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ON ONE PROBLEM FOR A VISCOELASTIC QUADRANGULAR PLATE (RHOMBUS) WITH A CIRCULAR HOLE

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Abstract. A plane problem for a viscoelastic quadrangular plate (rhombus) with a circular hole is considered. It is assumed that the inner boundary of the plate is under the action of uniformly distributed normal compressive forces (pressure or the values of constant normal displacements are prescribed), while to the outer boundary are applied absolutely rigid smooth linear stamps with the principal vectors of compressive forces; the stamps move only vertically relative to the boundary.

Using the methods of conformal mappings and the theory of boundary value problems of analytic functions, the problem is reduced to the Riemann–Hilbert boundary value problem for a circular ring and, using the Kelvin–Voigt model, the complex potentials, characterizing equilibrium of the plate, are constructed effectively (in analytical form). The limiting case for a rhombus that turns into a strip, is considered.

Introduction

Contact problems of the plane theory of elasticity are well developed (in the sense of effective solution) by the methods of conformal mappings and the theory of boundary value problems of analytic functions for a fairly large class of simply connected domains, which conformal mapping onto a unit disk is carried out by the rational functions. However, these methods (we mean the Kolosov–Muskhelishvili methods (see [6]) are less suitable for multiply connected (including doubly connected) domains. Formulas similar to the Christoffel–Schwartz formulas for the conformal mapping of doubly connected domains, bounded by polygons, onto a circular ring (see [4]) enable one to solve effectively the contact problems of the plane theory of elasticity for the above-mentioned domains and for their modifications obtained by passing to the limit. This approach is also efficient for solving the plane problems of the theory of viscoelasticity according to the Kelvin–Voigt model (see [3, 7]). In addition, it should be noted that in these cases, using the methods mentioned above, one can decompose the stated problems (with respect to the unknown complex potentials) into two Riemann–Hilbert problems for a circular ring. By solving the latter problems, the potentials are constructed in analytical form.

Statement of the problem. Let S be a viscoelastic quadrangular plate (rhombus) with a circular hole. By L_0 we denote the internal boundary $L_0 = \{|z| = R_0\}$, and by $L_1 = A_1A_2A_3A_4$ the rhombus

$$L_1 = \bigcup_{j=1}^{4} L_1^{(j)}$$

$\big(L_1^{(j)} = A_jA_{j+1};\ j = \overline{1,4},\ A_5 = A_1\big)$. Assume that the sides $l_1^{(j)}$ $(j = \overline{1,4})$ of the rhombus are under the action of rigid rectilinear smooth stamps with the known principal vectors of external compressive forces N (we consider the symmetric case). We also assume that L_0 is under the action of constant normal compressive forces (pressure) of intensity P_0 (or the constant normal displacements are prescribed). By $\alpha_j^0\pi$ we denote the angles at the vertices A_j $(j = \overline{1,4})$, i.e., $\alpha_1^0 = \alpha_3^0$; $\alpha_2^0 = \alpha_4^0 = 1 - \alpha_1^0$ (see Figure 1), and by $\alpha(\sigma)$ and $\beta(\sigma)$ the angles lying between the Ox-axis and the outer

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normal to the contour $L = L_0 \cup L_1$, i.e.,

$$\alpha(\sigma) = -\frac{\pi}{2}\left(1+\alpha_1^0\right), \quad \sigma \in L_1^{(1)}; \quad \alpha(\sigma) = \frac{\pi}{2}\left(-1+\alpha_1^0\right), \quad \sigma \in L_2^{(1)};$$
$$\alpha(\sigma) = \frac{\pi}{2}\left(1-\alpha_1^0\right), \quad \sigma \in L_3^{(1)}; \quad \alpha(\sigma) = \frac{\pi}{2}\left(1+\alpha_1^0\right), \quad \sigma \in L_4^{(1)};$$
$$\beta(\sigma) = \pi + \arg\sigma, \qquad \sigma \in L_0.$$

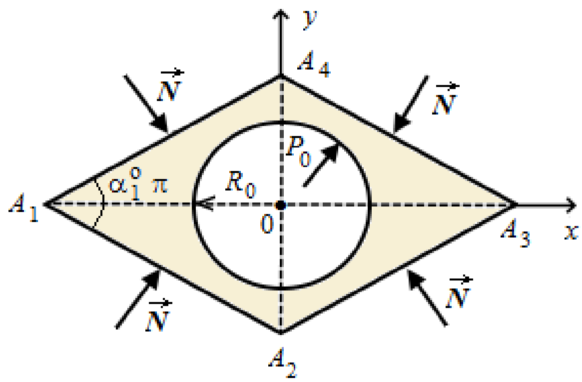

FIGURE 1

The purpose of the present work is to determine the complex potentials characterizing the distribution of stresses and displacements in the plate according to the Kelvin–Voigt model. Similar problems of the plane theory of elasticity have been considered in [1,2].

Solution of the problem. Let us mention some results from [4–6,8].

1. The conformal mapping of a doubly connected domain S^0 bounded by convex polygons (A) and (B), with internal angles $\pi\alpha_k^0$ and $\pi\beta_m^0$ at the vertices A_k $(k = (\overline{1,p}))$ and B_m $(m = (\overline{1,q}))$, respectively, onto a circular ring $D_=\{1 < |\zeta| < R_*\}$ is carried out by the function $z = \omega_*(\zeta)$, the derivative of which under the condition

$$\prod_{k=1}^{p}\left(\frac{a_k}{R_*}\right)^{\alpha_k^0-1}\prod_{m=1}^{q}(b_m)^{\alpha_m^0-1} = 1$$

has the form [4]

$$\omega'_*(\zeta) = K_*\prod_{j=-\infty}^{\infty} G\left(R_*^{2j}\zeta\right) g\left(R_*^{2j}\zeta\right) R^{2\delta_j}, \tag{1}$$

where

$$G(\zeta) = \prod_{k=1}^{p}(\zeta-a_k)^{\alpha_k^0-1}; \quad g(\zeta) = \prod_{m=1}^{q}(\zeta-b_m)^{\alpha_m^0-1}; \quad \delta^j = \begin{cases} 0, & j \ge 0, \\ 1, & j \le -1. \end{cases} \tag{2}$$

Note that the function $\omega'_*(\zeta)$ for a circular ring D_0 is a solution to the Riemann–Hilbert problem

$$\operatorname{Re}\left[i\eta e^{-i\alpha(\eta)}\omega'_*(\eta)\right] = 0, \ \ \eta \in l_1; \qquad \operatorname{Re}\left[i\eta e^{-i\beta(\eta)}\omega'_*(\eta)\right] = 0, \ \ \eta \in l_0, \tag{3}$$

where l_1 and l_0 are the preimages of the boundaries L_1 and L_0; $l_1 = \{|\zeta| = R_*\}$; $l_0 = \{|\zeta| = 1\}$.

If we now consider a regular n-angle inscribed into a circle and denote by S_n the area thus obtained, then as $n \to \infty$, for the function $\omega'(\zeta)$, in view of (1) and (2), we obtain the formula

$$\omega'(\zeta) = K\prod_{k=1}^{4}\left(1-\frac{\zeta}{a_k}\right)^{\alpha_k^0-1}\prod_{j=1}^{\infty}\prod_{k=1}^{4}\left(1-\frac{\zeta}{R^{2j}a_k}\right)^{\alpha_k^0-1}\left(1-\frac{a_k}{R^{2j}\zeta}\right)^{\alpha_k^0-1},$$

and the boundary conditions (3) will be written in the form

$$\operatorname{Re}\Big[i\eta e^{-i\alpha(\eta)}\omega'(\eta)\Big]=0,\ \ \eta\in l_1;\qquad \operatorname{Re}\left[i\omega'(\eta)\right]=0,\ \ \eta\in l_0. \tag{4}$$

2. The boundary conditions of the first and second basic problems for a viscoelastic plate S according to the Kelvin–Voigt model have the form ([5,6,8])

$$\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}=i\int\limits_0^\sigma\big(X_n(\sigma,t)+iY_n(\sigma,t)\big)ds+c_1+ic_2, \tag{5}$$

$$\Gamma\varphi(\sigma,t)-M\big[\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}\big]=2\mu^*\big(u(t)+iv(t)\big),\quad \sigma\in L=L_0\bigcup L_1, \tag{6}$$

where t is the time parameter, and Γ and M are the operators

$$\begin{gathered}\Gamma\varphi(\sigma,t)=\int\limits_0^t\big[\varkappa^*e^{k(\tau-t)}+2e^{m(\tau-t)}\big]\varphi(\sigma,\tau)d\tau,\\ M\Big[\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}\Big]=i\int\limits_0^t e^{m(\tau-t)}\Big[\varphi(\sigma,\tau)+\sigma\overline{\varphi'(\sigma,\tau)}+\overline{\psi(\sigma,\tau)}\Big]d\tau,\quad \sigma\in L.\end{gathered} \tag{7}$$

Since the domain S is doubly connected, it is advisable to use the functions $\Phi(z,t)=\varphi'(z,t)$ and $\Psi(z,t)=\psi'(z,t)$, which are single-valued in the case of a multiply connected domain, as well. Taking into account the equality

$$X_n(\sigma,t)+iY_n(\sigma,t)=\big(N(\sigma,t)+iT(\sigma,t)\big)e^{i\alpha(\sigma)}=-i\Big(N(\sigma,t)+iT(\sigma,t)\frac{d\sigma}{ds}\Big),$$

differentiating (5) with respect to σ and passing to the complex-conjugate value, we obtain

$$\Phi(\sigma,t)+\overline{\Phi(\sigma,t)}+\sigma_s'^{\,2}\big[\overline{\sigma}\Phi'(\sigma,t)+\Psi(\sigma,t)\big]=N(\sigma,t)-iT(\sigma,t),\quad \sigma\in L. \tag{8}$$

Also, taking into account the equalities

$$\begin{gathered}u+iv=i(v_n+iv_\tau)e^{i\alpha(\sigma)};\quad v_\tau=v_n^{(j)}=\text{const},\quad T(\sigma,t)=0,\quad \sigma\in L_1;\\ V_n=v_n^{(0)}=\text{const},\quad v_\tau=0,\quad T(\sigma,t)=0,\quad N(\sigma,t)=P_0,\quad \sigma\in L_0,\end{gathered}$$

from (6), by the differentiation with respect to σ, in view of (8), we get

$$\Gamma\Phi(\sigma,t)-M\big[N(\sigma,t)-iT(\sigma,t)\big]=\begin{cases}2\mu^*v_n^{(0)}R_0^{-1}, & \sigma\in L_0,\\ 0, & \sigma\in L_1.\end{cases} \tag{9}$$

Since $N=P_0$, $\sigma\in L_0$; $T=0$, $\sigma\in L_1$, we obtain from (9) the following boundary value problem:

$$\operatorname{Re}\Gamma\Phi(\sigma,t)=P(t),\quad \sigma\in L_0;\quad \operatorname{Im}\Gamma\Phi(\sigma,t)=0,\quad \sigma\in L_1 \tag{10}$$

where

$$P(t)=P_0F(t)+2\mu^*R_0^{-1}v_n^{(0)};\quad F(t)=\frac{1}{m}\big[1-e^{-mt}\big].$$

Mapping the domain S onto a circular ring $D=\{1<|\zeta|<R\}$ and introducing the notation $\Phi[\omega(\zeta),t]=\Phi_0(\zeta,t)$, from (10), for the circular ring D, we obtain the following Riemann–Hilbert boundary value problem:

$$\operatorname{Re}\big[\Gamma\Phi_0(\eta,t)-P(t)\big]=0,\ \ \eta\in l_0;\qquad \operatorname{Im}\big[\Gamma\Phi_0(\eta,t)-P(t)\big]=0,\ \ \eta\in l_1. \tag{11}$$

Since the boundary problem (11) has only a trivial solution, from (7) and (11), we obtain the integral equation with respect to $\Phi_0(\zeta,t)$,

$$\int\limits_0^t\big[\varkappa^*e^{k(\tau-t)}+2e^{m(\tau-t)}\big]\Phi_0(\zeta,\tau)d\tau=P(t). \tag{12}$$

From (12), by the differentiation with respect to t, we have

$$-ke^{-kt}\int\limits_0^t \varkappa^* e^{k\tau}\Phi_0(\zeta,\tau)d\tau + \varkappa^*\Phi_0(\zeta,t) - 2me^{-mt}\int\limits_0^t e^{m\tau}\Phi_0(\zeta,\tau)d\tau + 2\Phi_0(\zeta,t) = \dot{P}(t), \tag{13}$$

here, $\dot{P}(t)$ denotes a derivative with respect to t.

Multiplying (12) by m and summing with (13), we get

$$(m-k)\varkappa^*\int\limits_0^t e^{k\tau}\Phi_0(\zeta,\tau)d\tau + (\varkappa^*+2)e^{kt}\Phi_0(\zeta,t) = \big(P_0 + 2\mu^* m R_0^{-1} v_n^{(0)}\big)e^{kt}. \tag{14}$$

From (14), by the differentiation with respect to t, for the function $\Phi_0(\zeta,t)$, we arrive at the differential equation

$$\dot{\Phi}_0(\zeta,t) + a\Phi_0(\zeta,t) = b, \tag{15}$$

where

$$a = \frac{m\varkappa^* + 2k}{\varkappa^*+2}; \qquad b = \frac{k(P_0 + 2\mu^* m R_0^{-1} v_n^{(0)})}{\varkappa^*+2}. \tag{16}$$

In addition, from (12), in view of (14), we have the following initial condition:

$$\Phi_0(\zeta,0) = \frac{b}{k}. \tag{17}$$

The solution of equation (15) for the initial condition (17) takes the form

$$\Phi_0(\zeta,t) = b\Big[\frac{1}{a} + \Big(\frac{1}{k} - \frac{1}{a}\Big)e^{-at}\Big], \tag{18}$$

where a and b are defined by formula (16).

Having defined $\Phi_0(\zeta,t)$, to find the function $\Psi_0(\zeta,t) = \Psi[\omega(\zeta),t]$, we take advantage of equality (8) which after the conformal mapping is written in the form

$$\begin{aligned}\Phi_0(\eta,t) + \overline{\Phi_0(\eta,t)} - \frac{\eta^2}{\rho^2\overline{\omega'(\eta)}}\Big[\overline{\omega(\eta)}\,\Phi'_0(\eta,t) + \omega'(\eta)\Psi_0(\eta,t)\Big] \\ = N(\eta,t) - iT(\eta,t), \quad \eta\in l = l_0\cup l_1,\end{aligned} \tag{19}$$

where $\rho = R$, $\eta\in l_1$ and $\rho = 1$, $\eta\in l_0$. Relying on (9) and (19), we get

$$\begin{aligned}\Gamma\overline{\Phi_0(\eta,t)} - M\Big\{\Phi_0(\eta,t) + \overline{\Phi_0(\eta,t)} - \frac{\eta^2}{\rho^2\overline{\omega'(\eta)}}\Big[\overline{\omega(\eta)}\Phi'_0(\eta,t) + \omega'(\eta)\Psi_0(\eta,t)\Big]\Big\} \\ = \begin{cases} 2\mu^* R_0^{-1} v_n^{(0)}, & \eta\in l_0, \\ 0, & \eta\in l_1. \end{cases}\end{aligned} \tag{20}$$

Taking into account (18), from (20), we obtain the following Riemann–Hilbert boundary value problem:

$$\begin{aligned}&\operatorname{Re} M\Big[\eta^2\frac{\omega'(\eta)}{\overline{\omega'(\eta)}}\Psi_0(\eta,t)\Big] = f_0(\eta,t), \quad \eta\in l_0 \\ &\operatorname{Im} M\Big[\frac{\eta^2}{R^2}\frac{\omega'(\eta)}{\overline{\omega'(\eta)}}\Psi_0(\eta,t)\Big] = 0, \quad \eta\in l_1\end{aligned} \tag{21}$$

where

$$f_0(\eta,t) = -\Gamma^*\Phi_0(\eta,t) + 2\mu R_0^{-1} v_n^{(0)}, \quad \eta\in l_0 \tag{22}$$

$$\Gamma^*\Phi_0(\zeta,t) = \int\limits_0^t \varkappa^* e^{k(\tau-t)}\Phi_0(\zeta,\tau)d\tau, \quad \zeta\in D. \tag{23}$$

From (4) we get

$$\omega'(\eta) = \overline{\omega'(\eta)}, \ \ \eta \in l_0, \qquad \omega'(\eta) = \frac{\overline{\eta}^2}{R^2}\,\overline{\omega'(\eta)}, \ \ \eta \in l_1,$$

and thus (21) will take form

$$\mathrm{Re}\,\big[M[\eta^2\Psi_0(\eta,t)]\big] = f_0(\eta,t); \ \ \eta \in l_0; \qquad \mathrm{Im}\,[M\Psi_0(\eta,t)] = 0, \ \ \eta \in l_1.$$

Let us now consider the function

$$T(\zeta) = \Big(1-\frac{\zeta}{R}\Big)^{-2}\prod_{j=1}^{\infty}\Big(1-\frac{\zeta}{RR^{2j}}\Big)^{-2}\Big(1-\frac{R}{R^{2j}\zeta}\Big)^{-2}.$$

It is easy to check that the equalities hold

$$\frac{T(\eta)}{\overline{T(\eta)}} = 1, \ \ \eta \in l_0, \qquad \frac{T(\eta)}{\overline{T(\eta)}} = \frac{\eta^2}{R^2}, \ \ \eta \in l_1,$$

and, thus, conditions (21) regarding the function

$$\Omega_0(\zeta,t) = \frac{\zeta^2\Psi_0(\zeta,t)}{T^2(\zeta)} \tag{24}$$

will be written in the form

$$\mathrm{Re}\,[M\Omega_0(\eta,t)] = \frac{f_0(\eta,t)}{T^2(\eta)}, \ \ \eta \in l_0; \qquad \mathrm{Im}\,[M\Omega_0(\eta,t)] = 0, \ \ \eta \in l_1. \tag{25}$$

The solution to problem (25) has the form (see [2])

$$M[\Omega_0(\zeta,t)] = f_1(\zeta,t),$$

where

$$f_1(\zeta,t) = \frac{1}{\pi i}\sum_{j=-\infty}^{\infty}(-1)^j\int_{l_0}\frac{f_0(\eta,t)T^{-2}(\eta)}{\eta - R^{2j}z}\,d\eta$$

and thus from (24) with respect to the function $\Psi_0(\zeta,t)$ we obtain the equation

$$M\Psi_0(\zeta,t) = f(\zeta,t), \tag{26}$$

where

$$f(\zeta,t) = \frac{f_1(\zeta,t)T^2(\zeta)}{\zeta^2}\,. \tag{27}$$

From (26) taking into account (7) we easily obtain

$$\Psi_0(\zeta,t) = mf(\zeta,t) + \dot f(\zeta,t),$$

where $f(\zeta,t)$ is defined by formula (27).

Remarks.

1. Since when formulating the problem there appears one of the values P_0 or $v_n^{(0)}$, but in the expressions of the functions $\Phi_0(\zeta,t)$ and $\Psi_0(\zeta,t)$ there are both values P_0 and $v_n^{(0)}$, it is necessary to write the dependence between these values. If at the initial moment on L_0 there appears P_0, then we will have

$$X_x = Y_y = P_0; \quad X_x + Y_y = 4\,\mathrm{Re}\,[\Phi_0(\sigma,0)]\,; \quad \mathrm{Re}\,[\Phi_0(\sigma,0)] = \frac{P_0}{2}, \quad \sigma \in L_0$$

and from (16) and (17) it follows that

$$\frac{P_0}{2} = \frac{P_0 + 2\mu^* m R_0^{-1} v_n^{(0)}}{\varkappa^* + 2},$$

whence we have

$$v_n^{(0)} = \frac{\varkappa^* P_0 R_0}{2\mu^* m}.$$

2. Let us consider a limiting case for the rhombus whose vertices A_1 and A_3 tend to infinity, that is, the domain S turns into a strip S^* with a circular hole. In this case, for the numbers α_j^0 $(j=\overline{1,4})$, we have $\alpha_1^0=\alpha_3^0=0$; $\alpha_2^0=\alpha_4^0=1$ and by putting $\omega(-\infty)=-R$, $\omega(\infty)=R$ for the function $\omega'(\zeta)$, we obtain the formula

$$\omega'(\zeta)=K_*\Big(1-\frac{\zeta^2}{R^2}\Big)^{-1}\prod_{j=1}^{\infty}\Big(1-\frac{\zeta^2}{R^{4j}R^2}\Big)^{-1}\Big(1-\frac{R^2}{R^{4j}\eta^2}\Big)^{-1}.$$

Consequently, the solution of the problem for the domain S^* can be carried out in a way similar to the one given above.

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