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# PROPER EMBEDDING FOR MORREY–LORENTZ SPACES

NAOYA HATANO

**Abstract.** The embeddings for the Morrey–Lorentz spaces have been previously introduced by M. A. Ragusa. This paper describes the major differences between these embedding types.

## 1. Introduction

In an extant study, Ragusa [5] has described the embeddings for the Morrey–Lorentz spaces. These function spaces can be defined as follows by recalling the expressions for the Lorentz spaces.

**Definition 1.1.** When $t > 0$ and $f$ is a measurable function on $\mathbb{R}^n$, the rearrangement function $f^*(t)$ can be expressed as

$$f^*(t) := \inf\{\lambda > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \lambda\}| \le t\}.$$

In the above expression, it is understood that $\inf \emptyset = \infty$. Let $0 < p, q \le \infty$. The Lorentz space $L^{p,q}(\mathbb{R}^n)$ can be defined as the linear space comprising all measurable functions $f$ with a finite quasi-norm given by

$$\|f\|_{L^{p,q}} := \begin{cases} \left( \int\limits_0^\infty \left[t^{\frac{1}{p}} f^*(t)\right]^q \frac{\mathrm{d}t}{t} \right)^{\frac{1}{q}}, & 0 < p, q < \infty, \\ \sup\limits_{t>0} t^{\frac{1}{p}} f^*(t), & 0 < p \le q = \infty. \end{cases}$$

Notably, the space $L^{\infty,q}(\mathbb{R}^n)$ for $0 < q < \infty$ (for details, refer to [1, Example 1.4.8]) is not considered in the above expression.

The Morrey–Lorentz spaces can be expressed as follows.

**Definition 1.2.** Let $0 < q \le p < \infty$ and $0 < r \le \infty$. Accordingly, the Morrey–Lorentz space $\mathcal{M}^p_{q,r}(\mathbb{R}^n)$ can be expressed as the space comprising all measurable functions $f$ with the finite quasi-norm

$$\|f\|_{\mathcal{M}^p_{q,r}} := \sup_{Q \in \mathcal{Q}(\mathbb{R}^n)} |Q|^{\frac{1}{p} - \frac{1}{q}} \|f \chi_Q\|_{L^{q,r}},$$

where $\mathcal{Q}(\mathbb{R}^n)$ is denoted by the set of all cubes in $\mathbb{R}^n$ that are parallel to the coordinate axes, and $\chi_E$ is an indicator function for a measurable set $E$.

Additionally, the function spaces can be considered extensions of the Lorentz and Morrey spaces as follows.

**Remark 1.3.** By definition, it is observed that

$$\mathcal{M}^p_{p,r}(\mathbb{R}^n) = L^{p,r}(\mathbb{R}^n) \quad \text{and} \quad \mathcal{M}^p_{q,q}(\mathbb{R}^n) = \mathcal{M}^p_q(\mathbb{R}^n)$$

with coincidence quasi-norms for $0 < q \le p < \infty$ and $0 < r \le \infty$. Moreover, $\mathcal{M}^p_q(\mathbb{R}^n)$ represents a Morrey space, which is endowed with the quasi-norm

$$\|f\|_{\mathcal{M}^p_q} := \sup_{Q \in \mathcal{Q}(\mathbb{R}^n)} |Q|^{\frac{1}{p} - \frac{1}{q}} \left( \int\limits_Q |f(x)|^q \, \mathrm{d}x \right)^{\frac{1}{q}}.$$

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G. Lorentz [3] defined the Lorentz spaces, and the separability of Lorentz spaces is proven. The Morrey spaces were introduced by Morrey [4] to investigate the solutions of second-order elliptic partial differential equations.

Ragusa introduced the embeddings for the Morrey–Lorentz spaces (see [5, Theorem 3.1]) as follows.

**Proposition 1.4.** *The following assertions hold:*

(1) *If* $0 < q \le p < \infty$ *and* $0 < r_1 \le r_2 \le \infty$,

$$\mathcal{M}^p_{q,r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q,r_2}(\mathbb{R}^n).$$

(2) *If* $0 < q_2 < q_1 \le p < \infty$ *and* $0 < r_1, r_2 \le \infty$,

$$\mathcal{M}^p_{q_1,r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2,r_2}(\mathbb{R}^n).$$

The main result obtained in this study can be expressed as follows.

**Theorem 1.5.** *Let* $0 < q < p < \infty$ *and* $0 < r < \tilde{r} \le \infty$. *Moreover, let* $R > 1$, *thereby satisfying* $(1+R)^{n/p-n/q}2^{n/q} = 1$. *Consider*

$$F := \bigcup_{m=1}^{\infty} F_m, \quad F_m := \bigcup_{a \in A(R,m)} \{y + a \in \mathbb{R}^n \,:\, y \in [0,1]^n\}, \tag{1.1}$$

$$A(R,m) := \left\{ a \in \mathbb{R}^n \,:\, a = \sum_{k=1}^{m} R(1+R)^{k-1} e_k \ \text{ for some } \ \{e_k\}_{k=1}^m \in (\{0,1\}^n)^m \right\}$$

*and*

$$V_k := \begin{cases} \emptyset, & k = 0, \\ \{x \in \mathbb{R}^n \,:\, (1+R)^{-k} x \in F\}, & k \in \mathbb{N}, \end{cases} \tag{1.2}$$

*and define*

$$f := \begin{cases} \displaystyle\sum_{k=1}^{\infty} \frac{1}{k^{\frac{1}{r}} (1+R)^{\frac{nk}{p}}} \chi_{V_k \setminus V_{k-1}}, & \tilde{r} < \infty, \\ \displaystyle\sum_{k=1}^{\infty} \frac{1}{(1+R)^{\frac{nk}{p}}} \chi_{V_k \setminus V_{k-1}}, & \tilde{r} = \infty. \end{cases} \tag{1.3}$$

*Thence*

$$f \in \mathcal{M}^p_{q,\tilde{r}}(\mathbb{R}^n) \setminus \mathcal{M}^p_{q,r}(\mathbb{R}^n).$$

We remark the sets $A(R,m)$, $F_m$ and $V_k$ as follows.

**Remark 1.6.** (1) The set $A(R,m)$ is increasing for the parameter $m \in \mathbb{N}$, that is,

$$A(R,1) \subset A(R,2) \subset \cdots \subset A(R,m) \subset \cdots.$$

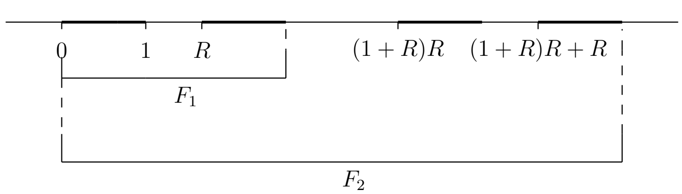
(2) It follows from (1) that $\{F_m\}_{m=1}^{\infty}$ is also an increasing family, that is,

$$[0,1]^n \subset F_1 \subset F_2 \subset \cdots \subset F_m \subset \cdots.$$

(3) The family $\{V_k\}_{k=0}^{\infty}$ stands for the expansion of $F$ and satisfies

$$V_0 \subset F \subset V_1 \subset V_2 \subset \cdots \subset V_k \subset \cdots.$$

The following figure is the family $\{F_m\}_{m=1}^{\infty}$ in the case $n = 1$:



Contrary to what the family $\{V_k\}_{k=0}^{\infty}$ stands for the expansion of $F$, the family $\{E_j\}_{j=0}^{\infty}$, which is defined in [9, Proposition 2.1], stands for the reduction of $F$, that is,

$$E_j = \{x \in \mathbb{R}^n : (1+R)^j x \in F_j\} \subset [0,1]^n.$$

The above Theorem 1.5 represents the proper embedding expressed as

$$\mathcal{M}^p_{q,r}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q,\tilde{r}}(\mathbb{R}^n)$$

for $0 < q < p < \infty$ and $0 < r < \tilde{r} \le \infty$. The other cases have been explored previously.

**Remark 1.7.** (1) The embedding $L^{p,r_1}(\mathbb{R}^n) \hookrightarrow L^{p,r_2}(\mathbb{R}^n)$ is proper for the cases, where $0 < p < \infty$ and $0 < r_1 < r_2 \le \infty$ (see, e.g., [1, Exercise 1.4.8]).

(2) Gunawan et al. [2] reported the proper embedding expressed as $\mathcal{M}^p_q(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q,\infty}(\mathbb{R}^n)$ for $0 < q < p < \infty$.

(3) In cases, where $0 < \tilde{q} < q \le p < \infty$, Sawano [6] revealed that $\mathcal{M}^p_q(\mathbb{R}^n)$ represents a non-dense subspace in $\mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$. Therefore, if $0 < q_2 < q_1 \le p < \infty$ and $0 < r_1, r_2 \le \infty$, by virtue of the embedding

$$\mathcal{M}^p_{q_1,r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{\frac{2q_1+q_2}{3}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{\frac{q_1+2q_2}{3}}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2,r_2}(\mathbb{R}^n),$$

the embedding $\mathcal{M}^p_{q_1,r_1}(\mathbb{R}^n) \hookrightarrow \mathcal{M}^p_{q_2,r_2}(\mathbb{R}^n)$ is proper.

From this perspective, this study attempts to validate Theorem 1.5.

Here and below, we employ the notation $A \sim B$ instead of $c^{-1}B \le A \le cB$ for some $c \ge 1$.

We organize the remaining part of the paper as follows: To prove Theorem 1.5, we prepare two lemmas in Section 2. In Section 3, we provide the proof of Theorem 1.5.

## 2. Preliminaries

This section describes the lemma required to prove Theorem 1.5.

**Lemma 2.1.** *Let* $0 < q < p < \infty$. *Similar to* (1.1) *in Theorem* 1.5 *above, set* $F \subset \mathbb{R}^n$. *Then* $\chi_F \in \mathcal{M}^p_q(\mathbb{R}^n)$. *Moreover, when* $\tilde{r} = \infty$ *in Theorem* 1.5*,*

$$f \sim \sup_{k \in \mathbb{N}} \frac{\chi_F((1+R)^{-k}\cdot)}{\|\chi_F((1+R)^{-k}\cdot)\|_{\mathcal{M}^p_q}}, \tag{2.1}$$

*where* $f$ *can be expressed as described in* (1.3).

The set $F$ introduced by (1.1) in Theorem 1.5 satisfies the fact that $\chi_F \in \mathcal{M}^p_q(\mathbb{R}^n)$ given in [8]. Moreover, the indicator function $\chi_F$ is not in the $\mathcal{M}^p_q(\mathbb{R}^n)$-closure of $\mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$ (see [6]). Especially, the statement

$$\chi_F \in \mathcal{M}^p_q(\mathbb{R}^n) \setminus \mathcal{M}^p_{\tilde{q}}(\mathbb{R}^n)$$

holds when $0 < q < \tilde{q} \le p < \infty$.

*Proof of Lemma* 2.1. By [9, Proposition 2.1],

$$\|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim (1+R)^{-\frac{jn}{p}}.$$

Then using the Fatou property for the Morrey (quasi-)norm $\|\cdot\|_{\mathcal{M}^p_q}$, we obtain

$$\|\chi_F\|_{\mathcal{M}^p_q} = \lim_{j\to\infty} \|\chi_{F_j}\|_{\mathcal{M}^p_q}$$

and

$$\|\chi_{E_j}((1+R)^{-j}\cdot)\|_{\mathcal{M}^p_q} = (1+R)^{\frac{jn}{p}} \|\chi_{E_j}\|_{\mathcal{M}^p_q} \sim 1,$$

as desired. □

In addition, to simplify the proof of Theorem 1.5, the Morrey–Lorentz quasi-norms for the functions $f$ given in (1.3) could be rewritten as follows.

**Lemma 2.2.** *Let $0 < q < p < \infty$. Similar* (1.2) *in Theorem* 1.5 *above, set $\{V_k\}_{k=0}^{\infty}$ and define*

$$g := \sum_{k=1}^{\infty} b_k \chi_{V_k \setminus V_{k-1}},$$

*where $\{b_k\}_{k=1}^{\infty}$ is a non-increasing positive sequence, where, for any $r_0 \in (0, \infty]$, we have*

$$\|g\|_{\mathcal{M}^p_{q,r_0}} \sim \sup_{j \in \mathbb{N} \cup \{0\}} |[0, (1+R)^j]^n|^{\frac{1}{p} - \frac{1}{q}} \|g\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}},$$

*where the implicit constant in "$\sim$" is independent of $r_0$.*

*Proof.* It is clear from the definition of $\|\cdot\|_{\mathcal{M}^p_{q,r_0}}$ that

$$\|g\|_{\mathcal{M}^p_{q,r_0}} \geq \sup_{j \in \mathbb{N} \cup \{0\}} |[0, (1+R)^j]^n|^{\frac{1}{p} - \frac{1}{q}} \|g\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}}.$$

If $Q \in \mathcal{Q}(\mathbb{R}^n)$ satisfies $|Q| \leq 1$, based on the monotonicity of $\{b_k\}_{k=1}^{\infty}$, it follows that

$$\begin{aligned} |Q|^{\frac{1}{p} - \frac{1}{q}} \|g\chi_Q\|_{L^{q,r_0}} &\leq |Q|^{\frac{1}{p} - \frac{1}{q}} \left\| \left( \sum_{k=1}^{\infty} b_1 \chi_{V_k \setminus V_{k-1}} \right) \chi_Q \right\|_{L^{q,r_0}} \\ &\leq |Q|^{\frac{1}{p} - \frac{1}{q}} \|b_1 \chi_Q\|_{L^{q,r_0}} \leq |[0,1]^n|^{\frac{1}{p} - \frac{1}{q}} \|b_1 \chi_{[0,1]^n}\|_{L^{q,r_0}} \\ &= |[0,1]^n|^{\frac{1}{p} - \frac{1}{q}} \|g\chi_{[0,1]^n}\|_{L^{q,r_0}}. \end{aligned} \tag{2.2}$$

Meanwhile, considering $j \in \mathbb{N}$, it can be supposed that $Q \in \mathcal{Q}(\mathbb{R}^n)$ satisfies $(1+R)^{(j-1)n} < |Q| \leq (1+R)^{jn}$. Note that for each $k \in \mathbb{N} \cap [1, j]$,

$$V_k = \bigcup_{l=1}^{\infty} \bigcup_{e \in \{0,1\}^n} (R(1+R)^{l-1+j} e + V_k^j),$$

where

$$V_k^j := V_k \cap [0, (1+R)^j]^n. \tag{2.3}$$

Considering the simple geometric observation, we see that

$$\sharp\{k \in \mathbb{N} : Q \cap (V_k \setminus V_{k-1}) \neq \emptyset\} \leq j.$$

In fact, since near the set $V_k^j$ is high density, if the cube $Q \in \mathcal{Q}(\mathbb{R}^n)$ takes $[0, (1+R)^j]^n$, for example, the left-hand side is larger, that is,

$$\sharp\{k \in \mathbb{N} : Q \cap (V_k \setminus V_{k-1}) \neq \emptyset\} \leq \sharp\{k \in \mathbb{N} : [0, (1+R)^j]^n \cap (V_k \setminus V_{k-1}) \neq \emptyset\} = j.$$

On the other hand, by the mapping $g : \mathbb{R}^n \to \{b_k\}_{k=1}^{\infty} \cup \{0\}$ and the monotonicity of $\{b_k\}_{k=1}^{\infty}$, we let a subsequence $\{b_{k(l)}\}_{l=1}^{\infty} \subset \{b_k\}_{k=1}^{\infty}$ by

$$b_{k(1)} := \max_{x \in Q} g(x) \leq b_1,$$

$$b_{k(l+1)} := \begin{cases} \max_{x \in Q \setminus g^{-1}(\{b_{k(\tilde{l})}\}_{\tilde{l}=1}^{l})} g(x), & Q \setminus g^{-1}(\{b_{k(\tilde{l})}\}_{\tilde{l}=1}^{l}) \neq \emptyset, \\ 0, & Q \setminus g^{-1}(\{b_{k(\tilde{l})}\}_{\tilde{l}=1}^{l}) = \emptyset \end{cases} \leq b_{l+1}$$

for each $l \in \mathbb{N}$, and then for all $l > j$, $b_{k(l)} = 0$ by the previous observation. Therefore,

$$\begin{aligned} \|g\chi_Q\|_{L^{q,r_0}} &\leq \left\| \left( b_1 \chi_{g^{-1}(\{b_{k(1)}\})} + \sum_{l=1}^{j} b_{l+1} \chi_{g^{-1}(\{b_{k(\tilde{l})}\})\}_{\tilde{l}=1}^{l+1}) \setminus g^{-1}(\{b_{k(\tilde{l})}\}_{\tilde{l}=1}^{l+1})} \right) \chi_Q \right\|_{L^{q,r_0}} \\ &\leq \|g\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}}, \end{aligned}$$

where we use the translation invariant for the Lorentz quasi-norm $\|\cdot\|_{L^{q,r_0}}$. It follows that

$$|Q|^{\frac{1}{p} - \frac{1}{q}} \|g\chi_Q\|_{L^{q,r_0}} \leq |[0, (1+R)^{j-1}]^n|^{\frac{1}{p} - \frac{1}{q}} \|g\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}}. \tag{2.4}$$

By gathering the estimates described in (2.2) and (2.4), it can be inferred that

$$|Q|^{\frac{1}{p}-\frac{1}{q}}\|g\chi_Q\|_{L^{q,r_0}} \le (1+R)^{-\frac{n}{p}+\frac{n}{q}} \sup_{j\in\mathbb{N}\cup\{0\}} |[0,(1+R)^j]^n|^{\frac{1}{p}-\frac{1}{q}}\|g\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}},$$

for all $Q \in \mathcal{Q}(\mathbb{R}^n)$. □

## 3. Proof of Theorem 1.5

First, the case $\tilde{r} < \infty$ can be proved as follows. In accordance with Lemma 2.2,

$$\sup_{j\in\mathbb{N}} |[0,(1+R)^j]^n|^{\frac{1}{p}-\frac{1}{q}}\|f\chi_{[0,(1+R)^j]^n}\|_{L^{q,r_0}} \sim \|f\|_{\mathcal{M}^p_{q,r_0}}$$

for all $r_0 \in (0,\infty)$. Next, considering $V^j_k$ as in (2.3) for $j,k \in \mathbb{N}$, it follows that

$$\begin{aligned}|V^j_k| &= \|\chi_F((1+R)^{-k}\cdot)\chi_{[0,(1+R)^j]^n}\|_{L^1} = (1+R)^{nk}\|\chi_F\chi_{[0,(1+R)^{j-k}]^n}\|_{L^1}\\ &= (1+R)^{nk}2^{(j-k)n}\end{aligned}$$

for all $k \in \mathbb{N}\cap[1,j]$. In accordance with [1, Example 1.4.2],

$$\left(f\chi_{[0,(1+R)^j]^n}\right)^* = \left(\sum_{k=1}^{j}\frac{1}{k^{\frac{1}{r}}(1+R)^{\frac{nk}{p}}}\chi_{V^j_k\setminus V^j_{k-1}}\right)^* = \sum_{k=1}^{j}\frac{1}{k^{\frac{1}{r}}(1+R)^{\frac{nk}{p}}}\chi_{[|V^j_{k-1}|,|V^j_k|)}$$

for $j \in \mathbb{N}$. Therefore,

$$\begin{aligned}\|f\chi_{[0,(1+R)^j]^n}\|^{r_0}_{L^{q,r_0}} &= \int_0^\infty\left[t^{\frac{1}{q}}\sum_{k=1}^{j}\frac{1}{k^{\frac{1}{r}}(1+R)^{\frac{nk}{p}}}\chi_{[|V^j_{k-1}|,|V^j_k|)}\right]^{r_0}\frac{\mathrm{d}t}{t}\\
&\sim \frac{\{(1+R)^n2^{(j-1)n}\}^{\frac{r_0}{q}}}{r_0(1+R)^{\frac{nr_0}{p}}} + \sum_{k=2}^{j}\frac{\{(1+R)^{nk}2^{(j-k)n}\}^{\frac{r_0}{q}} - \{(1+R)^{n(k-1)}2^{(j-k+1)n}\}^{\frac{r_0}{q}}}{r_0k^{\frac{r_0}{r}}(1+R)^{\frac{nkr_0}{p}}}\\
&= \frac{2^{\frac{njr_0}{q}}}{r_0}\left[1+\sum_{k=2}^{j}\frac{1-2^{\frac{nr_0}{q-p}}}{k^{\frac{r_0}{r}}}\right],\end{aligned}$$

for all $j \in \mathbb{N}\setminus\{1\}$ and $r_0 \in (0,\infty)$. Hence

$$\|f\|_{\mathcal{M}^p_{q,r_0}} \sim \left(\sum_{k=1}^{\infty}\frac{1}{k^{\frac{r_0}{r}}}\right)^{\frac{1}{r_0}} = \begin{cases}\infty, & r_0 = r,\\ \zeta\left(\frac{\tilde{r}}{r}\right)^{\frac{1}{\tilde{r}}}, & r_0 = \tilde{r},\end{cases}$$

where $\zeta(s)$ $(s > 1)$ denotes the Riemann zeta function. This proves that

$$f \in \mathcal{M}^p_{q,\tilde{r}}(\mathbb{R}^n)\setminus\mathcal{M}^p_{q,r}(\mathbb{R}^n).$$

Next, we prove the case $\tilde{r} = \infty$. Similar to the approach followed in the case $\tilde{r} < \infty$, we have

$$\|f\chi_{[0,(1+R)^j]^n}\|^r_{L^{q,r}} \sim \sum_{k=1}^{j}2^{\frac{jnr}{q}} = j2^{\frac{jnr}{q}}$$

for all $j \in \mathbb{N}$. Subsequently,

$$\|f\|_{\mathcal{M}^p_{q,r}} = \infty.$$

Meanwhile, in accordance with (2.1) and [7, Example 17], it follows that

$$f \in \mathcal{M}^p_{q,\infty}(\mathbb{R}^n).$$

This proves Theorem 1.5.

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Department of Mathematics, Chuo University, 1-13-27, Kasuga, Bunkyo-ku, Tokyo 112-8551, Japan
*Email address*: n.hatano.chuo@gmail.com