(\frac{\partial^4 w}{\partial\varphi^4}\Big) = 0. \tag{1.10}$$

A solution of system (1.3) for harmonic oscillations will be sought in the form

$$u = U(\xi)\sin n\varphi\cos wt,$$
$$v = V(\xi)\cos n\varphi\cos wt,$$
$$w = W(\xi)\sin n\varphi\cos wt.$$

The first two equations of system (1.3) yield

$$n^2 U = [1-2(2-\nu)\delta]W', \tag{1.11}$$
$$nV = (1+2\nu\delta)W. \tag{1.12}$$

Note some simplifications of boundary conditions of rigid fixing for the shells (when $\xi = \text{const}$) which are written as follows:

$$u = v = w = w'_\xi = 0. \tag{1.13}$$

By virtue of equality (1.12), we find that the fulfilment of the condition $w = 0$ implies that of the condition $v = 0$. Thus relying now on (1.11), the fulfilment of the condition $v = 0$ results to that of the condition $u = 0$.

Thus if the conditions $w = w'_\xi = 0$ (for $\xi = \text{const}$) are fulfilled, then all the conditions (1.13) are fulfilled, as well.

Let the shell edges be rigidly fixed. In this case, a solution must satisfy the periodicity condition with respect to $\varphi$ and the following boundary conditions along the $\xi$ coordinate:

$$w = 0 \ \Big(\xi = \pm\Big(\frac{l}{r}\Big)\Big), \qquad w'_\xi = 0 \ \Big(\xi = \pm\Big(\frac{l}{r}\Big)\Big). \tag{1.14}$$

As it has been mentioned above, a solution $w$ of equation (1.10) for harmonic oscillations (for $q = q_0$) is sought in the form

$$w = W(\xi)\sin n\varphi\cos\omega t. \tag{1.15}$$

From (1.10) and (1.15), it follows that

$$W^{(4)} - \Big(45n^2 - \frac{\alpha T}{1-\nu}n^4\Big)W^{(2)} - n^4\Big\{\frac{\rho r^2}{E}\omega^2 + \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big]g^{-1}n^2 - \varepsilon n^4 - 4\overline{\delta}^2\Big\}W = 0,$$
$$\overline{\delta}^2 = \delta^2 + \Big(\frac{\gamma}{4}\Big). \tag{1.16}$$

Assuming $W = Ce^{\overline{\alpha}\xi}$, we obtain the characteristic equation

$$\overline{\alpha}^4 - \Big(4\delta n^2 - \frac{\alpha T}{1-\nu}n^4\Big)\overline{\alpha}^2 - n^4\Big\{\frac{\rho r^2}{E}\omega^2 + \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big]g^{-1}n^2 - \varepsilon n^4 - 4\overline{\delta}^2\Big\} = 0$$

which can be written as follows:

$$p^2 - ap - b = 0, \qquad \overline{\Omega} = \frac{\rho r^2}{E} \tag{1.17}$$

$$p = \overline{\alpha}^2, \quad a = 4\delta n^2 - \frac{\alpha T}{1-\nu}n^4, \quad b = n^4\Big\{\overline{\Omega}\omega^2 + \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big]g^{-1}n^2 - \varepsilon n^4 - 4\overline{\delta}^2\Big\}. \tag{1.18}$$

We will proceed from the condition $b > 0$. Then from (1.17) and (1.18), we have

$$\overline{\alpha}_{1,2} = \pm\sqrt{p_1}, \qquad \overline{\alpha}_{3,4} = \pm\sqrt{-p_2},$$

$$p_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} + b} > 0, \qquad p_2 = \frac{a}{2} - \sqrt{\frac{a^2}{4} + b} < 0. \tag{1.19}$$

A general solution of equation (1.16) takes the form

$$W = Achk_1\xi + B\,\mathrm{sh}\,k_1\xi + C\cos k_2\xi + D\sin k_2\xi,$$

$$k_1 = \sqrt{p_1}, \quad k_2 = \sqrt{-p_2}.$$

By satisfying the boundary conditions (1.14), we obtain a system of four homogeneous equations. From the condition that the determinant of this system is equal to zero, we obtain

$$\mathrm{th}\,k_1\overline{l} = \frac{k_1}{k_2}\,\mathrm{tg}\,k_2\overline{l} = \frac{k_2}{k_1}\,\mathrm{tg}\,k_2\overline{l}, \qquad \overline{l} = \frac{l}{r}\,. \tag{1.20}$$

Consequently, this system splits into two independent systems and, accordingly, the solution splits into odd and even functions. An even function corresponds to the $\xi$-symmetric oscillations and an odd function corresponds to the skew-symmetric oscillations. Thus we obtain

$$W = D\Big(\sin k_2\xi - \frac{\sin k_2\overline{l}}{\mathrm{sh}\,k_1\overline{l}}\,\mathrm{sh}\,k_1\xi\Big),$$

$$W = C\Big(\cos k_2\xi - \frac{\cos k_2\overline{l}}{\mathrm{ch}\,k_1\overline{l}}\,\mathrm{ch}\,k_1\xi\Big).$$

Consider first the case $\delta = 0$, $\overline{q} = \gamma = T = 0$. For $p_1 = -p_2 = \sqrt{b}$, $k_1 = k_2 = \sqrt[4]{b} = k$. Equation (1.20) corresponding to the skew-symmetric modes of oscillations, admits the form

$$\mathrm{th}\,k\overline{l} = \mathrm{tg}\,k\overline{l}.$$

The lowest root of this equation corresponds to the value

$$k = 3,927\frac{r}{l}.$$

Whereas equation (1.20) corresponding to symmetric modes of oscillations takes for $\delta = 0$, $\overline{q} = \gamma = T = 0$ the form

$$\mathrm{th}\,k\overline{l} = \mathrm{tg}\,k\overline{l}.$$

The lowest root of this equation corresponds to the value

$$k = 2,365\frac{r}{l} = 0,75\frac{\pi r}{l} \tag{1.21}$$

that is, the lowest value $k$ corresponds to symmetric modes of oscillation. Therefore, in the sequel, we will consider oscillations with symmetric deflection with respect to $\xi$. Taking into account that for $\delta = 0$, $\overline{q} = \gamma = T = 0$,

$$-p_1p_2 = b, \quad b = n^4(\overline{\Omega}\omega^2 - \varepsilon n^4)$$

we obtain

$$k^4 = n^4(\overline{\Omega}\omega^2 - \varepsilon n^4).$$

This implies that the lowest root (1.21) for fixed $n$ corresponds to the smallest value of eigenfrequency which is defined by the expression

$$\overline{\Omega}\omega^2 = \varepsilon n^4 + (d_1\lambda_1)^4 n^{-4}, \quad d_1 = 1,55, \quad \lambda_1 = \frac{\pi r}{2l}\,. \tag{1.22}$$

The least frequency value, depending on $n$, is realized for

$$n_0^2 = d_1\lambda_1\varepsilon^{-1/4}. \tag{1.23}$$

For $n = n_0$, from (1.22), for the least frequency of cylindrical average length shell with rigidly fixed edges, we obtain the well-known formula [2]

$$\overline{\Omega}\omega_{0_1}^2 = 2d_1^2\lambda_1^2\varepsilon^{1/2}.$$

For freely supported edges of cylindrical shell, the lowest frequency is defined, as is known, by the formula

$$\overline{\Omega}\omega_0^2 = 2\lambda_1^2\varepsilon^{1/2}. \tag{1.24}$$

Let us pass now to a general case and consider symmetric in axia direction forms of oscillation corresponding to the lowest frequencies. On the basis of (1.19), we have

$$-p_2 = p_1 - a, \quad a = \Big(4\delta - \frac{\alpha T}{1-\nu}n^2\Big)n^2,$$

which for $x = \bar{l}\sqrt{p_1}$ yield

$$-p_2\bar{l}^2 = x^2 - \beta, \quad \beta = 4n^2\frac{\delta_0}{r} - \frac{\alpha T}{1-\nu}\Big(\frac{l}{r}\Big)^2 n^4. \tag{1.25}$$

Then equation (1.20) corresponding to symmetric modes of oscillation can be represented as follows:

$$x\,\mathrm{th}\,x = -\sqrt{x^2-\beta}\ \mathrm{tg}\sqrt{x^2-\beta}. \tag{1.26}$$

Relying on the first equality (1.19), it follows that $p_1(p_1 - a) = b$, from which we find that

$$\overline{\Omega}\omega^2 = \varepsilon n^4 + x^2(x^2-\beta)\Big(\frac{r}{l}\Big)^4 n^{-4} + 4\overline{\delta}^2 - \Big[\overline{q}(1-2\nu\delta) + \frac{\alpha T\alpha_0}{1-\nu}\Big]g^{-1}n^2. \tag{1.27}$$

Consequently, in a general case, the eigenfrequencies $\omega$ for the shells under consideration are defined by formula (1.27), where $x$ is any root of equation (1.26). The least frequency $\omega$ is obtained as a result of minimization of the right-hand side of equality (1.27) with respect to $n$ when $x$ is assumed to be a smallest root of equation (1.26) which we denote by $n_\omega$. Owing to (1.25) and (1.26), it is not difficult to see that $x_\omega$ depends on $\frac{\delta_0}{r}$, $T$, $\gamma_0$ as well as on $n$. Such a minimization is realized by sorting out natural values $n$ in the neighbourhood of $n_0$, defined by equality (1.23). Below, we present the results of calculations for the shell with geometric sizes $l = r$, $\frac{h}{r} = 10^{-2}$, $\nu = 0,3$ under different values of $\frac{\delta_0}{r}$ (for $\overline{q} = \gamma_0 = T = 0$). In Figure 1, we can see the graph of dependence of $x_\omega$ on $\frac{\delta_0}{r}$ (curve (1) corresponds to the rigid fastening of the shell edges; straight line (0) corresponds to the free support of the shell edges). Figure 2 presents the graph of dependence of $x_\omega$ on $\frac{\delta_0}{r}$ ((1) corresponds to the rigid fastening of the shell edges; (0) corresponds to the free support of the shell edges). Figure 3 shows the curves of dependence of the least dimensionless frequencies $\omega^2/\omega_0^2 n_\omega$ on $\frac{\delta}{r}$ ((1) is the case of rigidly fixed edges; (0) is the case of freely supported edges [3]; $\omega_0^2$ is defined by expression (1.24)).

For $\omega = 0$, from equality (1.27), we obtain

$$\overline{q}_*(1-2\nu\delta) = \Big[\varepsilon n^2 + x^2(x^2-\beta)n^{-6}\Big(\frac{r}{l}\Big)^4 + 4\overline{\delta}^2 n^{-2}\Big]g - \frac{\alpha T\gamma_0}{1-\nu}, \quad g = 1+\gamma_0. \tag{1.28}$$

The least value $\overline{q}_*$ is obtained by minimizing the right-hand side of equality (1.28) just as it has been done for the frequency $\omega$.

**2.** Consider now the case, where

$$q = q_0 + q_t\cos\Omega t.$$

A solution of equation (1.3) will be sought in the form

$$w = \mathscr{F}(\xi,t)\sin n\varphi,$$

where $\mathscr{F}(\xi,t) = f(t)W(\xi)$.

Substituting the above-obtained solution into (1.3) and introducing notation (1.6′), we get

$$\varepsilon n^8 f(t)W(\xi) + W^{(4)}(\xi)f(t) - 4\delta W^{(2)}(\xi)f(t)n^2 + 4\delta^2 W(\xi)f(t)n^4$$
$$-\Big(\overline{q}(t)(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big)g^{-1}n^6 W(\xi)f(t) + \frac{\alpha T}{1-\nu}W^{(2)}(\xi)f(t)n^4$$
$$+\gamma n^4 W(\xi)f(t) + W(\xi)f^{(2)}(t)n^4\frac{\rho r^2}{E} = 0. \tag{2.1}$$

Equation (2.1) can be divided with respect to the variables $\xi$ and $t$. Thus we obtain

$$\frac{W^{(4)}(\xi) - \Big(4\delta n^2 - \frac{\alpha T}{1-\nu}n^4\Big)W^{(2)}(\xi)}{W(\xi)}$$
$$= \frac{-\Big\{\frac{\rho r^2}{E}f^{(2)}(t)n^4 + \Big[\varepsilon n^8 + 4\overline{\delta}^2 n^4 - \Big(\overline{q}(t)(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big)g^{-1}n^6\Big]f(t)\Big\}}{f(t)}.$$

Introduce the parameter $\lambda^2$. We have

$$\frac{W^{(4)} - \Big(4\delta n^2 - \frac{\alpha T}{1-\nu}n^4\Big)W^{(2)}}{W} = \lambda^2,$$
$$-\frac{\frac{\rho r^2}{E}f^{(2)}n^4 + \Big[\varepsilon n^8 + 4\overline{\delta}^2 n^4 - \Big(\overline{q}(t)(1-2\nu\delta) + \frac{\alpha T\gamma_0}{1-\nu}\Big)g^{-1}n^6\Big]f}{f} = \lambda^2,$$

whence we obtain

$$W^{(4)} - \Big(4\delta - \frac{\alpha T}{1-\nu}n^2\Big)n^2 W^{(2)}(\xi) - \lambda^2 W(\xi) = 0, \tag{2.2}$$

$$\frac{\rho r^2}{E}f^{(2)}(t) + \Big[\varepsilon n^4 + 4\overline{\delta}^2 - \frac{\alpha T\gamma_0}{1-\nu}g^{-1}n^2 - \overline{q}(t)(1-2\nu\delta)g^{-1}n^2 + \lambda^2 n^{-4}\Big]f(t) = 0. \tag{2.3}$$

Define now the value $\lambda$ on the basis of equation (2.2). Letting $W = Ce^{\overline{\alpha}\xi}$, we get the corresponding characteristic equation

$$\overline{\alpha}^4 - a\overline{\alpha}^2 - \lambda^2 = 0, \quad a = 4\delta n^2 - \frac{\alpha T}{1-\nu}n^4,$$

which can be represented in the form

$$p^2 - ap - \lambda^2 = 0, \quad p = \overline{\alpha}^2. \tag{2.4}$$

Then from (2.4) it follows that

$$\overline{\alpha}_{1,2} = \pm\sqrt{p_1}, \qquad \overline{\alpha}_{3,4} = \pm i\sqrt{-p_2},$$
$$p_1 = \frac{a}{2} + \sqrt{\frac{a^2}{4} + \lambda^2} > 0, \qquad p_2 = \frac{a}{2} - \sqrt{\frac{a^2}{2} + \lambda^2} < 0. \tag{2.5}$$

A general solution of equation (2.2) takes the form

$$W = A\operatorname{ch}k_1\xi + B\operatorname{sh}k_1\xi + C\cos k_2\xi + D\sin k_2\xi$$
$$k_1 = \sqrt{p_1}, \quad k_2 = \sqrt{-p_2}. \tag{2.6}$$

Satisfying the boundary conditions (1.14), we obtain a system of four homogeneous equations. From the condition that the determinant of this system is equal to zero, we obtain

$$\operatorname{th}k_1\overline{l} = \frac{k_1}{k_2}\operatorname{tg}k_2\overline{l} = -\frac{k_2}{k_1}\operatorname{tg}k_2\overline{l}, \quad \overline{l} = \frac{l}{r}. \tag{2.7}$$

Equality (2.7) in its form coincides with (1.20), however, the expressions for $k_1$ and $k_2$ look differently because of the presence of the parameter $\lambda^2$, according to (2.5).

In connection with the above-said, let us consider a solution corresponding to symmetric form of deflection with respect to $\xi$. In addition,

$$\operatorname{th} k_1\bar{l} = -\frac{k_1}{k_2} \operatorname{tg} k_2\bar{l}. \tag{2.8}$$

In view of (2.5), we have

$$-p_2 = p_1 - a, \quad a = 4\delta n^2 - \frac{\alpha T}{1-\nu} n^4,$$

whence, letting $x = \bar{l}\sqrt{p_1}$, we obtain

$$-p_2\bar{l}^2 = x^2 - \beta, \quad \beta = a\bar{l}^2, \quad a\bar{l}^2 = 4\frac{\delta_0}{r} n^2 - \frac{\alpha T}{1-\nu}\left(\frac{l}{r}\right)^2 n^4.$$

Then equation (2.8) can be presented as

$$x \operatorname{th} x = -\sqrt{x^2-\beta}\, \operatorname{tg} \sqrt{x^2-\beta}. \tag{2.9}$$

Moreover, due to (2.5),

$$x^2\bar{l}^{-2} - \frac{a}{2} = \sqrt{\frac{a^2}{4} + \lambda^2}$$

whence we find that

$$\lambda^2 = x^2\bar{l}^{-2}(x^2\bar{l}^{-2} - a) = x^2(x^2-\beta)\bar{l}^{-4}, \tag{2.10}$$

where $x$ is any root of equation (2.9).

Next, we substitute $\lambda^2$, according to (2.10), into equation (2.3) and obtain

$$f^{(2)}(t) + \frac{E}{\rho r^2}\left\{\varepsilon n^2 + 4\bar{\delta}^2 - \frac{\alpha T \gamma_0}{1-\nu} g^{-1} n^2 - \bar{q}(t)(1-2\nu\delta) g^{-1} n^2 + x^2(x^2-\beta)\bar{l}^{-4} n^{-4}\right\} f(t) = 0, \tag{2.11}$$

where $\bar{q}(t) = \bar{q}_0 + \bar{q}_t \cos \Omega t$.

This equation (2.11) can be easily reduced to the Mathieu equation [6]

$$\frac{d^2 f}{d\tau^2} + \frac{4\omega^2}{\Omega^2}(1 - 2\varepsilon \cos 2\tau) f = 0, \tag{2.12}$$

where

$$\tau = \frac{\Omega t}{2}, \quad \varepsilon - \frac{\bar{q}_t}{2(\bar{q}_* - \bar{q}_0)}\,; \tag{2.13}$$

here, $\omega$ and $\bar{q}_*$ are, respectively, frequency of eigenoscillations and critical pressure (their expressions have been obtained above).

Frequency of eigenoscillations of the shells under consideration (for $\bar{q} = \bar{q}_0$) is defined from equation (2.11), if we assume that $f(t) = C \sin \omega t$, and expressed by formula (1.27).

Critical pressure $\bar{q}_*$ is defined from equation (2.11) under the assumption $f(t) = \text{const}$ and expressed by formula (1.28) which in expanded representation has the form

$$\bar{q}_* = \frac{1+\gamma_0}{1-2\nu\delta}\left[\varepsilon n^2 + 4\bar{\delta}^2 n^{-2} + x^2(x^2-\beta)\left(\frac{r}{l}\right) n^{-6}\right] - \frac{\alpha T \gamma_0}{(1-\nu)(1-2\nu\delta)}\,. \tag{2.14}$$

The solution of equation (2.12) has been studied in a number of works, where it was noted that under certain relations between $\varepsilon$, $\Omega$, $\omega$ and $t \to \infty$ this solution will infinitely increase in the regions of instability. Generalizing the results of works [6] and [1] to the shell under consideration, we present below the formulas to reveal the influence of boundary conditions of rigid fastening and temperature on the location of redions of dynamical instability. First, consider the case, where $q_t \to 0$ ($\varepsilon \to 0$). We find that these regions lie in the vicinity of exciting frequencies

$$\Omega_* = \frac{2\omega}{k}\,.$$

Depending on a number $k$, we distinguish the first, second, third and so on regions of dynamical instability. The region of instability ($k = 1$) lying in the vicinity $\Omega_* = 2\omega$ when $\omega$ is expressed by formula (1.27) and takes the lowest value, is the most dangerous and has therefore the greatest practical importance. This region is called the principal region of dynamical instability.

For $q_t$, other than zero, for the boundaries of the principal region of instability, bounded by the values of critical frequencies, we obtain the following formula:

$$\Omega_* = 2\omega\sqrt{1 \pm \varepsilon}.$$

Taking into account resistance forces, proportional to the first derivative of displacement in time (with damping coefficient $\overline{\beta}$), the formula for finding the boundaries of the principal region of instability takes the form

$$\Omega_* = 2\omega\sqrt{1 \pm \sqrt{\varepsilon^2 - \left(\frac{\Delta}{\pi}\right)^2}}, \qquad \Delta = \frac{2\pi\overline{\beta}}{\omega} \tag{2.15}$$

where the terms with higher powers $\frac{\Delta}{\pi}$ are discarded, given the damping decrement $\Delta$ is usually very small compared to unity.

The values $\omega$, $\overline{q}_*$, $\varepsilon$ are determined by formulas (1.27), (2.14), (2.13), where $x_\omega$ and $n$ correspond to the least value $\omega$.

It follows from formula (2.15) that the minimal (critical) value of the excitation coefficient $\varepsilon$ for which undamped oscillations are still possible, is determined by the equality

$$\varepsilon_{*_1} = \frac{\Delta}{\pi}.$$

For boundaries of the second region of instability ($k = 2$), the formula

$$\Omega_* = \omega\sqrt{1 + \varepsilon^2 \pm \sqrt{\varepsilon^4 - \left(\frac{\Delta}{\pi}\right)^2 (1 - \varepsilon^2)}}$$

holds. In the given case, critical value of the excitation coefficient is determined approximately by the equality $\varepsilon_{*_2} = \left(\frac{\Delta}{\pi}\right)^{1/2}$. In a similar way, generalizing the results [1], we can derive formulas for the boundaries of the third region of dynamical instability which is, practically, seldom realized.

Relying on the above-given formulas, it is not difficult to determine variation intervals of critical frequencies $\Omega_*$ (depending on $\delta$, $q_0$, $q_t$, $T$) which fall into the regions of dynamical instability under rigid fastening of shell edges.

The above formulas allow one to determine easily how significantly temperature, rigid fastening of shell edges and acting load may affect the boundaries of regions of dynamical instability.

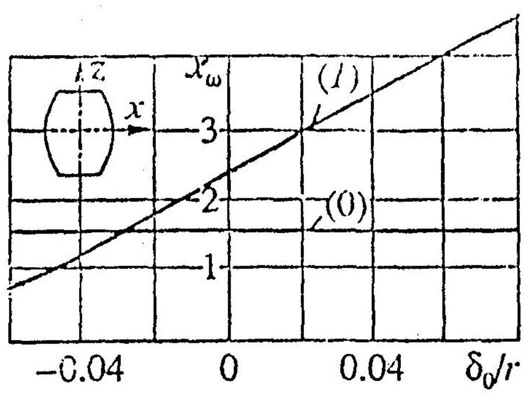


FIGURE 1

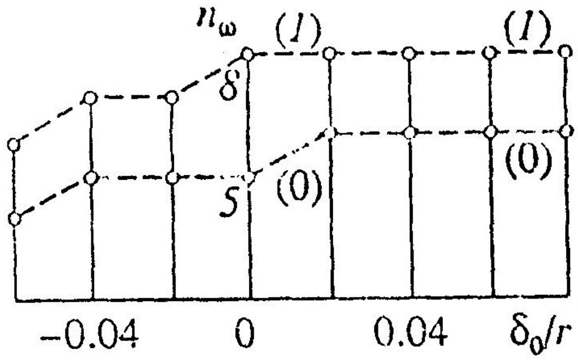


FIGURE 2

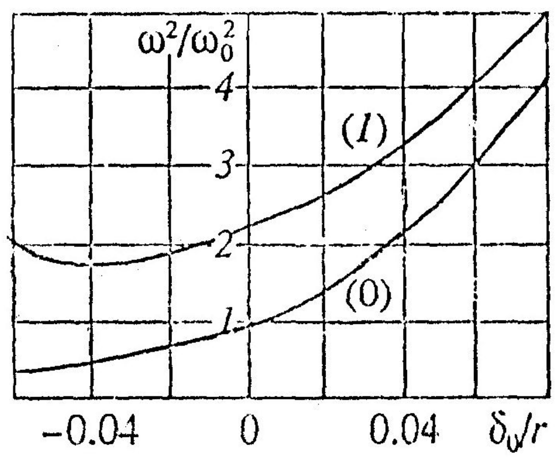


FIGURE 3

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