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A PRECIS OF DARK MATTER FOR MATHEMATICIANS

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Abstract. We contribute a mini-review of Dark Matter (DM), especially at galactic halo scales, to a mathematician reader, highlighting the logic and empirical statements which confirm each of our current beliefs and serve as the basis for the experimental Dark Matter searches. We pay attention to the shape of the DM halo around galaxies, the way the halo is elongated is a natural explanation of the orbits around the galaxy with the surprising constant velocity and how torsion of stellar streams emerges as a possible observable phenomenon that may test the shape of the halo.

1. Nine Decades of Dark Matter and Still Searching

Leverrier in 1846 explained the irregularities in the orbit of Uranus, respect to the Newtonian computation, by assuming the existence of an additional planet, Neptune, whose mass, orbit and position could be calculated, although it had theretofore not been detected. This initiated a physical tradition in which small discrepancies with the established mathematics lead either to modifications of the physical laws (such as the Michelson–Morley experiment or the small precession of Mercury's perihelion which gave rise to the theory of relativity) or to the addition of new material substances or states (such as the Hoyle resonance of Carbon 12 to explain stellar nuclear fusion).

And here we are halfway through the ninth decade since F. Zwicky [34] noticed that a mass more than the visible one is needed to explain the motion of what we call today "galaxy clusters", and yet that "Dark Matter" has not been detected to date.

From a suite of different observations we know the amount of it, its overall distribution in the universe, and its local distribution near galaxies, which we are addressing later in this work. We know that it is not made of conventional matter (anything from the periodic table of elements). What we ignore, such as its granularity (how much mass is there per DM unit), the intensity of its interaction with nuclei on Earth, the frequency of its annihilation in the cosmos and many others, we are constraining with a vigorous experimental effort. But a direct detection such as the historic observation of Neptune remains elusive.

It is natural then that a part of the community has focused on modifying the laws of gravitation that fail to match observations without DM, with the best known such theory being Modified Newtonian Dynamics (MOND). In this brief review we will expose why we believe that DM still seems the best alternative of the two. Combined theories with both modified gravity and additional matter content have also been proposed (but they are less minimal).

Section 2 presents a primer on dark matter observations in statement form, with each paragraph referring to the literature for the theory background. Section 3 lists a few types of DM searches that constrain its properties, but have failed to provide evidence for a material candidate to be the DM "atom". Section 4 discusses one widely known theory of DM as a modification of gravity and not as a sort of a substance. Section 5 turns to our own contribution, assessing from a theoretical perspective that galaxy rotation is naturally described by elongated DM haloes. In view of that observation, we are proposing in Section 6 a new observable, tailored to find deformations of that galactic halo from its spherical form, the geometric torsion of stellar streams. These are rivers of stars that are being newly identified [5] because of the extraordinary data quality and amount, thanks to, for example, the Gaia collaboration. They are actively being deployed to study Dark Matter distributions [23]. Finally, Section 7 completes the discussion.

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In addition to DM, the Hubble expansion of the universe is influenced by a parameter Λ, the cosmological constant, a second gravitational constant additional to Cavendish's G, that is sometimes interpreted as a "dark energy" fluid. It is entirely left out of this presentation that focuses on the parts that are known to have matter-like behaviour.

2. A Suite of Positive Astrophysical Observables and what they Entail

From the smallest to the largest scale, the positive evidence for Dark Matter in the universe can be organised as follows. There are many more astrophysical observables that we leave out for conciseness, but these here already support the standard picture that a form of inertial matter which does not interact electromagnetically dominates the matter content of the universe at scales between 10 kiloparsec and the 4.2 Gigaparsec of the entire observable cosmos.

Empirical Statement 1. *Galactic rotation curves fail to follow Kepler's third law (Newton's second law from the visible matter).*

Among the irrefutable pieces of evidence for something like Dark Matter, this is the one which appears at the smallest length scales, of order 1 to 50 kiloparsec. First of all, stellar velocities in galaxies are measured from the Doppler shift of atomic spectral lines. Typical velocities are 30-300 km/s, which are at most a thousandth of the speed of light. Newtonian mechanics is therefore a good framework to deal with galaxy rotation, with relativity bringing in modest corrections.

For circular motion of a satellite around either a pointlike or a spherical source of mass M, satisfying the orbital equilibrium condition

$$m\frac{v^2}{r} = \frac{GMm}{r^2} \tag{2.1}$$

leads to a version of Kepler's third law for the rotation velocity of the satellite, $v \propto \frac{1}{\sqrt{r}}$. In a Legendre expansion of the potential, this is the dominant term, with more complicated angular shapes, falling off with stronger powers of r in the denominator.

But the observed galaxy rotation is not informed at all by that classical theory. Instead, at sizeable distances from the galactic center, well beyond the visible matter, velocities turn to constants [26] $v \to$ constant. Figure 1 illustrates the point with data from SPARC [17].

The usual interpretation of this phenomenon is in terms of a "halo" of DM surrounding the visible galaxy. This will be further discussed in Section 5 below.

Empirical Statement 2. *Galactic rotation curves are not due to electromagnetism.*

One could conceive that in addition to gravity, there may be another force, such as the electric Coulomb repulsion of like charges, which helps maintain galactic rotation curves at a faster velocity. This is however untenable. The only long-distance force other than known gravity is electromagnetism, and it affects electric charges.

To rule the hypothesis out, rotation curves have been measured not for stellar light, but for neutral Hydrogen gas. The consistency of the velocity measurements of these uncharged clouds of gas [9] with those based on stars makes clear that the force supporting the rotation curves is due to gravity.

Empirical Statement 3. *Galactic velocities in clusters of galaxies are larger than predicted from the virial theorem and visible matter.*

This is probably the oldest clear manifestation of Dark Matter [34]. It arises from comparing data to the Virial Theorem of Classical Mechanics, which for a system of bodies can be condensed in the expression

$$2\langle T\rangle + \langle V\rangle = \frac{1}{2}(G(\tau) - G(0)) \xrightarrow[\tau\to\infty]{} 0$$

relating the averaged kinetic T and potential V energies to the virial quantity $G(\tau) = \sum_k \mathbf{p}_k \cdot \mathbf{r}_k|_{t=\tau}$ that vanishes for periodic orbits or large times. If this difference of Gs in the right hand side cancels, which is reasonably achieved by looking for smaller (1 Megaparsec size) clusters of galaxies that have contracted at least a factor of 2 in each spatial dimension (as is measured by the gas density), and with

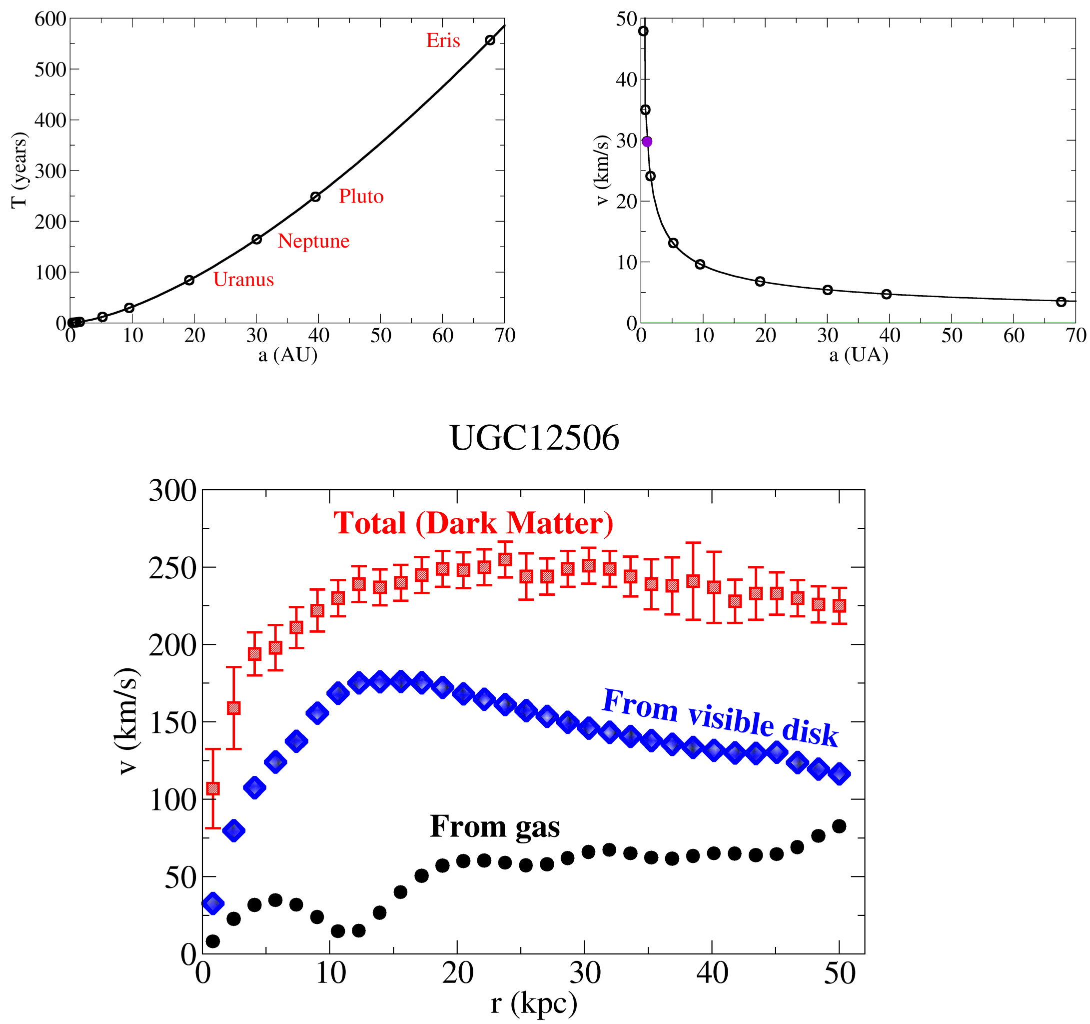

FIGURE 1. The top two plots represent Kepler's third law; on the left, in its best known form between the period and the semiaxis of each body's orbit. Induction from that data is a historical success valid to the outer reaches of the solar system. The right plot represents it as a body velocity v against distance to the sun (semiaxis) to expose the $v \propto \sqrt{r}$ dependence. The bottom plot depicts a typical galactic rotation curve, that of UGC12506 (data from the SPARC database). Two observations are relevant: visible matter contributes only a fraction of the total needed centripetal force to maintain the observed velocity (the velocities have to be added in quadrature due to $v^2 \propto M$, so the effect is larger than depicted) and the observed velocity (red squares) does not fall outside the visible part of the galaxy, unlike Kepler's law in the top right graph.

the well-known expressions for V and T, one can find the mass of the cluster in terms of the average squared velocity and inverse distance to the cluster's center of the individual galaxies as follows:

$$M = \frac{2\langle \mathbf{v^2} \rangle}{G\langle \frac{1}{r} \rangle} \to \frac{3\pi \langle \mathbf{v^2}_{\text{radial}} \rangle}{G\langle \frac{1}{r_\perp} \rangle}$$

(the second expression arises from a more detailed study when it turned out that the cluster is visible in the sky projected onto an almost planar area, perpendicular to the line of sight, whereas the velocities are measured radially by the Doppler shift of atomic emission lines).

Typical values of the mass to luminosity ratio M/L normalized to that of the Sun's are 200 ± 50 whereas for a galaxy the ratio is typically 10 ± 5. This means that the source of gravity tying galaxies to clusters is much darker than the matter in galaxies themselves.

A part of this excess matter is a conventional gas, which can be photographed, for example in X-ray wavelengths, while the rest is unaccounted; in fact, this virial observable alone, while requiring extra gravity, does not call for exotic Dark Matter, it just requires less luminous matter which could technically still be in the form of pebbles or dust (how that would arise in the depths of space between galaxies is another issue).

Empirical Statement 4. *Lensing of light from the background sources around galaxy clusters yields amounts of dark matter similar to the observation in Statement* 3.

In Einstein's theory of relativity, not only mass but also light is deflected by gravitational fields. Around mass distributions, light does not propagate in a straight line, as happens in a refractive medium, thence the name "lensing".

Two main forms of gravitational lensing are used to study the matter distribution. A light beam around a spherical or point mass source M with the closest approach r will deflect an angle

$$\alpha(r) = \frac{4GM}{rc^2}$$

(detailed computations for more difficult cases require ng the geodesic differential equations). In the sky, we typically do not know the actual position of light sources bending around a galaxy or cluster, so in a *strong lensing*, one looks for multiple clear images of that background source to constrain M with lens equations.

In a *weak lensing* [15], statistical studies of the geometric distortion of perceived galaxy shapes informs about the gravitational potential through which their light travels.

The method cannot distinguish between dark matter and baryonic matter, but it does show a total amount consistent with other approaches.

Empirical Statement 5. *The X-ray emission of the gas in a galaxy cluster is insufficient given the strength of its gravity.*

The X-ray luminosity per unit volume is proportional to the square of the ordinary (luminous) baryon density ρ_B,

$$L'_x(V) \propto \rho_B^2 \ ;$$

but the hydrostatic equilibrium equation needs to equilibrate the pressure by the gravitational pull of the total mass of the cluster, as controlled by the ordinary differential equation [31]

$$\frac{d}{dr}\left(\frac{r^2}{\rho_B(r)}\frac{d}{dr}\left(\frac{kT(r)\rho_B(r)}{m_B}\right)\right) = -(4\pi G)r^2\rho_M(r), \tag{2.2}$$

(k_B and G are the usual physical constants, and m_B is the mass of the typical DM object or particle), with an analogous equation holding for the equilibrium of the dark matter density. The analysis thereof [27] suggests that the ordinary gas is in the proportion of 1:8 to the total mass .

Empirical Statement 6. *Structure formation in the universe is compatible with the dominance of inertial matter, such as Cold Dark Matter.*

Gravitational attraction destabilizes an initially homogeneous cosmic medium with small fluctuations. Matter flows into the overdensities (Matthew's effect: excess matter attracts more matter) forming sheets and then filaments of density above average, as seen in Figure 2 from the Digital Sloan Sky Survey (see, *e.g.,* [12]).

The ratio of overdensity to average cosmological density for baryons [33] is $\delta\rho_b/\rho_b \gg 1$ today. However, the Cosmic Microwave Background, frozen when the universe's length scale was about 1100 times smaller than today, shows fluctuations at a level $\delta\rho_b/\rho_b|_{\rm CMB} \sim 10^{-5}$ which, if they grew up in

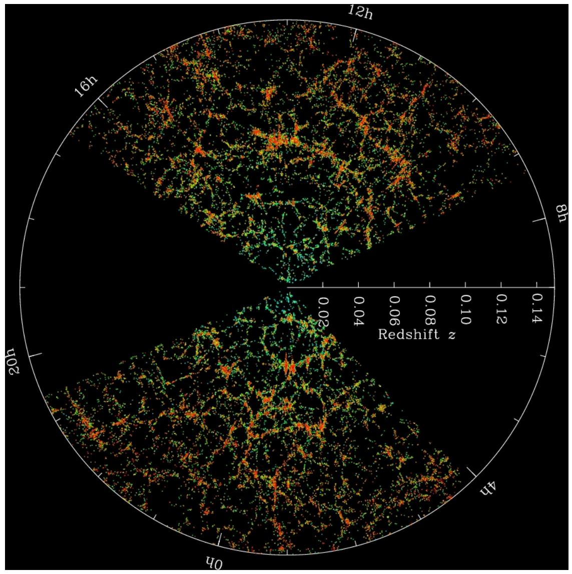

FIGURE 2. The SDSS [12] distribution of local galaxies shows a bubble-like structure of large-scale filaments and voids. Courtesy of the Sloan Digital Sky Survey III.

the linear regime, should be of order 10^{-2} today (much smaller than observed). Dark Matter is needed to accelerate gravitational collapse of a quasihomogeneous fluid into the seen structures by nonlinear mechanisms. The necessary DM has inertia but not pressure, thus pointing to nonconventional matter.

Empirical Statement 7. *Fluctuations in the Cosmic Microwave Background are best produced by a fluid with a large Dark Matter composition.*

The cosmic microwave background temperature fluctuations (measurable from the intensity and frequency of the microwave radiation) are described [19] by a brightness function (excess temperature over the average of 2.73 Kelvin degrees) with the argument of the unitary vector **n** pointing to the sky position from which the radiation comes, $\Theta(\hat{\mathbf{n}}) = \frac{\delta T}{T}$ and is decomposed into spherical harmonics as $\Theta(\hat{\mathbf{n}}) = \sum_{lm} Y_{lm}\Theta_{lm}$.

The correlation function of the l modes is parametrized by C_l,

$$\langle \Theta^*_{lm}\Theta_{l'm'}\rangle = \delta_{ll'}\delta_{mm'}C_l$$

a function of the integer variable l which has been by now very precisely measured and shows strong oscillations. The intensity of the peaks depends separately on the ordinary baryon matter (which has both inertia and pressure due to its electromagnetic interactions) and on the dark matter (which is taken as a purely inertial component). The data fit by the Planck collaboration [1] yields a ratio of dark to conventional matter of 5.4 : 1 so, a dark matter is by far dominant.

3. Failed Searches (so far)

With the overwhelming evidence for a form of nonbaryonic dark matter in large-scale astrophysics and cosmology, it is probably fair to say that the problem of the century in physics is to identify what that substance might actually be. Extensive searches have been conducted, all returning empty-handed so far. We here list a few, the first two belonging for the search of subatomic DM, the next ones trying to constrain macroscopic DM.

Empirical Statement 8. *Collider searches* (*laboratory production of DM particles*) *and direct searches* (*catching DM particles from space*) *have not yet found a particle candidate.*

If DM interacts via the weak nuclear force (or the Higgs force) with any of the Standard Model particles, it might be produced at accelerators [4]. Particularly intense have been searches for

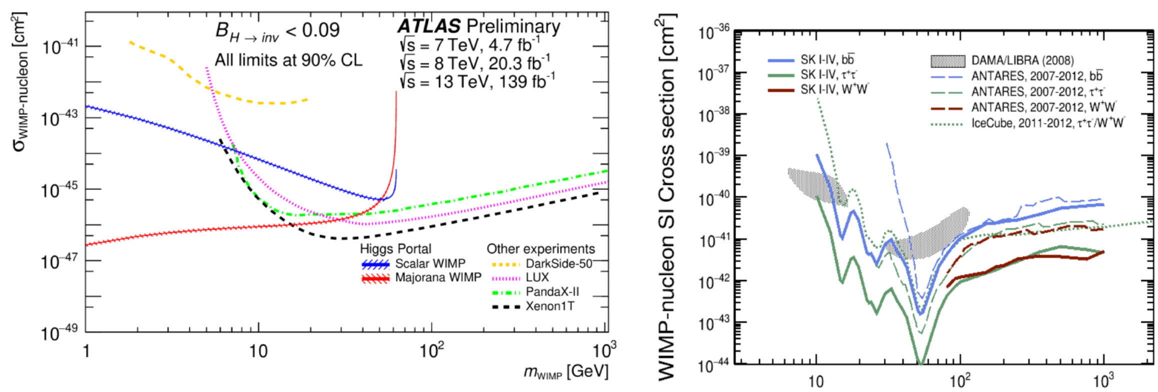

FIGURE 3. Left: collider and direct matter searches. The OX axis represents the mass of the presumed DM particle (not yet discovered). The OY axis depicts the equivalent probability of this DM particle hitting a nucleon whose recoil is detected (expressed as a cross-section). Each colored line, by a different experiment, excludes the parameter space above it in this plot. (Reproduced from [4] under the creative commons license: CC-BY-NC-ND-4.0.) Right: Same plot from indirect DM searches at Superkamiokande, a Japanese neutrino detector that looks for neutrinos coming from the center of the Earth, used as an annihilation pool for captured putative DM particles. (Reproduced from [20] under the creative commons license: CC-BY-DEED-3.0.)

teraelectronvolt-mass particles because of an aesthetically pleasing coincidence in the decoupling of DM in the early universe. These earthly production experiments are called "collider searches". There is also a vibrant experimental program of low–noise measurements trying to detect the recoil of an atomic nucleus (or an electron) hit by a passing DM particle from space. These are called "direct searches". No accepted DM candidate has been produced by these efforts. Then, a statistical analysis is employed to exclude parts of the parameter space.

A common graphical representation is the mass to cross-section diagram[1]. We exemplify with one such exclusion diagram in Figure 3.

For DM particle masses in the gigaelectronvolt-teraelectronvolt range, the maximum allowed cross-section above which DM would have been detected is of order 10^{-46} squared centimeters. For comparison, neutrinos, which interact only via the weak nuclear force, hit nucleons with a cross-section around 10^{-38} cm^2 at the 1 gigaelectronvolt energy. Thus, DM particles in the accelerator mass range would be very weakly interacting with conventional ones.

Empirical Statement 9. *Indirect searches (for Dark Matter particles decaying or pair-annihilating in astrophysical environments) are inconclusive.*

Indirect DM searches follow a different strategy. The DM particles are not supposed to annihilate inside experimental detectors; they may do this out in an astrophysical source [13] and the resulting $\gamma-$rays or other cosmic particles are then detected at Earth. Or alternatively, the core of the Earth can be used as the would-be stopper for the presumed DM particles and then secondary neutrinos can be looked for at the existing neutrino facilities [20]. This last case is shown in the right plot of Figure 3.

Every now and then, false positive results may occur; an example is the grey band in that plot, corresponding to a believed annual modulation of annihilation flux that was interpreted as stemming from DM, but that now has been ruled out owing to the advancing precision, as can be seen by the

[1]A cross-section is a physicist's way to express the probability of the DM-nucleus interaction taking place, expressing it as the equivalent plane-projected or shadow area of a rigid, solid sphere colliding with a pointlike target with the same probability.

solid lines that now leave that band in the exclusion zone (such exclusion bands are typically obtained at 95% confidence in view of the DM hypothesis, the statistical uncertainty of the experiment and an estimate of the systematic uncertainty based, for example, on the detection of the known radiation sources).

Empirical Statement 10. *Sky searches for macroscopic DM objects have not discovered an accepted candidate yet, although hints at black hole dark matter have been presented.*

The hypothesis that Dark Matter is in particulate form remains unproven. Two alternative hypotheses are popular, a traditional one in terms of compact objects (think of planetlike or starlike structures agglomerating DM) or a fluidlike substance with ultralight quanta that are not individually detectable, but that have to be observed in the bulk.

Let us start with the Massive Compact Halo Objects. These would be large boulders of dark matter scattered in the halo of the galaxy. When they cross the visual to a background star, their gravity should focuse starlight into Earthly telescopes like a magnifying glass, making the star shine brighter. If those compact objects were very heavy, they could evade these searches because of the unlikeliness of their crossing any visual by their smaller number (to reproduce the same amount of DM). But these multi-sun-mass dark stars would then gravitationally disrupt binary systems (pairs of stars) out of their orbit. The combination of both microlensing and binary searches excludes most if not all of the possible range of compact halo object masses [21], as illustrated in Figure 4. In spite of these analyses, some recent authors still find place among the constraints so, black holes in the 100 solar-mass range could be significant components of DM [7]. An important hint here is that the microlensing events from sources in M31 (Andromeda galaxy) do show the presence of some dark objects in its halo. Such population of black holes has been found in binary mergers by the aLIGO gravitational wave detector.

An alternative to DM forming compact objects is to think of ultralight DM candidates that form a fluid under different names like “wave dark matter”, “fuzzy dark matter”, “superfluid dark matter” etc. [6]. These ultralight DM particles behave as quantum even at very large scales, and can only be localized, due to Heisenberg’s uncertainty principle, in galactic-sized structures. Hence they are best treated as a fluid, without much hope of isolating their particle components. In the case of superfluids [16], for example, the waves in the fluid (collective phonons) may be subjected to interactions that disprove MOND results at the galactic scale (although this is not predicted, but merely accommodated). This is one of the most active lines of research in the last years. For example, freely infalling fluids can form large overdensities called caustics and known from optics [28]. Unfortunately, they do not seem to be visible in the SPARC database [10], nor other smoking guns for this fluid behaviour have really been established yet in observations.

Proposition 3.1. *Spherical DM haloes require a $\rho \propto r^{-2}$ radial dependence of the density, which is fine-tuned.*

In order for spherical DM haloes to explain the flatness of the galactic rotation curves, such as the one depicted in Figure 1, the density needs to fall off from the center exactly as $\rho = \text{constant } r^{-2}$. The proof is the simple observation that substituting $M \propto r$ in equation (2.1) leads to the desired $v = \text{constant}$. Because $M = V\rho$ and the spherical volume is $V \propto r^3$, the density needs to fall off with two powers of r.

This precise fall off is a case of “fine tuning”, as this is a very special dependence. It can be generated from an isothermal halo in which the hydrostatic equilibrium equation (2.2) is solved with constant temperature. This would provide an explanation for that precise law: but it would then behoove one to explain how exactly is dark matter supposed to thermalize, it being at best weakly interacting. This thermalization would restrict the scattering properties of DM which we are currently studying, a work that will be presented elsewhere.

In addition to the ones here discussed, many other ways to try and constrain Dark Matter have been devised, such as constraining neutron star cores, employing gravitational waves or trying to detect “light shining through a wall”, that is, the appearance of particles from where none should be coming.

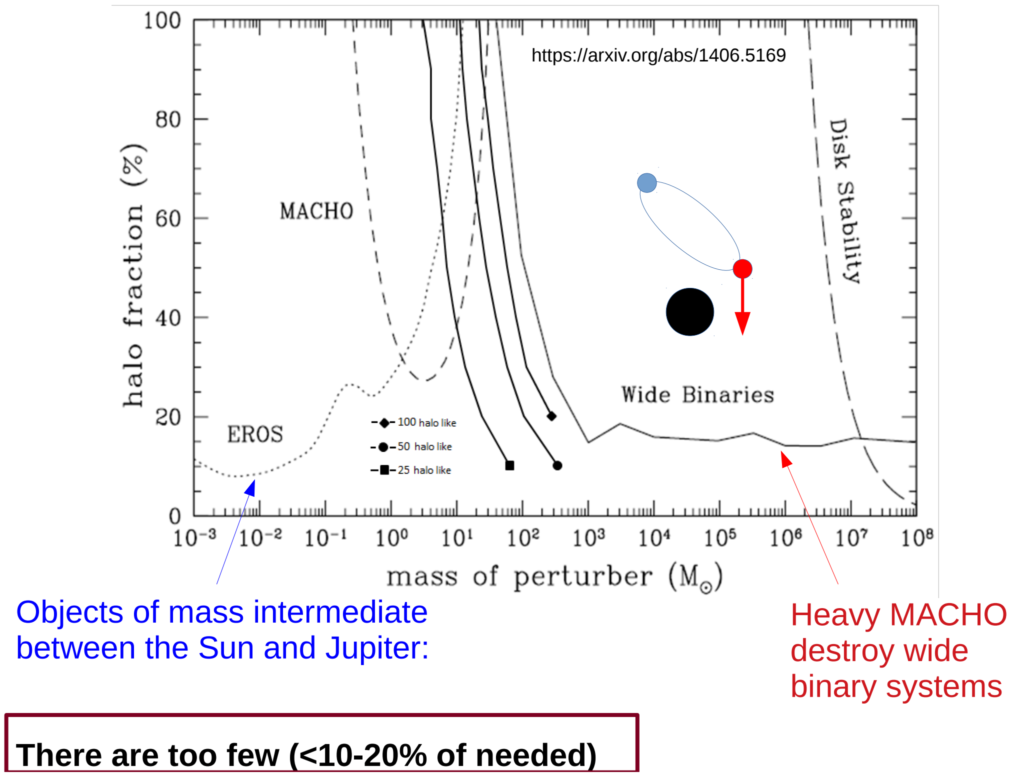

Figure 4. Searches for compact objects in the galactic halo by microlensing of light exclude Jupiter-type Dark Matter aggregates (left). The study of undisrupted binary star systems in the halo (right part of the plot) excludes multisolar mass objects. Thus, compact dark objects are disfavoured as DM candidates. (Data rendering reprinted from [21] with permission from C. Allen and the Institute of Physics, ©AAS; we have superimposed the sketch of the binary system and the orienting comments for the reader).

The literature is already very sizeable, as manifested by the Inspirehep database where almost 21000 publications are recorded to contain "Dark Matter" in the title. We hope that this discussion has given the reader a taste for the field's activity and proceed to the best known alternative to DM, namely, modified gravity.

4. Modified Newtonian Dynamics is Problematic at Larger-than-galactic Scales

We then turn to an alternative to Dark Matter, which is to modify the law of Newtonian gravitation. In MOND, a new acceleration scale $a_0 \sim 1.2\times 10^{-12}$meter/second^2 is postulated: above it, Newtonian mechanics applies, but at very small accelerations, a square root modification is hypothesized, yielding

$$\begin{aligned} a > a_0, &\quad a_{\rm Newton} = \frac{MG}{r^2} \\ a < a_0, &\quad a = \sqrt{a_0 a_{\rm Newton}} \end{aligned} \tag{4.1}$$

and thereafter the theory is formulated as an Effective Field Theory in the gravitational potential that reproduces that behaviour.

Proposition 4.1. *MOND can reproduce galactic rotation curves* $v = \text{constant}$*.*

Indeed, the net effect of the square root in equation (4.1) is to lower the power of Newton's $1/r^2$ force in equation (2.1) above by one unit, to $1/r$. Then, instead of a galactic rotation velocity $v \propto 1/\sqrt{r}$ falling with the distance to the galactic center, $v = \text{constant}$, is obtained from orbital equilibrium.

Empirical Statement 11. *Modified Newtonian Dynamics fails to fully reproduce cosmological observables at larger than galactic scales, particularly structure formation and the CMB peaks.*

The success of MOND at a galactic scale is not accompanied by empirical support at larger scales [11]. The reason MOND fails because both the structure formation and the primordial plasma oscillations seem to require more inertia and less electromagnetic interaction than conventional matter presents; but in MOND, all matter is conventional matter, since it is gravity in the weak regime that is modified. Also MOND underpredicts the velocity dispersion in galaxy clusters.

Other modified gravity theories that can come around some of these problems are less minimal and not more compelling (from the point of view of Occam's razor) than the DM theories that they seek to replace.

Empirical Statement 12. *It appears that Dark Matter and conventional matter can be separated.*

Galaxy cluster collisions differentiate between galaxies, that cross without large scattering, and gas clouds that can interact more strongly. The bullet cluster is a salient example [8]. There, the shocked hot gas is measured in X-rays in the central regions of the collision. Ordinary galaxies left that gas behind and are seen moving further away. The key point is that gravitational lensing maps of the distribution of gravitational potential see it accompanying the galaxies, already leaving the collision area. The fact that the gravitational field is not accompanying the shocked gas (most of the visible matter of the cluster) is unnatural in MOND: gravity is sourced by conventional matter. On the contrary, it is an indicator that DM can easily flow past ordinary matter.

A further evidence is the claim that there are galaxies without DM [30]. The first examples were NGC 1052-DF2 and DF4, which are diffuse galaxies for which the velocity v was measured from globular star clusters. They appear to have no DM at all (which does not make sense if there is no DM and instead a modification of gravity, always accompanying matter, is the culprit of the disagreement with Newtonian mechanics). Several more examples of this behaviour seem to have been found later. The irony of these findings is that the galaxies without dark matter establish their existence by showing that it is something that can be put there or taken away.

Finally, we can raise a theoretical objection to MOND on aesthetic grounds: the construction of Newtonian gravity with its $1/r^2$ force law that precisely matches the growth of the surface of a sphere as $S \propto r^2$ to leave a constant total flux of the force field, introducing a second order differential equation that accounts for the (Galilean) invariance under the changes in reference frame, is too precious to be easily dispensed with by an ad-hoc solution to a particular dynamical problem such as galactic rotation (and yes, this is an opinion that need not be universally shared).

5. Prolate Dark Matter Haloes

We now turn to our modest contribution to the discussion [18, 24, 25]. Our first observation is that the lowering of the r-power in the effective Newtonian interaction that MOND exerts on the galactic scale is achievable by reducing dimensions.

Proposition 5.1. *Kepler's third law in a two-dimensional* (*instead of three-dimensional*) *space is* $v = \text{constant}$*.*

The demonstration is very simple: again, equating the centrifugal force mv^2/r perceived by a star orbiting the galactic center against the Newtonian force corresponding to two dimensions, $F = Gm/r$, which can be obtained by spreading the flux of the force field over the circumference of length $2\pi r$, but not over a sphere of radius r^2.

Of course, we inhabit in a three-dimensional space, but such a dimensional reduction can be simply achieved by making one dimension redundant with a translational invariance.

Proposition 5.2. *Outside a cylindrical source of gravity (such as a filament or a cylinder of finite width) the rotational velocity is also* $v = \text{constant}$.

This proposition is well known from the theory of electromagnetism, as the potentials from a straight line of electric charge or current both give rise to an exterior electric or magnetic field, respectively, falling with $1/r$ in the direction, perpendicular to the line. In the case of interest here [29,32], a line of mass generates a gravitational field $g \propto 1/r$ and immediately, just like in Proposition 4.1, we obtain $v = \text{constant}$ which is the correct empirical rotation curve, the red squares at the top right of Figure 1.

In that plot, however, we see that the part attributable to DM (the top row of data) is monotonously increasing between 0 and about 15 kiloparsec. This is suggestive of the measured velocity corresponding to the interior of the mass distribution (since by the Gauss theorem, the exterior one does not contribute to the gravitational field of a symmetric body and the inside field is monotonically increasing with r).

Thus, the picture arising is that of a cylinder or otherwise elongated (prolate) distribution of Dark Matter, approximately perpendicular to the galactic plane, and of finite width in the radial direction to accommodate an initial growth of $v(r)$.

Empirical Statement 13. *The galactic rotation database favors prolate DM haloes over purely spherical or oblate ones. The ratio of major to minor axes (ellipticity) has an arithmetic average around* 0.69 *and a geometric average below* 0.1 *(many rotation curves are consistent with almost cylindrical haloes).*

This statement is the main conclusion of our major publication [25]. First, as a preliminary investigation, we explored several models of the DM halo and noticed that both spherical and cylindrical ones could fit the rotation data with similar χ^2 per degree of freedom. The best fit, though, came from a potential with logarithmic dependence $V \propto \log r$, that is, a cylindrical-type model, since the gradient of V leads to $F \propto 1/r$.

We then proceeded to a systematic investigation of the shape of the halo in terms of a multipole expansion (an expansion in associated Legendre functions of θ, with $\theta = \pi/2$ corresponding to the galactic plane). Two expansions were confronted with data: an expansion of the potential itself, and an expansion of the DM density from which the potential was later numerically calculated, with reasonably similar results.

For example, the second one employed a softened step function for the outward radial dependence,

$$\rho(r,\theta,\phi) = \frac{\rho_0}{1 + e^{(r-R(\theta,\phi))/a_0}} \ ,$$

with the angular shape of the equidensity surfaces, expanded as

$$R(\theta) = R_0 \cdot \left[1 + \sum_{l=1}^{\infty} \beta_{l0} Y_{l0}(\theta)\right] .$$

The balance of the galaxies preferred prolate haloes; the outliers were analyzed and classified, and only in very exceptional circumstances (*e.g.,* not enough data for a given galaxy) were removed from the fit.

We then analyzed the ellipticity b/a (with a along OZ is the major and b in the XY plane is the minor half-axes) of the spheroidal halo, which is replot in Figure 5.

Proposition 5.3. *The appropriate statistical estimator for averaging the ellipticity over the galaxy database is the geometric mean.*

A quick example to understand it is the arithmetic average of two numbers, 0.25 and 4, the first prolate and the second oblate with equal intensity, yielding $2.125 > 1$ which is biasing the average towards oblateness. But this pair of galaxies should yield an estimated spherical shape, with $\langle s \rangle = \langle a/b \rangle = 1$. It stands to reason that the correct averaging procedure for a variable distributed over $(0, \infty)$ with neutral point at $s = 1$ is to average on a log scale.

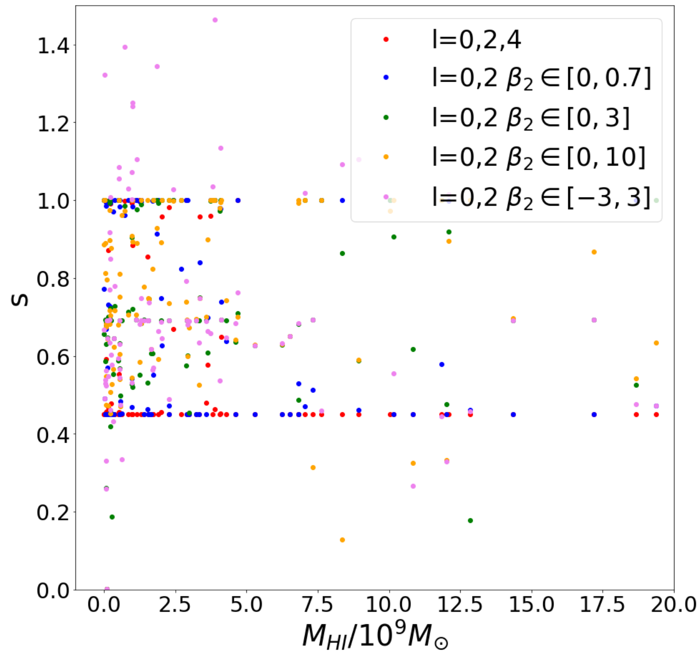

FIGURE 5. Scatterplot for the ellipticity (ratio between the minor and major axes) with range $s \in (0, \infty)$. Values below 1.0 correspond to prolate haloes (above 1.0, to oblate ones, with 1.0 perfectly spherical haloes). The accumulation points at 1.0 and about 0.45 stem from some of the fits implementing constraints over the spherical harmonic coefficient β_2. (For example, the haloes that in an unconstrained fit, the pink dots, turn up to be oblate, cluster at s = 1).

TABLE 1. Overview of averaged $s = b/a$ for ellipsoidal haloes. Additionally, the geometric average that we advocate. (If the original number is disaggregated by halo mass, we choose large values that make the condensation of typical spiral galaxies easier).

Method	s	$\log s$	Reference
Weak lensing	0.66(0.07)	−0.41(0.11)	[14]
Fit galactic $V(r)$		−3.6(4.7)	[25], uncleaned sample
Fit galactic $V(r)$	$0.14^{+0.7}$	−1.4	[25], curated sample
Fit galactic $V(r)$	$0.5^{+1.2}$	−4	[25], curated sample
Simulations at $z = 0$	$\simeq 0.6$		[2]

We then employ the geometric mean, coincident with the arithmetic mean of the log distribution of s, $\langle \log(s) \rangle$ and obtain, in a certain model with an exponential ellipsoid parametrization [25] the following:

$$e^{\langle \log(s) \rangle} = 0.026 << 1,$$

indicating a very prolate galaxy distribution, though with a large spread $s \in [0.0002, 2.783]$.

A second pair of analysis based on the spherical-harmonic expansion up to $l = 0, 2, 4$, yielded medians $s \simeq 0.01^{+0.20}$ and $s \simeq 0.16^{+0.60}$.

Table 1 then collects both arithmetic and geometric averages of galactic shape analysis; apparently, even the lensing data prefer prolate halos [14].

6. Torsion of Stellar Streams

If we adopt the hypothesis that Dark Matter haloes may be elongated, we should wish to test it against a new observable for which a definite prediction can be made. We have focused [3] on a geometric invariant in the theory of curves, torsion, for the reasons that we reveal here. In terms of the time derivatives of the position $\mathbf{r}$ of a body, $\mathbf{r}'$ (velocity), $\mathbf{r}''$ (acceleration) and $\mathbf{r}'''$ (jerk), the torsion can be computed by using the well known expression

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} \ . \tag{6.1}$$

Proposition 6.1. *The orbit of a body around a spherical source of gravity is torsionless.*

This can easily be demonstrated from equation (6.1) and Newton's second law for the motion [3]. A simple argument is that the conservation of the third component of the angular momentum confines the orbit to a plane, making it torsionless.

In reality, a spiral galaxy presents a disk structure in addition to the presumed DM halo. Even if this halo is taken as spherical, the sphericity of the overall source of gravity is broken by the visible disk. In our estimate with typical numbers for our own Milky Way galaxy, this leads to expected torsions for orbits near the plane, smaller than 10^{-3} kiloparsec^{-1}.

Thus, to establish that the halo is distorted from spherical shape, numbers well above this level need to be found.

Proposition 6.2. *Orbits around cylindrical objects do have nonvanishing torsion.*

If the source of gravity has perfect cylindrical symmetry, the torsion can easily be computed in terms of the azimuthal velocity v_φ of the body around the cylinder, the vertical velocity v_z and its radial distance ρ,

$$\tau = \frac{z'\varphi'}{(\rho\varphi')^2 + z'^2} = \frac{1}{\rho}\frac{v_z v_\varphi}{v_\varphi^2 + v_z^2} \ .$$

It is nonvanishing for helicoidal trajectories that wrap around the cylinder.

For more complicated elongated bodies, the differential equation for the trajectory can conveniently be numerically solved by an elementary integrator such as Euler's or Runge–Kutta's.

Empirical Statement 14. *Stellar streams provide observational surrogates for the orbits of bodies around a galaxy.*

Unlike comets, stars orbiting the galaxy do not leave clear trails which mark their orbit, and a human lifetime is too short to follow a significant stretch of such orbits when characteristic times are millions of years, especially for far structures that are more sensitive to the entire galactic matter distribution.

Fortunately, when a dwarf galaxy or a globular cluster of stars enters the galactic field, tidal forces stretch it forming a stellar stream [22]. Like runners on a track, stars closer to the galactic center advance a bit faster in angular position than the ones further out, so, an initial spherical cluster becomes an elongated structure, a river of stars. This is illustrated in Figure 6.

We therefore advocate the computation of torsion for the successive stellar streams that may be discovered. If $\log_{10}(\tau \times (1 \text{ kiloparsec}))$ is systematically larger than about -3, this would provide good evidence of nonspherical haloes.

We have examined a few stellar streams in our galaxy, at distances from the center of order 30 kiloparsec (far enough to be sensitive to the DM halo) and we extract nonvanishing torsion. We refer to [3] for further detail.

7. Conclusion

The presence of Dark Matter or a similar phenomenon in cosmology and galactic astrophysics is unavoidable to account for serious deviations in the predictions of conventional theories of gravity sourced by the conventional luminous matter alone.

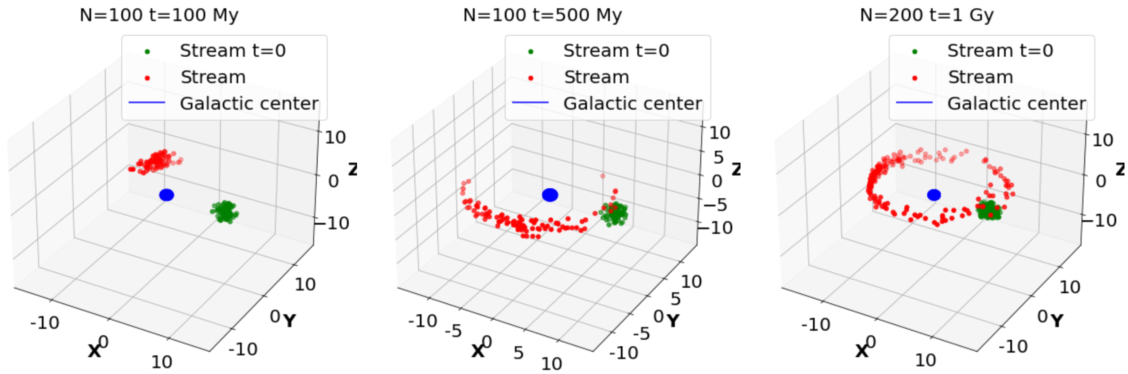

FIGURE 6. Formation of a tidal stream. Time advances from left to right as indicated. A blob of order 100 point stars (green) at $t = 0$ is stretched by the tidal field of a sphere+cylinder gravitational source with the parameters of our Milky Way galaxy. Because of a small initial v_z vertical velocity, the simulated stream is seen to acquire a torsion.

Whereas several observables at the galactic and cluster of galaxies scale need only a matter that is not radiating, which could be simply an ordinary baryonic matter, at larger scales the formation of structure in the universe and the cosmic microwave background do need purely inertial, pressureless matter: hence dark matter is some exotic substance. Likewise, the attempts to substitute it by a modification of gravity may work at galactic scales, but run against a number of obstacles at larger scales.

Presently, we have no compelling theory for the composition or nature of that dark matter. Particulate DM has failed to manifest itself in many experiments trying to detect it. Macroscopic aggregates (compact halo objects) have only small windows of mass; primordial black holes with 10-100 solar mass could be a possible candidate. Another research line focuses on ultralight particles forming a fluid of sorts, perhaps a superfluid at galactic scales.

Regardless of its nature, by studying the motion of stars we can infer the distribution of DM. We have exposed that elongated DM haloes naturally account for the rotation curves of galaxies, and proposed a new observable quantity–stellar streams torsion–to further study that hypothesis. We believe that the field will remain very active in the years to come.

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References

1. N. Aghanim, Y. Akrami, F. Arroja, M. Ashdown, J. Aumont, C. Baccigalupi, M. Ballardini, A. J. Banday, R. B. Barreiro, N. Bartolo, S. Basak, *Planck 2018 results-I. Overview and the Cosmological Legacy of Planck.* Astronomy & Astrophysics, vol. 641, A1, 2020.
2. B. Allgood, R. A. Flores, J. R. Primack, A. V. Kravtsov, R. H. Wechsler, A. Faltenbacher, J. S. Bullock, The shape of dark matter haloes: dependence on mass, redshift, radius and formation. *Mon. Not. Roy. Astron. Soc.* **367** (2006), no. 4, 1781–1796.
3. A. Bariego-Quintana, F. J. Llanes-Estrada, The torsion of stellar streams and the overall shape of galactic gravity's source, 2023. `https://arXiv:2307.07402`
4. J. K. Behr, Searches for Dark Matter with the ATLAS and CMS Experiments using LHC Run 2 (2015–2018) Data. *Pros Sci.* ATL-PHYS-PROC-2021-096. PANIC2021, Deutsches Elektronen-Synchrotron, 2021.

5. N. W. Borsato, S. L. Martell, J. D. Simpson, Identifying stellar streams in Gaia DR2 with data mining techniques. *Monthly Notices of the Royal Astronomical Society* **492** (2020), no. 1, 1370–1384.
6. J. A. R. Cembranos, C. Álvarez-Luna, R. L. Delgado, L. J. Garay, A. L. Maroto, S. J. Núñez Jareño, J. M. Sánchez Velázquez, J. J. Sanz-Cillero, H. Villarrubia-Rojo, Ultralight dark matter candidates. In: *Particle Physics at the Year of 150th Anniversary of the Mendeleev's Periodic Table of Chemical Elements: Proceedings of the Nineteenth Lomonosov Conference on Elementary Particle Physics*, 420–423, 2021.
7. S. Clesse, J. García-Bellido. Seven hints for primordial black hole dark matter. *Phys. Dark Univ.* **22** (2018), 137–146.
8. D. Clowe, A. Gonzalez, M. Markevitch, Weak-lensing mass reconstruction of the interacting cluster 1E 0657–558: Direct evidence for the existence of dark matter. *Astrophys. J.* **604** (2004), no. 2, 596–603.
9. L. Coccato, R. A. Swaters, V. C. Rubin, S. D'Odorico, S. S. McGaugh, VIMOS-VLT integral field kinematics of the giant low surface brightness galaxy ESO 323-G064. *Astron. Astrophys.* **490**, (2008), no. 2, 589–600.
10. D. Davydov, S. Troitsky, Testing universal dark-matter caustic rings with galactic rotation curves. *Phys. Lett. B* **839** (2023), 137798pp.
11. S. Dodelson, The real problem with MOND. *Int. J. Mod. Phys. D* **20** (2011), no. 14, 2749–2753.
12. D. J. Eisenstein, D. H. Weinberg, E. Agol, H. Aihara, C. A. Prieto, S. F. Anderson, J. A. Arns, É. Aubourg, S. Bailey, E. Balbinot, R. Barkhouser, SDSS-III: Massive spectroscopic surveys of the distant universe, the Milky Way, and extra-solar planetary systems. *The Astronomical Journal* **142** (2011), no. 3, 72pp.
13. S. Funk, Indirect Detection of Dark Matter with gamma rays. *Proc. Nat. Acad. Sci.* **112** (2015), no. 40, 12264–12271.
14. H. Hoekstra, H. K. Yee, M. D. Gladders, Properties of galaxy dark matter halos from weak lensing. *The Astrophysical Journal* **606** (2004), no. 1, 67pp.
15. N. Kaiser, G. Squires, Mapping the dark matter with weak gravitational lensing. *Astrophys. J.* **404** (1993), no. 2, 441–450.
16. J. Khoury, Dark Matter Superfluidity. *SciPost Phys. Lect. Notes* **42** (2022), 22pp.
17. F. Lelli, S. S. McGaugh, J. M. Schombert, SPARC: Mass models for 175 disk galaxies with Spitzer photometry and accurate rotation curves. *The Astronomical Journal* **152:157** (2016), no. 6, 14pp.
18. F. J. Llanes-Estrada, Elongated Gravity Sources as an Analytical Limit for Flat Galaxy Rotation Curves. *Universe* **7** (2021), no. 9, 346pp.
19. D. H. Lyth, A. R. Liddle, *The Primordial Density Perturbation: Cosmology, Inflation and the Origin of Structure*. Cambridge university press, 2009.
20. P. Mijakowski, Dark Matter Searches at Super-Kamiokande. *J. Phys. Conf. Ser.* **1342** (2020), no. 1, 012075pp.
21. M. A. Monroy-Rodríguez, C. Allen, The end of the MACHO era, revisited: new limits on MACHO masses from halo wide binaries. *Astrophys. J.* **790** (2014), no. 2, 159pp.
22. A. Noreña, J. C. Muñoz-Cuartas, L. F. Quiroga, N. Libeskind, Substructure formation in tidal streams of galactic minor mergers. *Rev. mex. astron. astrofis* **55** (2019), no. 2, 273–288.
23. C. G. Palau, J. Miralda-Escudé, The oblateness of the Milky Way dark matter halo from the stellar streams of NGC 3201, M68, and Palomar 5. *Monthly Notices of the Royal Astronomical Society* **524** (2023), no. 2, 2124–2147.
24. A. B. Quintana, F. J. Llanes-Estrada, O. M. Carretero, Dark-matter halo shapes from fits to SPARC galaxy rotation curves. *PoS EPS-HEP2021*, 2022. `https://arXiv:2109.11153`.
25. A. B. Quintana, F. J. Llanes-Estrada, O. M. Carretero, Galaxy rotation favors prolate dark matter haloes. *Phys. Rev. D* **107** (2023), no. 8, 083524pp.
26. V. C. Rubin, Jr. W. K. Ford, Rotation of the Andromeda nebula from a spectroscopic survey of emission regions. *Astrophys. J.* **159** (1970), 379–403.
27. S. Schindler, Ω_m-different ways to determine the matter density of the universe. In: *Matter in the Universe: Proceedings of an ISSI Workshop 19–23 March 2001*, 299–309, Bern, Switzerland. Springer Netherlands, 2002.
28. P. Sikivie, W. Kinney, Dark matter caustics. In: *3rd International Conference on Particle Physics and the Early Universe*, 54–61, 2000.
29. B. A. Slovick, *Cylindrical model for the dark matter halo of disk galaxies.* (preprint) 2010. `https://doi.org/10.48550/arXiv.1009.1113`.
30. P. van Dokkum, S. Danieli, Y. Cohen, A. Merritt, A. J. Romanowsky, R. Abraham, J. Brodie, C. Conroy, D. Lokhorst, L. Mowla, E. O'Sullivan, A galaxy lacking dark matter. *Nature* **555** (2018), no. 7698, 629–632.
31. S. Weinberg, *Cosmology.* Oxford University Press, Oxford, UK, 2008.
32. A. Widom, Y. N. Srivastava, N. Redington, Weyl cosmic strings and their consequences. *Phys. Rev. D* **48** (1993), no. 2, 554–556.
33. R. L. Workman et al., Review of particle physics. *PTEP* 2022, 083C01.
34. F. Zwicky, On the Masses of Nebulae and of Clusters of Nebulae. In: *A Source Book in Astronomy and Astrophysics, 1900–1975*, 729–737, Harvard University Press, 1979.

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