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ON SETS IN BINARY TOPOLOGICAL SPACES

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Abstract. The aim of this paper is to introduce some new types of sets in binary topological spaces in order to give a new characterization of them. Also, we discuss the relationships between them and find a connection between some of these new sets with the extremally disconnected and submaximal binary topological spaces. We compare these results with some results obtained in bioperation topological spaces and illustrate them with some examples.

1. Introduction

Structures dealing with two universal sets are important in mathematics because they allow us to study the relationships and interactions between two distinct mathematical objects or systems. These kinds of structures have applications in many fields, including mathematics, physics, computer sciences and engineering. Two common structures that deal with two universal sets are fuzzy sets and soft sets [1, 9, 12]. These structures provide a way to classify information, allowing for more nuanced and flexible reasoning and analysis. It is the case that the notion of binary topology introduced by Jothi and Thangavelu [6] plays an important role in providing notions of proximity between structures dealing with two universal sets. A binary topology $\mathscr{M}$ from a nonempty set X to a nonempty set Y is a subset of $\mathscr{P}(X) \times \mathscr{P}(Y)$ that satisfies some conditions of stability for operations related to unions and intersections. The binary topology was defined recently and has left the door open for the development of works related to this concept. To name a few instances, in [3–5] some notions of weaker binary open sets were introduced and some characterizations were obtained. In [7, 10, 11], separation axioms associated with binary topological spaces, binary sift sets and weaker notions of binary open sets were studied respectively. Within this work, we establish additional properties of closure and interior for a binary open sets and provide examples that these kinds of operations don't behave in the same way as the closure and interior operations of a topological space. After that, we introduce the notions of extremally disconnected and submaximal spaces in binary topological spaces and provide some properties. Finally, we introduce new classes of binary sets, so-called: $\mathscr{A}$-sets, $\mathscr{B}$-sets, $\mathscr{C}$-sets, $\mathscr{S}$-sets, α^*-open sets, s-sets, t-sets, β-set, β^{**}-set and locally closed sets. We study the relationship between them and illustrate with some examples.

2. Preliminaries

Throughout this section, $\mathscr{P}(X)$ and $\mathscr{P}(Y)$ denote the power set of X and Y, respectively. For $(A, B), (C, D) \in \mathscr{P}(X) \times \mathscr{P}(Y)$, the following notations will be assumed:

(1) $(A, B) \subseteq (C, D)$ means that $A \subseteq C$ and $B \subseteq D$. It can be expressed as (A, B) is a subset of (C, D).
(2) $(X, Y) \setminus (A, B) = (X \setminus A, Y \setminus B)$.

All definitions and results given in this section were introduced and proved in [6].

Definition 2.1. Let X and Y be any two nonempty sets. A binary topology from X to Y is a binary structure $\mathscr{M} \subset \mathscr{P}(X) \times \mathscr{P}(Y)$ that satisfies the following properties:

(1) $(\emptyset, \emptyset)$ and (X, Y) belong to $\mathscr{M}$.

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(2) If $(A_1, B_1) \in \mathscr{M}$ and $(A_2, B_2) \in \mathscr{M}$, then
$$(A_1 \cap A_2, B_1 \cap B_2) \in \mathscr{M}.$$
(3) If $\{(A_i, B_i)\}_{i \in I}$ is a family of elements of $\mathscr{M}$, then
$$(\cup_{i \in I} A_i, \cup_{i \in I} B_i) \in \mathscr{M}.$$

If $\mathscr{M}$ is a binary topology from X to Y, then the triple $(X, Y, \mathscr{M})$ is called a binary topological space, and the members of $\mathscr{M}$, denoted by (A, B), are called the binary open sets of the binary topological space $(X, Y, \mathscr{M})$. The pair (A, B) is called binary closed in $(X, Y, \mathscr{M})$ if $(X \setminus A, Y \setminus B) \in \mathscr{M}$.

Note that a binary topology $\mathscr{M}$ from X to Y isn't a topology on $X \times Y$. Firstly, because $\mathscr{M}$ is a subset of $\mathscr{P}(X) \times \mathscr{P}(Y)$, while a topology on $X \times Y$ is a subset of $\mathscr{P}(X \times Y)$. Secondly, one must be very careful with the union of sets operation. Although the elements of $\mathscr{M}$ can be understood as a Cartesian product, it is well known that the union and the Cartesian product do not distribute. Therefore the union of elements of $\mathscr{M}$ is not necessarily an element of $\mathscr{M}$.

Example. Let $X = Y = \mathbf{R}$ be the real number set, then
$$\mathscr{M} = \{(C, [0, +\infty)) : C \subseteq \mathbf{Q}\} \cup \{(\emptyset, \emptyset), (\mathbf{R}, \mathbf{R})\}$$
is a binary topology from X to Y.

Definition 2.2. Let $(X, Y, \mathscr{M})$ be a binary topological space, $(A, B) \subseteq (X, Y)$ and
$$(A, B)^{1^\diamond} = \cup \{A_\alpha : (A_\alpha, B_\alpha) \in \mathscr{M} \quad \text{and} \quad (A_\alpha, B_\alpha) \subseteq (A, B)\},$$
$$(A, B)^{2^\diamond} = \cup \{B_\alpha : (A_\alpha, B_\alpha) \in \mathscr{M} \quad \text{and} \quad (A_\alpha, B_\alpha) \subseteq (A, B)\}.$$
Then the pair $\left((A, B)^{1^\diamond}, (A, B)^{2^\diamond}\right)$ is called the binary interior of (A, B) and denoted by $\mathrm{Int}((A, B))$.

Definition 2.3. Let $(X, Y, \mathscr{M})$ be a binary topological space, $(A, B) \subseteq (X, Y)$ and
$$(A, B)^{1^*} = \cap \{A_\alpha : (X \setminus A_\alpha, Y \setminus B_\alpha) \in \mathscr{M} \quad \text{and} \quad (A, B) \subseteq (A_\alpha, B_\alpha)\},$$
$$(A, B)^{2^*} = \cap \{B_\alpha : (X \setminus A_\alpha, Y \setminus B_\alpha) \in \mathscr{M} \quad \text{and} \quad (A, B) \subseteq (A_\alpha, B_\alpha)\}.$$
Then the pair $\left((A, B)^{1^*}, (A, B)^{2^*}\right)$, is called the binary closure of (A, B) and denoted by $\mathrm{Cl}((A, B))$.

Some properties of binary interior and binary closure have been shown in [6], and some of them are listed below.

Theorem 2.1. *Let $(X, Y, \mathscr{M})$ be a binary topological space and (A, B), (C, D) be the subsets of (X, Y). Then the following statements hold:*

(1) $\mathrm{Int}((A, B)) \in \mathscr{M}$.
(2) $\mathrm{Int}((A, B)) \subseteq (A, B)$.
(3) $(A, B) \in \mathscr{M}$ *if and only if* $\mathrm{Int}((A, B)) = (A, B)$.
(4) *If* $(A, B) \subseteq (C, D)$, *then* $\mathrm{Int}((A, B)) \subseteq \mathrm{Int}((C, D))$.
(5) $\mathrm{Int}(\mathrm{Int}((A, B))) = \mathrm{Int}((A, B))$.
(6) $\mathrm{Cl}((A, B))$ *is a binary closed set.*
(7) $(A, B) \subseteq \mathrm{Cl}((A, B))$.
(8) *If* $(A, B) \subseteq (C, D)$, *then* $\mathrm{Cl}((A, B)) \subseteq \mathrm{Cl}((C, D))$.
(9) (A, B) *is a binary closed set if and only if* $\mathrm{Cl}((A, B)) = (A, B)$.
(10) $\mathrm{Cl}(\mathrm{Cl}((A, B))) = \mathrm{Cl}((A, B))$.

Like in general topology, many authors have defined the sets in terms of interior and closure, as shown in the following

Definition 2.4. Let $(X, Y, \mathscr{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Then, (A, B) is said to be:

(1) A binary regular open set [8] if $(A, B) = \mathrm{Int}(\mathrm{Cl}((A, B)))$.
(2) A binary regular closed set [8] if $(A, B) = \mathrm{Cl}(\mathrm{Int}((A, B)))$.
(3) A binary semi pre open set [5] if $(A, B) \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}((A, B))))$.

(4) A binary semi pre closed set [5] if $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}((A,B)))) \subseteq (A,B)$.
(5) A binary semi open set [3] if $(A,B) \subseteq \mathrm{Cl}(\mathrm{Int}((A,B)))$.
(6) A binary semi closed set [3] if $\mathrm{Int}(\mathrm{Cl}((A,B))) \subseteq (A,B)$.
(7) A binary pre open set [4] if $(A,B) \subseteq \mathrm{Int}(\mathrm{Cl}((A,B)))$.
(8) A binary pre closed set [4] if $\mathrm{Cl}(\mathrm{Int}((A,B))) \subseteq (A,B)$.
(9) A binary α-open set if $(A,B) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}((A,B))))$.

3. More About Binary Interior and Binary Closure

Now, we will show additional properties about closure and interior in a binary topology. The first of them is that the notions of binary interior and binary closure in binary topological spaces are dual to each other, as we can see in the next theorem.

Theorem 3.1. *Let $(X,Y,\mathscr{M})$ be a binary topological space and $(A,B) \subseteq (X,Y)$. Then the following statements hold:*

(1) $(X,Y) \setminus \mathrm{Cl}((A,B)) = \mathrm{Int}((X \setminus A, Y \setminus B))$.
(2) $(X,Y) \setminus \mathrm{Int}((A,B)) = \mathrm{Cl}((X \setminus A, Y \setminus B))$.

Proof.
(1) Let $(x,y) \in (X,Y) \setminus \mathrm{Cl}((A,B))$, then $x \in X \setminus (A,B)^{1^*}$ and $y \in Y \setminus (A,B)^{2^*}$ so, there exists $A_\alpha \subseteq X$ such that $x \notin A_\alpha$ for some (A_α, B_α) binary closed set and $(A,B) \subseteq (A_\alpha, B_\alpha)$, that means $x \in X \setminus A_\alpha$, $(X \setminus A_\alpha, Y \setminus B_\alpha) \in \mathscr{M}$ and $(X \setminus A_\alpha, Y \setminus B_\alpha) \subseteq (X \setminus A, Y \setminus B)$, hence $x \in X \setminus (A,B)^{1^\diamond}$. In a similar way, there exists $B_\beta \subseteq Y$ such that $y \notin B_\beta$ for some (A_β, B_β) binary closed set and $(A,B) \subseteq (A_\beta, B_\beta)$, that means $y \in Y \setminus B_\beta$, $(X \setminus A_\beta, Y \setminus B_\beta) \in \mathscr{M}$ and $(X \setminus A_\beta, Y \setminus B_\beta) \subseteq (X \setminus A, Y \setminus B)$, hence $y \in Y \setminus (A,B)^{2^\diamond}$. Therefore $(x,y) \in \mathrm{Int}((X \setminus A, Y \setminus B))$. Now, let $(x,y) \in \mathrm{Int}((X \setminus A, Y \setminus B))$, then $x \in (X \setminus A, Y \setminus B)^{1^\diamond}$ and $y \in (X \setminus A, Y \setminus B)^{2^\diamond}$ so, there exists $A_\alpha \subseteq X$ such that $x \in A_\alpha$ for some (A_α, B_α) binary open set and $(A_\alpha, B_\alpha) \subseteq (X \setminus A, Y \setminus B)$, that means $(X \setminus A_\alpha, Y \setminus B_\alpha)$ is a binary closed set $x \in X \setminus A_\alpha$ and $(A,B) \subseteq (X \setminus A_\alpha, Y \setminus B_\alpha)$, hence $x \notin (A,B)^{1^*}$. In a similar way, there exists $B_\beta \subseteq Y$ such that $y \in B_\beta$ for some (A_β, B_β) binary open set and $(A_\beta, B_\beta) \subseteq (X \setminus A, Y \setminus B)$, that means $(X \setminus A_\beta, Y \setminus B_\beta)$ is a binary closed set $y \in Y \setminus B_\beta$ and $(A,B) \subseteq (X \setminus A_\beta, Y \setminus B_\beta)$, hence $y \notin (A,B)^{2^*}$. Therefore $(x,y) \in (X,Y) \setminus \mathrm{Cl}((A,B))$.

(2) Let $(x,y) \in \mathrm{Cl}((X \setminus A, Y \setminus B))$, then $x \in (X \setminus A, Y \setminus B)^{1^*}$ and $y \in (X \setminus A, Y \setminus B)^{2^*}$ so, $x \in C_\alpha$ for all (C_α, D_α) binary closed set and $(X \setminus A, Y \setminus B) \subseteq (C_\alpha, D_\alpha)$. So, if $(A_\beta, B_\beta) \in \mathscr{M}$ and $(A_\beta, B_\beta) \subseteq (A,B)$, then $(X \setminus A_\beta, Y \setminus B_\beta)$ is a binary closed set and $(X \setminus A, Y \setminus B) \subseteq (X \setminus A_\beta, Y \setminus B_\beta)$, that means $x \in X \setminus A_\beta$ for all β and therefore $x \notin (A,B)^{1^\diamond}$. In a similar way, it can be concluded that $y \notin (A,B)^{2^\diamond}$; hence $(x,y) \in (X \setminus (A,B)^{1^\diamond}, Y \setminus (A,B)^{2^\diamond}) = (X,Y) \setminus ((A,B)^{1^\diamond}, (A,B)^{2^\diamond}) = (X,Y) \setminus \mathrm{Int}((A,B))$. Now, let $(x,y) \in (X,Y) \setminus \mathrm{Int}((A,B))$, then $x \in X \setminus (A,B)^{1^\diamond}$ and $y \in Y \setminus (A,B)^{2^\diamond}$ so, $x \notin A_\alpha$ for all (A_α, B_α) binary open set such that $(A_\alpha, B_\alpha) \subseteq (A,B)$. Let (C_β, D_β) be a binary closed set such that $(X \setminus A, Y \setminus B) \subseteq (C_\beta, D_\beta)$, then $(X \setminus C_\beta, Y \setminus D_\beta)$ is a binary open set and $(X \setminus C_\beta, Y \setminus D_\beta) \subseteq (A,B)$, and therefore $x \notin X \setminus C_\beta$ so, $x \in (X \setminus A, Y \setminus B)^{1^*}$. In a similar way, it can be concluded that $y \in (X \setminus A, Y \setminus B)^{2^*}$; hence $(x,y) \in (X \setminus A, Y \setminus B)^{1^*}, (X \setminus A, Y \setminus B)^{2^*}) = \mathrm{Cl}((X \setminus A, Y \setminus B))$. □

In topological spaces, even in generalized topological spaces, the closure points can be characterized as follows: $x \in \mathrm{Cl}(A)$ if and only if for every neighborhood U of x, $U \cap A \neq \emptyset$. In binary topological spaces, this is, in general, not true. Next, we will give a necessary condition and a counterexample.

Theorem 3.2. *Let $(X,Y,\mathscr{M})$ be a binary topological space, $(A,B) \subseteq (X,Y)$. If $(x,y) \in \mathrm{Cl}((A,B))$, then for every binary open set (U,V) such that $x \in U$ and $y \in V$, it is true that $U \cap A \neq \emptyset$ or $V \cap B \neq \emptyset$.*

Proof. Suppose that there exist a binary open set (U,V) such that $x \in U$ and $y \in V$ and $U \cap A = \emptyset$ and $V \cap B = \emptyset$, then $(X \setminus U, Y \setminus V)$ is a binary closed, $(A,B) \subseteq (X \setminus U, Y \setminus V)$ and $x \notin X \setminus U$ and $y \notin Y \setminus V$, hence $x \notin (A,B)^{1^*}$ and $y \notin (A,B)^{2^*}$. Therefore $(x,y) \notin \mathrm{Cl}((A,B))$. □

The reciprocal of the previous proposition is, in general, not true, as we will show in the following

Example. Let $X = Y = \mathbf{R}$ be the real number set and consider the following binary topology:

$$\mathscr{M} = \{(C, [0, +\infty)) : C \subseteq \mathbf{Q}\} \cup \{(\emptyset, \emptyset), (\mathbf{R}, \mathbf{R})\}.$$

For $A = \{0\}$ and $B = (-\infty, 0)$, we have

$$(A, B)^{1^*} = (\mathbf{R} \setminus \mathbf{Q}) \cup \{0\},$$
$$(A, B)^{2^*} = (-\infty, 0)$$

and therefore $\mathrm{Cl}((A, B)) = ((\mathbf{R} \setminus \mathbf{Q}) \cup \{0\}, (-\infty, 0))$. However, for every binary open set (U, V) such that $0 \in U$ and $0 \in V$, we obtain that $U \cap A \neq \emptyset$, but $(0, 0) \notin \mathrm{Cl}((A, B))$.

From Proposition 3.7 and Proposition 3.15 of [6], some results related to the union and intersection of closures and interiors are established.

Theorem 3.3. *Let $(X, Y, \mathscr{M})$ be a binary topological space and let (A, B) and (C, D) be the subsets of (X, Y). Then the following statements hold:*

1. $\mathrm{Cl}((A \cup C, B \cup D)) \supseteq \mathrm{Cl}((A, B))$.
2. $\mathrm{Cl}((A \cap C, B \cap D)) \subseteq \mathrm{Cl}((A, B))$.
3. $\mathrm{Int}((A \cup C, B \cup D)) \supseteq \mathrm{Int}((A, B))$.
4. $\mathrm{Int}((A \cap C, B \cap D)) \subseteq \mathrm{Int}((A, B))$.

Proof.
(1) Since $(A, B) \subseteq (A \cup C, B \cup D)$, by item (8) of Theorem 3.3 [6, item (v), Proposition 3.7], it follows that $\mathrm{Cl}((A, B)) \subseteq \mathrm{Cl}((A \cup C, B \cup D))$.
(2) Since $(A \cap C, B \cap D) \subseteq (A, B)$, by item (8) of Theorem 3.3 [6, item (v), Proposition 3.7], it follows that $\mathrm{Cl}((A \cap C, B \cap D)) \subseteq \mathrm{Cl}((A, B))$.
(3) Since $(A, B) \subseteq (A \cup C, B \cup D)$, by item (4) of Theorem 3.3 [6, item (v), Proposition 3.15], it follows that $\mathrm{Int}((A, B)) \subseteq \mathrm{Int}((A \cup C, B \cup D))$.
(4) Since $(A \cap C, B \cap D) \subseteq (A, B)$, by item (4) of Theorem 3.3 [6, item (v), Proposition 3.15], it follows that $\mathrm{Int}((A \cap C, B \cap D)) \subseteq \mathrm{Int}((A, B))$. □

The equality in the above theorem need not be true as shown in the following

Example. Let $X = \{1, 2, 3\}$, $Y = \{5, 6\}$ and

$$\mathscr{M} = \{(\emptyset, \emptyset), (X, Y), (\{1\}, \{6\}), (\{2\}, Y), (\{1, 2\}, Y)\}.$$

The binary closed sets are

$$\{(\emptyset, \emptyset), (X, Y), (\{2, 3\}, \{5\}), (\{1, 3\}, \emptyset), (\{3\}, \emptyset)\}.$$

1. Take $(A, B) = (\{1\}, \emptyset)$ and $(C, D) = (\{2\}, \emptyset)$, then
$$\mathrm{Cl}(A, B) = (\{1, 3\}, \emptyset),$$
but $\mathrm{Cl}((A \cup C, B \cup D)) = (X, Y)$.
2. Take $(A, B) = (\{1\}, \{5\})$ and $(C, D) = (\{2\}, \{6\})$, then
$$\mathrm{Int}((A, B)) = (\emptyset, \emptyset),$$
but $\mathrm{Int}((A \cup C, B \cup D)) = (\{1, 2\}, Y)$.
3. Take $(A, B) = (\{2, 3\}, \{5\})$ and $(C, D) = (\{3\}, \{6\})$, then
$$\mathrm{Cl}((A, B)) = (\{2, 3\}, \{5\}),$$
but $\mathrm{Cl}((A \cap C, B \cap D)) = (\{3\}, \emptyset)$.
4. Take $(A, B) = (\{1\}, \{6\})$ and $(C, D) = (\{2\}, Y)$, then
$$\mathrm{Int}((A, B)) = (\{1\}, \{6\}),$$
but $\mathrm{Int}((A \cap C, B \cap D)) = (\emptyset, \emptyset)$.

4. Extremally Disconnected and Submaximal Binary Spaces

Extremally disconnected spaces are topological spaces with the property that the closure of every open set is open. This means that the open sets in the space are completely separated from each other. In binary topological spaces, this notion can be studied.

Definition 4.1. Let $(X, Y, \mathscr{M})$ be a binary topological space. We said that $(X, Y, \mathscr{M})$ is an extremally disconnected binary space if the binary closure of every binary open set is a binary open set.

The following result provides a characterization of extremely disconnected binary spaces in terms of binary semi-open and binary pre-open sets.

Theorem 4.1. *Let $(X, Y, \mathscr{M})$ be a binary topological space. Then $(X, Y, \mathscr{M})$ is an extremally disconnected binary space if and only if every binary semi-open set is a binary pre-open set.*

Proof. Let (A, B) be a binary semi open set, since $(X, Y, \mathscr{M})$ is an extremally disconnected binary space, we have

$$\mathrm{Cl}(\mathrm{Int}((A,B))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}((A,B))))$$

and therefore

$$(A,B) \subseteq \mathrm{Cl}(\mathrm{Int}((A,B))) = \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}((A,B)))) \subseteq \mathrm{Int}(\mathrm{Cl}((A,B))),$$

hence (A, B) is a binary pre-open set. Now, let (A, B) be a binary open set, then $(A, B) \subseteq \mathrm{Cl}(\mathrm{Int}((A, B)))$, that means (A, B) is a binary semi-open set and by the hypothesis, (A, B) is a binary pre-open set. It follows that

$$\mathrm{Cl}(\mathrm{Int}((A,B))) \subseteq \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}((A,B))))).$$

So, $\mathrm{Cl}(\mathrm{Int}((A, B)))$ is a binary semi-open set and by the hypothesis, is a binary pre-open set, therefore

$$\begin{aligned}\mathrm{Cl}(A,B) &= \mathrm{Cl}(\mathrm{Int}((A,B)))\\ &\subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Cl}(\mathrm{Int}((A,B)))))\\ &= \mathrm{Int}(\mathrm{Cl}((A,B))).\end{aligned}$$

Hence $(X, Y, \mathscr{M})$ is an extremally disconnected binary space. □

A submaximal topological space is a topological space satisfying that every dense subset is open. Now, we are going to define a binary dense set and, consequently, a submaximal binary space. A subset (A, B) of a binary topological space is said to be a binary dense set, if $(X, Y) = \mathrm{Cl}((A, B))$.

Definition 4.2. Let $(X, Y, \mathscr{M})$ be a binary topological space. We say that $(X, Y, \mathscr{M})$ is a submaximal binary space if every binary dense set is a binary open set.

Now, we are going to characterize the submaximal binary spaces.

Theorem 4.2. *Let $(X, Y, \mathscr{M})$ be a binary topological space. $(X, Y, \mathscr{M})$ is a submaximal binary space if and only if for $(A, B) \subseteq (X, Y)$ such that* $\mathrm{Int}(A, B) = (\emptyset, \emptyset)$*, it is true that (A, B) is a binary closed set.*

Proof. Suppose that $(X, Y, \mathscr{M})$ is a submaximal binary space and let (A, B) be a subset of (X, Y) such that $\mathrm{Int}(A, B) = (\emptyset, \emptyset)$, then $\mathrm{Cl}(X \setminus A, Y \setminus B) = (X, Y)$ and by the hypothesis, $(X \setminus A, Y \setminus B) \in \mathscr{M}$, hence (A, B) is a binary closed set.

Now, let (A, B) be a subset of (X, Y) such that $\mathrm{Cl}(A, B) = (X, Y)$, then $\mathrm{Int}(X \setminus A, Y \setminus B) = (\emptyset, \emptyset)$ and by the hypothesis, it is true that $\mathrm{Cl}(X \setminus A, Y \setminus B) = (X \setminus A, Y \setminus B)$ so, $\mathrm{Int}(A, B) = (A, B)$ and therefore (X, Y) is a binary submaximal space. □

5. New Classes of Binary Sets and their Properties

In this section, using binary closure and binary interior, we are going to introduce a new class of binary sets and to study the relationships between them.

Definition 5.1. Let $(X, Y, \mathscr{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Then (A, B) is said to be:

(1) A binary α^*-set, if $\text{Int}(A, B) = \text{Int}(\text{Cl}(\text{Int}((A, B))))$.
(2) A binary t-set, if $\text{Int}((A, B)) = \text{Int}(\text{Cl}((A, B)))$.
(3) A binary s-set, if $\text{Int}((A, B)) = \text{Cl}(\text{Int}((A, B)))$.
(4) A binary β^*-set, if $\text{Int}((A, B)) = \text{Cl}(\text{Int}(\text{Cl}((A, B))))$.

Now, the existing relationships between the defined binary sets are presented in

Theorem 5.1. *Let $(X, Y, \mathscr{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Then the following statements hold:*

(1) *If (A, B) is a binary t-set, then (A, B) is a binary α^*-set.*
(2) *If (A, B) is a s-set, then (A, B) is a binary α^*-set.*
(3) *If (A, B) is a binary β^*-set, then (A, B) is both a binary t-set and s-set.*
(4) *If (A, B) is a binary regular open set, then (A, B) is a binary t-set.*
(5) *If (A, B) is a binary closed set, then (A, B) is a binary t-set.*

Proof.
(1) Note that $\text{Int}((A, B)) \subseteq \text{Int}(\text{Cl}(\text{Int}((A, B))))$. Suppose that (A, B) is a binary t-set, then

$$\text{Int}((A, B)) = \text{Int}(\text{Cl}((A, B))) \supseteq \text{Int}(\text{Cl}(\text{Int}((A, B)))),$$

hence $\text{Int}((A, B)) = \text{Int}(\text{Cl}(\text{Int}((A, B))))$, and therefore (A, B) is a binary α^*-set.
(2) Note that $\text{Int}((A, B)) \subseteq \text{Int}(\text{Cl}(\text{Int}((A, B))))$. Suppose that (A, B) is a binary s-set, then

$$\text{Int}((A, B)) = \text{Cl}(\text{Int}((A, B))) \supset \text{Int}(\text{Cl}(\text{Int}((A, B)))),$$

hence $\text{Int}(\text{Cl}(\text{Int}((A, B)))) = \text{Int}((A, B))$, and therefore (A, B) is a binary α^*-set.
(3) Let (A, B) be a binary β^*-set, then

$$\text{Int}((A, B)) = \text{Cl}(\text{Int}(\text{Cl}((A, B)))) \supset \text{Int}(\text{Cl}((A, B))),$$

hence $\text{Cl}(\text{Int}((A, B))) = \text{Int}((A, B))$, and therefore (A, B) is a binary β^*-set. In a similar way, since

$$\text{Int}((A, B)) = \text{Cl}(\text{Int}(\text{Cl}((A, B)))) \supset \text{Int}(\text{Cl}((A, B))),$$

hence $\text{Int}(\text{Cl}((A, B))) = \text{Int}((A, B))$, and therefore (A, B) is a binary t-set.
(4) Suppose that (A, B) is a binary regular open set, then $(A, B) = \text{Int}(\text{Cl}((A, B)))$, and therefore $\text{Int}((A, B)) = \text{Int}(\text{Cl}((A, B)))$, that is (A, B) is a binary t-set.
(5) Suppose that (A, B) is a binary closed set, then $\text{Int}((A, B)) = \text{Int}(\text{Cl}(A, B)))$, hence (A, B) is a binary t-set. □

The next examples show that the notions of a binary t-set and binary s-set are independent, as well the reciprocal propositions of Theorem 5.1, are not true.

Example. Let $X = Y = \mathbf{R}$ be the real number set and let us consider the following binary topology:

$$\begin{aligned}\mathscr{M} = &\{(C, [0, +\infty)) : C \subseteq \mathbf{Q}\} \cup \{(\emptyset, \emptyset), (\mathbf{R}, \mathbf{R})\} \\ &\cup \{(C, \emptyset) : C \subseteq \mathbf{Q}\} \cup \{(C, \mathbf{R}) : C \subseteq \mathbf{Q}\} \\ &\cup \{(\emptyset, [0, +\infty))\} \cup \{(\mathbf{R}, [0, +\infty))\}.\end{aligned}$$

For $A = \{0\}$ and $B = (-\infty, 0)$, we have

$$\begin{aligned}
\mathrm{Cl}((A,B)) &= ((\mathbf{R}\setminus\mathbf{Q})\cup\{0\}, (-\infty,0))\,,\\
\mathrm{Int}((A,B)) &= (\{0\},\emptyset)\,,\\
\mathrm{Int}(\mathrm{Cl}(((A,B))) &= (\{0\},\emptyset)\,,\\
\mathrm{Cl}(\mathrm{Int}(((A,B))) &= ((\mathbf{R}\setminus\mathbf{Q})\cup\{0\},\emptyset)\,,\\
\mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(((A,B)))) &= ((\mathbf{R}\setminus\mathbf{Q})\cup\{0\},\emptyset)\,,\\
\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(((A,B)))) &= (\{0\},\emptyset)\,.
\end{aligned}$$

Then (A, B):

(1) Is a binary t-set, but is not a binary s-set.
(2) Is a binary α^*-set, but is not a binary s-set.
(3) Is a binary t-set, but is not a binary regular closed set.
(4) Is a binary t-set, but is not a binary closed set.

Example. Let $X = \{1, 2\}$, $Y = \{5\}$, and consider

$$\mathscr{M} = \{(\emptyset,\emptyset),(X,Y),(\{1\},Y),(\{2\},\emptyset)\}.$$

Note that $(\{2\}, Y)$ is a binary s-set, but is not a binary t-set. In the same way, $(\{2\}, Y)$ is a binary α^*-set, but is not a binary t-set.

Example. Let $X = \{1, 2\}$, $Y = \{5, 6\}$, and consider

$$\mathscr{M} = \{(\emptyset,\emptyset),(X,Y),(\{1\},\{6\})\}.$$

Note that $(X, \emptyset)$ is both a binary t-set and a binary s-set, but is not a binary β^*-set.

The next result gives us the conditions for the reciprocal propositions of Theorem 5.1, came to be true.

Theorem 5.2. *Let $(X, Y, \mathscr{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Then the following statements hold:*

(1) *If (A, B) is a binary α^*-set and binary semi-open, then (A, B) is a binary t-set.*
(2) *If (A, B) is a binary α^*-set and $(X, Y, \mathscr{M})$ is a extremally disconnected binary space, then (A, B) is a binary s-set.*
(3) *If (A, B) is a binary t-set and a binary open set, then (A, B) is a binary regular open set.*

Proof.

(1) Note that $\mathrm{Int}((A, B)) \subseteq \mathrm{Int}(\mathrm{Cl}((A, B)))$. Suppose that (A, B) is a binary α^*-set and binary semi-open, then

$$\mathrm{Int}(\mathrm{Cl}((A,B))) \subseteq \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}((A,B)))) = \mathrm{Int}((A,B)).$$

It follows that $\mathrm{Int}(\mathrm{Cl}((A, B))) = \mathrm{Int}((A, B))$, and therefore (A, B) is a binary t-set.

(2) Suppose that (A, B) is a binary α^*-set, then $\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}((A, B)))) = \mathrm{Int}((A, B))$. Since $(X, Y, \mathscr{M})$ is an extremally disconnected space

$$\mathrm{Int}(\mathrm{Cl}(\mathrm{Int}((A,B)))) = \mathrm{Cl}(\mathrm{Int}((A,B))),$$

therefore $\mathrm{Cl}(\mathrm{Int}((A, B))) = \mathrm{Int}((A, B))$, that is, (A, B) is a binary s-set.

(3) Suppose that (A, B) is a binary t-set and binary open set, then

$$\mathrm{Int}(\mathrm{Cl}((A,B))) = \mathrm{Int}((A,B)) = (A,B),$$

hence (A, B) is a binary regular open set. □

Theorem 5.3. *Let $(X, Y, \mathscr{M})$ be a binary topological space and $(A, B) \subseteq (X, Y)$. Then the following statements hold:*

(1) *(A, B) is a binary α-open set and a binary α^*-set if and only if it is a binary regular open set.*
(2) *(A, B) is a binary α-open set if and only if it is a binary semi-open set and a binary pre-open set.*

Proof.

(1) Since (A, B) is a binary α-open set and a binary α^*-set, then (A, B) is a binary open set. It follows that $(A, B) \subseteq \text{Int}(\text{Cl}((A, B)))$, and then $\text{Int}(\text{Cl}((A, B))) = \text{Int}(\text{Cl}(\text{Int}((A, B)))) = \text{Int}((A, B)) = (A, B)$. Reciprocally, since (A, B) is a binary α-open set, it follows that $\text{Int}((A, B)) \subseteq \text{Int}(\text{Cl}(\text{Int}((A, B))))$. Now, since (A, B) is a binary regular open set, it follows that $\text{Int}((A, B)) = \text{Int}(\text{Cl}((A, B))) \supseteq \text{Int}(\text{Cl}(\text{Int}((A, B))))$ and then (A, B) is a binary α^*-set.

(2) It is clear that every binary α-open set is a binary semi open set and a binary pre-open set. Conversely, if (A, B) is a binary semi-open set and a binary pre-open set, then $(A, B) \subseteq \text{Cl}(\text{Int}((A, B)))$ and $(A, B) \subseteq \text{Int}(\text{Cl}((A, B)))$. It follows that

$$(A, B) \subseteq \text{Int}(\text{Cl}((\text{Cl}(\text{Int}((A, B))))))$$

and then $(A, B) \subseteq \text{Int}(\text{Cl}(\text{Int}((A, B))))$. □

Definition 5.2. Let $(X, Y, \mathscr{M})$ be a binary topological space and (A, B) be a subset of (X, Y). Then (A, B) is said to be:

(1) A binary $\mathscr{A}$-set if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary regular closed set.
(2) A binary $\mathscr{B}$-set if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary t- set.
(3) A binary $\mathscr{C}$-set if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary α^*- set.
(4) A binary locally closed set if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary closed set.
(5) A binary β-set if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary β^*-set.
(6) A binary $\mathscr{S}$-set if $(A, B) = (U \cap C, V \cap D)$, where $(U, V) \in \mathscr{M}$ and (C, D) is a binary s-set.

Theorem 5.4. *Let $(X, Y, \mathscr{M})$ be a binary topological space. The following statements hold:*

(1) *Every binary $\mathscr{A}$-set is a binary $\mathscr{B}$-set.*
(2) *Every binary $\mathscr{B}$-set is a binary $\mathscr{C}$-set.*
(3) *Every binary $\mathscr{S}$-set is a binary $\mathscr{C}$-set.*
(4) *Every binary locally closed set is a binary $\mathscr{B}$-set.*
(5) *Every binary $\mathscr{A}$-set is a binary locally closed set.*
(6) *Every binary β-set is both a binary $\mathscr{B}$-set and a binary $\mathscr{C}$-set.*
(7) *Every binary t-set is a binary $\mathscr{B}$-set.*
(8) *Every binary α^*-set is a binary $\mathscr{C}$-set.*
(9) *Every binary closed set is a binary $\mathscr{B}$-set.*
(10) *Every binary β^*-set is a binary β-set.*

Proof.

(1) Let (A, B) be a binary $\mathscr{A}$-set, then $(A, B) = (U \cap C, V \cap D)$ for some $(U, V) \in \mathscr{M}$ and some (C, D) binary regular closed set. As every binary regular closed set is a binary t-set, then $(A, B) = (U \cap C, V \cap D)$, for some $(U, V) \in \mathscr{M}$ and some (C, D) binary t-set, and therefore (A, B) is a binary $\mathscr{B}$-set.

(2) Let (A, B) be a binary $\mathscr{B}$-set, then $(A, B) = (U \cap C, V \cap D)$ for some $(U, V) \in \mathscr{M}$ and some (C, D) binary t-set. As every binary t-set is a binary α^*-set, then $(A, B) = (U \cap C, V \cap D)$, for some $(U, V) \in \mathscr{M}$ and some (C, D) binary α^*-set, and therefore (A, B) is a binary $\mathscr{C}$-set.

(3) Let (A, B) be a binary $\mathscr{S}$-set, then $(A, B) = (U \cap C, V \cap D)$ for some $(U, V) \in \mathscr{M}$ and some (C, D) binary s-set. As every binary s-set is a binary α^*-set, then $(A, B) = (U \cap C, V \cap D)$, for some $(U, V) \in \mathscr{M}$ and some (C, D) binary α^*-set, and therefore (A, B) is a binary $\mathscr{C}$-set.

(4) Let (A, B) be a binary locally closed set, then $(A, B) = (U \cap C, V \cap D)$, for some $(U, V) \in \mathscr{M}$ and some (C, D) binary closed set. As every binary closed set is a binary t-set, then $(A, B) = (U \cap C, V \cap D)$, for some $(U, V) \in \mathscr{M}$ and some (C, D) binary t-set, and therefore (A, B) is a binary $\mathscr{B}$-set.

(5) Let (A, B) be a binary $\mathscr{A}$-set, then $(A, B) = (U \cap C, V \cap D)$ for some $(U, V) \in \mathscr{M}$ and some (C, D) binary regular closed set. As every binary regular closed set is a binary closed set, then $(A, B) = (U \cap C, V \cap D)$, for some $(U, V) \in \mathscr{M}$ and some (C, D) binary closed set, and therefore (A, B) is a binary $\mathscr{B}$-set.

(6) Let (A, B) be a binary β-set, then $(A, B) = (U \cap C, V \cap D)$ for some $(U, V) \in \mathscr{M}$ and some (C, D) binary β^*-set. As every binary β^*-set is both a binary t-set and a binary s-set, and every binary s-set is α^*-set, then $(A, B) = (U \cap C, V \cap D)$, for some $(U, V) \in \mathscr{M}$ and some (C, D) binary t-set; $(A, B) = (U \cap C, V \cap D)$, for some $(U, V) \in \mathscr{M}$ and some (C, D) binary α^*-set. Therefore (A, B) is a binary $\mathscr{B}$-set and a binary $\mathscr{C}$-set.

(7) Let (A, B) be a binary t-set, then $(A, B) = (X \cap A, Y \cap B)$, note that $(X, Y) \in \mathscr{M}$ and therefore $(A, B) = (X \cap A, Y \cap B)$, for $(X, Y) \in \mathscr{M}$ and (A, B) binary t-set; hence (A, B) is a binary $\mathscr{B}$-set.

(8) Let (A, B) be a binary α^*-set, then $(A, B) = (X \cap A, Y \cap B)$, note that $(X, Y) \in \mathscr{M}$ and therefore $(A, B) = (X \cap A, Y \cap B)$, for $(X, Y) \in \mathscr{M}$ and (A, B) binary α^*-set; hence (A, B) is a binary $\mathscr{C}$-set.

(9) Let (A, B) be a binary closed set, then (A, B) is a binary t-set and $(A, B) = (X \cap A, Y \cap B)$, note that $(X, Y) \in \mathscr{M}$, and therefore $(A, B) = (X \cap A, Y \cap B)$, for $(X, Y) \in \mathscr{M}$ and (A, B) binary t-set; hence (A, B) is a binary $\mathscr{B}$-set.

(10) Let (A, B) be a binary β^*-set, then $(A, B) = (X \cap A, Y \cap B)$, note that $(X, Y) \in \mathscr{M}$, and therefore $(A, B) = (X \cap A, Y \cap B)$, for $(X, Y) \in \mathscr{M}$ and (A, B) binary β^*-set; hence (A, B) is a binary β-set. □

The converse of the items in Theorem 5.4, are not necessarily true, as we can see in the next example.

Example. Let $X = \{1, 2\}$, $Y = \{5, 6\}$ and consider

$$\mathscr{M} = \{(\emptyset, \emptyset), (X, Y), (\{1\}, Y), (\{2\}, \emptyset)\}.$$

(1) $(\{1\}, \{5\})$ is a binary $\mathscr{B}$-set, but is not a binary $\mathscr{A}$-set.
(2) $(\{2\}, \{6\})$ is a binary $\mathscr{C}$-set, but is not a binary $\mathscr{B}$-set.
(3) $(\{2\}, \{6\})$ is a binary $\mathscr{C}$-set, but is not a binary $\mathscr{S}$-set.
(4) $(\{2\}, \{6\})$ is a binary $\mathscr{C}$-set, but is not a binary locally closed set.
(5) $(\{2\}, \{5\})$ is a binary locally closed set, but is not a binary $\mathscr{A}$-set.
(6) $(X, \emptyset)$ is a binary $\mathscr{B}$-set, but is not a binary β-set.

For a binary topological space $(X, Y, \mathscr{M})$, the relationship between the different notions of binary open sets can be summarized in the following diagram:

$$\begin{array}{ccccccccc}
 & & & & \mathscr{A}\textit{-set} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \textit{locally closed} & & & & \\
 & & & & \downarrow & & & & \\
 & & & & \beta\textit{-set} & & & & \\
 & & & \swarrow & \uparrow & \searrow & & & \\
 & & & & \beta^{\star}\textit{-set} & & & & \\
 & \swarrow & & & & & & \searrow & \\
 & & & \swarrow & & \searrow & & & \\
\mathscr{S}\textit{-set} & \leftarrow & s\textit{-set} & & & & t\textit{-set} & \rightarrow & \mathscr{B}\textit{-set} \\
 & \searrow & & \searrow & & \swarrow & & \swarrow & \\
 & & & & \alpha^{\star}\textit{-set} & & & & \\
 & & & \searrow & \downarrow & \swarrow & & & \\
 & & & & \mathscr{C}\textit{-set} & & & &
\end{array}$$

Of interest is to know that the notions of open sets have been defined in the bioperation-topological space $(X, \tau, \gamma, \gamma')$ (see [2]) by using the operations on a topology, and the relationship between these notions are closely related to those established here.

Theorem 5.5. *Let $(X, Y, \mathscr{M})$ be a binary topological space. If every subset (A, B) of (X, Y) is a binary locally closed set, then $(X, Y, \mathscr{M})$ is a binary submaximal space.*

Proof. Let (A, B) be a binary dense set. By the hypothesis, there exists (U, V) binary open set and (C, D) binary closed set such that

$$(A, B) = (U \cap C, V \cap D).$$

Note that

$$(X, Y) = \mathrm{Cl}(A, B) = \mathrm{Cl}(U \cap C, V \cap D) \subseteq \mathrm{Cl}((C, D)) = (C, D).$$

So, $(A, B) = (U \cap X, V \cap Y) = (U, V) \in \mathscr{M}$, and therefore $(X, Y, \mathscr{M})$ is a binary submaximal space. □

In the topological spaces, submaximal spaces are characterized in terms of locally closed sets, that is, X is a submaximal space if and only if every subset of X is a locally closed set.

Question. Could it be possible to characterize a binary submaximal space in terms of binary locally closed sets?

6. Conclusions

In this work, making use of binary topological spaces and associated notions of interior and closure, new classes of binary sets are introduced, the relationship between them is studied, and the conditions such as extremely disconnected binary spaces or submaximal binary spaces are established in order to obtain characterizations of certain notions of binary sets. The impact of this study initially lies in the theoretical contribution of new concepts, however, it is highly likely to find applications in the theory of binary soft sets.

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