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BOUNDARY-TRANSMISSION DYNAMICAL PROBLEMS OF THE THERMO-PIEZO-ELECTRICITY THEORY WITHOUT ENERGY DISSIPATION

ANIKA TOLORAIA

Abstract. The Dirichlet, Neumann and mixed type interaction dynamical problems between thermo-elastic and thermo-piezo-elastic bodies are studied. The model under consideration is based on the Green–Naghdi theory of thermo-piezo-electricity without energy dissipation. This theory allows the thermal waves to propagate only with a finite speed. Using the Laplace transform, potential theory and the method of boundary pseudodifferential equations, the existence and uniqueness of solutions is proved and their smoothness is analyzed.

1. Introduction

In this paper, we investigate the boundary-transmission dynamical problems, i.e., the Dirichlet, Neumann and mixed type interaction dynamical problems between thermo-elastic and thermo-piezo-elastic bodies. The model under consideration is based on the Green–Naghdi theory of thermo-piezo-electricity without energy dissipation. This theory allows the thermal waves to propagate only with a finite speed.

Other models of thermo-piezo-electricity, in particular, the Voigt and Mindlin model, are well known. Our model is refined, it takes into account microrotation and microstretch of a particle.

Almost complete historical and bibliographical notes in this direction can be found in [23], where the dynamical equations of the thermo-piezo-electricity without energy dissipation are derived on the basis of the Green–Naghdi theory established in [21, 22] and on Eringen's results obtained in [19, 20]. In the present paper, we consider the pseudo-oscillation equations obtained by the Laplace transform from the dynamical equations derived by Ieşan in [23] for homogeneous isotropic solids possessing thermo-piezo-electricity properties without energy dissipation. The mixed and crack type pseudo-oscillation problem of thermo-piezo-electricity without energy dissipation is investigated in [5].

The basic dynamical problems of the classical elasticity and thermo-elasticity with either the Dirichlet or Neumann type boundary conditions on the whole boundary were developed in [24]. The mixed type dynamical problems of the classical elasticity for anisotropic bodies were studied in [25]. The mixed and crack type dynamical problems of the electro-magneto-elasticity can be found in [6] and the mixed boundary-transmission dynamical problems of generalized thermo-electro-magneto-elasticity theory for piecewise homogeneous composed structures are studied in [8].

In [16], a three-dimensional dynamical problem of fluid-solid interaction is considered, when an anisotropic elastic body occupying a bounded region is immersed in an inviscid fluid occupying an unbounded region, and the generalized Green–Lindsay's model of the thermo-electro-magneto-elasticity theory is considered in a solid redion. In this direction, one can see [9–15].

Using the Laplace transform, the potential theory and the method of boundary pseudodifferential equations, we prove the existence and uniqueness theorems of solutions in appropriate function spaces. We prove regularity results of the Dirichlet and Neumann boundary-transmission dynamical problems. Further, we analyze the regularity of solutions of a mixed type boundary-transmission dynamical problem near the exceptional curve, where different type boundary conditions collide. This regularity of solutions depends on the material constants and does not depend on the geometry of the exceptional

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curve. If these constants meet certain conditions, then the smoothness of solutions is $C^m([0,\infty), C^{\frac{1}{2}})$, $m \geq 2$ (see [2–5, 7]). Definition of this class see in Section 3 of this paper.

The Dirichlet, Neumann and mixed type boundary-transmission pseudo-oscillation problems of thermo-piezo-electricity without energy dissipation are studied in [17], and the mixed type boundary-transmission pseudo-oscillation, thermo-piezo-electricity problem with interior cracks and without energy dissipation can be found in [18].

2. Thermo-elastic Field Equations and Thermo-piezo-elastic Field Equations without Energy Dissipation

The model under consideration is based on the Green–Naghdi theory of thermo-piezo-electricity without energy dissipation.

Consider the disjoint bounded domains Ω_1 and Ω_2 in the Euclidean space $\mathbb{R}^3$ with sufficiently smooth boundaries $\partial\Omega_1 = S_1$ and $\partial\Omega_2 = S_1 \cup S_2$ ($S_1 \cap S_2 = \varnothing$). Throughout the paper, $n = (n_1, n_2, n_3)$ stands for the exterior unit normal vector to $\partial\Omega_1 = S_1$, and $\nu = (\nu_1, \nu_2, \nu_3)$ stands fot the exterior unit normal vector to $\partial\Omega_2 = S_1 \cup S_2$.

Suppose the domain Ω_1 is filled with a homogeneous thermo-elastic material, then the system of governing differential equations of dynamics with respect to the unknown vector function $U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top$, where $u^{(1)} = (u_1^{(1)}, u_2^{(1)}, u_3^{(1)})^\top$ is the displacement vector and $\vartheta^{(1)}$ is temperature, has the following form (see [24]):

$$(\mu^{(1)}+\varkappa^{(1)})\Delta u^{(1)}+(\lambda^{(1)}+\mu^{(1)})\operatorname{grad}\operatorname{div} u^{(1)}-\rho_1\partial_t^2 u^{(1)}-\beta_0^{(1)}\operatorname{grad}\partial_t\vartheta^{(1)}=(F_1^{(1)},F_2^{(1)},F_3^{(1)})^\top, \quad (2.1)$$

$$k^{(1)}\Delta\vartheta^{(1)} - a^{(1)}\partial_t^2\vartheta^{(1)} - \beta_0^{(1)}\partial_t \operatorname{div} u^{(1)} = F_4^{(1)}, \quad (2.2)$$

where $(F_1^{(1)}, F_2^{(1)}, F_3^{(1)})^\top$ is a mass force density, $F_4^{(1)}$ is a heat source density, ρ_1 is the mass density, $\mu^{(1)}$, $\varkappa^{(1)}$, $\lambda^{(1)}$, $\beta_0^{(1)}$, $k^{(1)}$ and $a^{(1)}$ are the thermo-elastic constants satisfying the conditions

$$\varkappa^{(1)} > 0, \ \ \varkappa^{(1)} + 2\mu^{(1)} > 0, \ \ \varkappa^{(1)} + 2\mu^{(1)} + 3\lambda^{(1)} > 0, \ \ k^{(1)} > 0, \ \ \rho_1 > 0, \ \ a^{(1)} > 0,$$
$$\beta_0^{(1)} > 0, \ \ \tau = \sigma + i\omega, \ \ \sigma > \sigma_0 > 0, \ \ \omega \in \mathbb{R}.$$

The stress operator for a homogeneous isotropic system of equations is defined as follows:

$$\begin{aligned} T^{(1)} &= T^{(1)}(\partial_x, n, \partial_t) = \big[T_{ij}^{(1)}(\partial_x, n, \partial_t)\big]_{4\times 4} \\ &:= \begin{bmatrix} [\lambda^{(1)} n_i\partial_j + \mu^{(1)} n_j\partial_i + \delta_{ij}(\mu^{(1)}+\varkappa^{(1)})n_k\partial_k]_{3\times 3}, & [-\beta_0^{(1)} n\,\partial_t]_{3\times 1} \\ [0]_{1\times 3} & k^{(1)} n_l\partial_l \end{bmatrix}_{4\times 4}. \end{aligned}$$

We can write the above system (2.1)–(2.2) of equations for pseudo-oscillations of the theory of homogeneous isotropic thermo-elasticity in the following matrix form:

$$A^{(1)}(\partial_x, \tau)U^{(1)} = F^{(1)},$$

where $U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top$, $F^{(1)} = (F_1^{(1)}, F_2^{(1)}, F_3^{(1)}, F_4^{(1)})^\top$, and $A^{(1)}(\partial_x, \partial_t)$ is the 4-dimensional matrix differential operator of the generalized thermo-elasticity:

$$\begin{aligned} A^{(1)}(\partial_x, \partial_t) &= \big[A_{ij}^{(1)}(\partial_x, \partial_t)\big]_{4\times 4} \\ &:= \begin{bmatrix} [\delta_{ij}(\mu^{(1)}+\varkappa^{(1)})\Delta + (\lambda^{(1)}+\mu^{(1)})\partial_i\partial_j - \rho_1\delta_{ij}\,\partial_t^2]_{3\times 3}, & -\beta_0^{(1)}\,\partial_t\,[\partial_i]_{3\times 1} \\ -\beta_0^{(1)}\,\partial_t\,[\partial_j]_{1\times 3} & -a^{(1)}\partial_t^2 + k^{(1)}\Delta \end{bmatrix}_{4\times 4}, \end{aligned}$$

where δ_{ij} is the Kronecker delta.

The domain Ω_2 is filled with a thermo-electro-elastic material. The corresponding system of differential equations of pseudo-oscillations with respect to the sought vector function $U^{(2)}$ has the following form (see [23]):

$$\begin{aligned} &(\mu^{(2)}+\varkappa^{(2)})\partial_j\partial_j u_i^{(2)} + (\lambda^{(2)}+\mu^{(2)})\partial_i\partial_j u_j^{(2)} - \rho_2\partial_t^2\, u_i^{(2)} + \varkappa^{(2)}\varepsilon_{ijk}\partial_j\phi_k^{(2)} \\ &\qquad + \lambda_0^{(2)}\partial_i\varphi^{(2)} - \beta_0^{(2)}\partial_t\,\partial_i\vartheta^{(2)} = -\rho_2 g_i^{(2)}, \quad i=1,2,3, \end{aligned} \quad (2.3)$$

$$k^{(2)}\partial_j\partial_j\vartheta^{(2)} - a^{(2)}\partial_t^2\,\vartheta^{(2)} - \beta_0^{(2)}\partial_t\,\partial_j u_j^{(2)} - c_0^{(2)}\partial_t\,\varphi^{(2)} + \nu_1^{(2)}\partial_j\partial_j\varphi^{(2)}$$
$$- \nu_3^{(2)}\partial_j\partial_j\psi^{(2)} = -\frac{1}{T_0}\,\rho_2 Q^{(2)}, \tag{2.4}$$
$$\gamma^{(2)}\partial_j\partial_j\phi_i^{(2)} + (\alpha^{(2)} + \beta^{(2)})\partial_j\partial_i\phi_j^{(2)} - I_0^{(2)}\partial_t^2\,\phi_i^{(2)} + \varkappa^{(2)}\varepsilon_{ijk}\partial_j u_k^{(2)}$$
$$- 2\varkappa^{(2)}\phi_i^{(2)} = -\rho_2 X_i^{(2)}, \quad i = 1,2,3, \tag{2.5}$$
$$(a_0^{(2)}\partial_j\partial_j - \xi_0^{(2)})\varphi^{(2)} - j_0^{(2)}\partial_t^2\,\varphi^{(2)} - \lambda_2^{(2)}\partial_j\partial_j\psi^{(2)} + \nu_1^{(2)}\partial_j\partial_j\vartheta^{(2)}$$
$$+ c_0^{(2)}\partial_t\,\vartheta^{(2)} - \lambda_0^{(2)}\partial_j u_j^{(2)} = -\rho_2 F^{(2)}, \tag{2.6}$$
$$\lambda_0^{(2)}\partial_j\partial_j\varphi^{(2)} + \chi^{(2)}\partial_j\partial_j\psi^{(2)} + \nu_3^{(2)}\partial_j\partial_j\vartheta^{(2)} = -g^{(2)}, \tag{2.7}$$

where $U^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \vartheta^{(2)}, \phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top$, $u^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)})^\top$ is the displacement vector, $\vartheta^{(2)}$ is temperature, $\phi^{(2)} = (\phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)})^\top$ is the vector of microrotation, $\varphi^{(2)}$ is the microstretch, $\psi^{(2)}$ is the electric field potential and $(g_1^{(2)}, g_2^{(2)}, g_3^{(2)})$ is the external body force per unit mass, $Q^{(2)}$ is the external rate of heat supply per unit mass, $X_i^{(2)}$ is the external body couple per unit mass, $F^{(2)}$ is the microstretch body force, $g^{(2)}$ is the density of free charge, T_0 is the initial reference temperature, ε_{ijk} is the Levi–Civita symbol and ρ_2 is the mass density.

Due to the positiveness of internal energy, the coefficients of system (2.3)–(2.7) must satisfy the following conditions:

$$\varkappa^{(2)} > 0, \quad \varkappa^{(2)} + 2\mu^{(2)} > 0, \quad \varkappa^{(2)} + 2\mu^{(2)} + 3\lambda^{(2)} > 0,$$
$$\xi_0^{(2)}(\varkappa^{(2)} + 2\mu^{(2)} + 3\lambda^{(2)}) > 3(\lambda_0^{(2)})^2,$$
$$\gamma^{(2)} > |\beta^{(2)}|, \quad a_0^{(2)}k^{(2)} - (\nu_1^{(2)})^2 > 0, \quad \beta^{(2)} + \gamma^{(2)} + 3\alpha^{(2)} > 0,$$
$$\chi^{(2)} > 0, \quad a^{(2)} > 0, \quad k^{(2)} > 0, \quad a_0^{(2)} > 0, \quad a_0^{(2)}(\gamma^{(2)} - \beta^{(2)}) > 2(b_0^{(2)})^2,$$
$$(\gamma^{(2)} - \beta^{(2)})[a_0^{(2)}k^{(2)} - (\nu_1^{(2)})^2] + 4b_0^{(2)}\nu_1^{(2)}\nu_2^{(2)} - 2a_0^{(2)}(\nu_2^{(2)})^2 - 2k^{(2)}(b_0^{(2)})^2 > 0,$$
$$\rho_2 > 0, \quad I_0^{(2)} > 0, \quad j_0^{(2)} > 0, \quad \beta_0^{(2)} > 0.$$

Denote by
$$A^{(2)}(\partial_x, \partial_t) = [A_{ij}^{(2)}(\partial_x, \partial_t)]_{9\times 9}$$
the matrix differential operator generated by the left-hand side expressions in (2.3)–(2.7),

$$A_{ij}^{(2)}(\partial_x, \partial_t) = \delta_{ij}(\mu^{(2)} + \varkappa^{(2)})\partial_l\partial_l + (\lambda^{(2)} + \mu^{(2)})\partial_i\partial_j - \rho_2\delta_{ij}\partial_t^2,$$
$$A_{i4}^{(2)}(\partial_x, \partial_t)) = -\beta_0^{(2)}\partial_t\,\partial_i, \quad A_{i,j+4}^{(2)}(\partial_x, \partial_t) = -\varkappa^{(2)}\varepsilon_{ijl}\partial_l,$$
$$A_{i8}^{(2)}(\partial_x, \partial_t) = \lambda_0^{(2)}\partial_i, \quad A_{i9}^{(2)}(\partial_x, \partial_t) = 0,$$
$$A_{4j}^{(2)}(\partial_x, \partial_t) = -\beta_0^{(2)}\partial_t\,\partial_j, \quad A_{44}^{(2)}(\partial_x, \partial_t) = k^{(2)}\partial_l\partial_l - a^{(2)}\partial_t^2,$$
$$A_{4,j+4}^{(2)}(\partial_x, \partial_t) = 0, \quad A_{48}^{(2)}(\partial_x, \partial_t) = \nu_1^{(2)}\partial_l\partial_j - c_0^{(2)}\partial_t, \quad A_{49}^{(2)}(\partial_x, \partial_t) = -\nu_3^{(2)}\partial_l\partial_l,$$
$$A_{i+4,j}^{(2)}(\partial_x, \partial_t) = -\varkappa^{(2)}\varepsilon_{ijl}\partial_l, \ \ A_{i+4,4}^{(2)}(\partial_x, \partial_t) = 0,$$
$$A_{i+4,j+4}^{(2)}(\partial_x, \partial_t) = \delta_{ij}\gamma^{(2)}\partial_l\partial_l + (\alpha^{(2)} + \beta^{(2)})\partial_i\partial_j - (2\varkappa^{(2)} + I_0^{(2)}\partial_t^2)\delta_{ij},$$
$$A_{i+4,8}^{(2)}(\partial_x, \partial_t) = 0, \quad A_{i+4,9}^{(2)}(\partial_x, \partial_t) = 0,$$
$$A_{8j}^{(2)}(\partial_x, \partial_t) = -\lambda_0^{(2)}\partial_j, \quad A_{84}^{(2)}(\partial_x, \partial_t) = \nu_1^{(2)}\partial_l\partial_l + c_0^{(2)}\partial_t,$$
$$A_{8,j+4}^{(2)}(\partial_x, \partial_t) = 0, \quad A_{88}^{(2)}(\partial_x, \partial_t) = a_0^{(2)}\partial_l\partial_l - (\xi_0^{(2)} + j_0^{(2)}\partial_t^2),$$
$$A_{89}^{(2)}(\partial_x, \partial_t) = -\lambda_2^{(2)}\partial_l\partial_l, \quad A_{9j}^{(2)}(\partial_x, \partial_t) = 0, \quad A_{94}^{(2)}(\partial_x, \partial_t) = \nu_3^{(2)}\partial_l\partial_l,$$
$$A_{9,j+4}^{(2)}(\partial_x, \partial_t) = 0 \quad A_{98}^{(2)}(\partial_x, \partial_t) = \lambda_2^{(2)}\partial_l\partial_l, \quad A_{99}^{(2)}(\partial_x, \partial_t) = \chi^{(2)}\partial_l\partial_l, \ \ i,j = 1,2,3.$$

The stress differential operator of thermo-electro-elasticity is defined as follows:

$$T^{(2)} = T^{(2)}(\partial_x, \nu, \partial_t) := [T_{ij}^{(2)}(\partial_x, \nu, \partial_t)]_{9\times 9},$$

where

$$T_{ij}^{(2)}(\partial_x, \nu, \partial_t) = \lambda^{(2)}\nu_i\partial_j + \mu^{(2)}\nu_j\partial_i + \delta_{ij}(\mu^{(2)} + \varkappa^{(2)})\nu_k\partial_k, \quad T_{i4}^{(2)}(\partial_x, \nu, \partial_t) = -\beta_0^{(2)}\nu_i\,\partial_t,$$
$$T_{i,j+4}^{(2)}(\partial_x, \nu, \partial_t) = -\varkappa^{(2)}\varepsilon_{ijk}\nu_k, \quad T_{i8}^{(2)}(\partial_x, \nu, \partial_t) = \lambda_0^{(2)}\nu_i, \quad T_{i9}^{(2)}(\partial_x, \nu, \partial_t) = 0,$$
$$T_{4,j}^{(2)}(\partial_x, \nu, \partial_t) = 0, \quad T_{44}^{(2)}(\partial_x, \nu, \partial_t) = k^{(2)}\nu_l\partial_l,$$
$$T_{4,j+4}^{(2)}(\partial_x, \nu, \partial_t) = -\nu_2^{(2)}\varepsilon_{ljk}\nu_l\partial_k, \quad T_{48}^{(2)}(\partial_x, \nu, \partial_t) = \nu_1^{(2)}\nu_k\partial_k,$$
$$T_{49}^{(2)}(\partial_x, \nu, \partial_t) = -\nu_3^{(2)}\nu_k\partial_k, \quad T_{i+4,j}^{(2)}(\partial_x, \nu, \partial_t) = 0,$$
$$T_{i+4,4}^{(2)}(\partial_x, \nu, \partial_t) = \nu_2^{(2)}\varepsilon_{lik}\nu_l\partial_k, \quad T_{i+4,j+4}^{(2)}(\partial_x, \nu, \partial_t) = \alpha^{(2)}\nu_i\partial_j + \beta^{(2)}\nu_j\partial_i + \delta_{ij}\gamma^{(2)}\nu_k\partial_k,$$
$$T_{i+4,8}^{(2)}(\partial_x, \nu, \partial_t) = b_0^{(2)}\varepsilon_{lik}\nu_l\partial_k, \quad T_{i+4,9}^{(2)}(\partial_x, \nu, \partial_t) = \lambda_1^{(2)}\varepsilon_{lik}\nu_l\partial_k,$$
$$T_{8j}^{(2)}(\partial x, \nu, \partial_t) = 0, \quad T_{84}^{(2)}(\partial_x, \nu, \partial_t) = \nu_1^{(2)}\nu_k\partial_k,$$
$$T_{8,j+4}^{(2)}(\partial_x, \nu, \partial_t) = -b_0^{(2)}\varepsilon_{lik}\nu_l\partial_k, \;\; T_{88}^{(2)}(\partial_x, \nu, \partial_t) = a_0^{(2)}\nu_k\partial_k, \quad T_{89}^{(2)}(\partial_x, \nu, \partial_t) = -\lambda_2^{(2)}\nu_k\partial_k,$$
$$T_{9j}^{(2)}(\partial x, \nu, \partial_t) = 0, \quad T_{94}^{(2)}(\partial_x, \nu, \partial_t) = \nu_3^{(2)}\nu_k\partial_k,$$
$$T_{9,j+4}^{(2)}(\partial_x, \nu, \partial_t) = -\lambda_1^{(2)}\varepsilon_{ljk}\nu_l\partial_k, \quad T_{98}^{(2)}(\partial_x, \nu, \partial_t) = \lambda_2^{(2)}\nu_k\partial_k,$$
$$T_{99}^{(2)}(\partial_x, \nu, \partial_t) = \chi^{(2)}\nu_k\partial_k, \;\; i, j = 1, 2, 3.$$

The system of equations (2.3)–(2.7) can be written in a matrix form

$$A^{(2)}(\partial_x, \partial_t)U^{(2)} = \Phi,$$

where

$$U^{(2)} = (u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \vartheta^{(2)}, \phi_1^{(2)}, \phi_2^{(2)}, \phi_3^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top,$$
$$\Phi = -\Big(\rho_2 g_1^{(2)}, \rho_2 g_2^{(2)}, \rho_2 g_3^{(2)}, \frac{1}{T_0}\rho_2 Q^{(2)}, \rho_2 X_1^{(2)}, \rho_2 X_2^{(2)}, \rho_2 X_3^{(2)}, \rho_2 F^{(2)}, g^{(2)}\Big)^\top$$

and $A^{(2)}(\partial_x, \tau)$ is the 9-dimensional matrix differential operator corresponding to system (2.3)–(2.7).

3. Formulation of the Boundary-transmission Dynamical Problems

3.1. Formulation of the Dirichlet boundary-transmission dynamical problem $(TD)_t$. We are looking for a solution

$$U^{(1)} = (u^{(1)}, \vartheta^{(1)})^\top = (u^{(1)}, u_4^{(1)})^\top,$$
$$U^{(2)} = (u^{(2)}, \vartheta^{(2)}, \phi^{(2)}, \varphi^{(2)}, \psi^{(2)})^\top = (u^{(2)}, u_4^{(2)}, u_5^{(2)}, \dots, u_9^{(2)})^\top$$

of the dynamical equations

$$A^{(1)}(\partial_x, \partial_t)U^{(1)} = \Phi_1 \quad \text{in } \Omega_1 \times [0, \infty),$$
$$A^{(2)}(\partial_x, \partial_t)U^{(2)} = \Phi_2 \quad \text{in } \Omega_2 \times [0, \infty),$$

which satisfy on the surface S_1 the following boundary-transmission conditions:

$$\{u_j^{(1)}\}^+ - \{u_j^{(2)}\}^+ = f_j^{(1)} \quad \text{on } S_1 \times [0, \infty), \;\; j = \overline{1, 4},$$
$$\{T^{(1)}(\partial_x, n, \partial_t)U^{(1)}\}_j^+ + \{T^{(2)}(\partial_x, \nu, \partial_t)U^{(2)}\}_j^+ = f_j^{(2)} \quad \text{on } S_1 \times [0, \infty), \;\; j = \overline{1, 4}, \;\; \nu = -n,$$

and the boundary conditions

$$\{u_j^{(2)}\}^+ = Q_j^{(2)}, \quad S_1 \times [0, \infty), \;\; j = \overline{5, 9},$$

while on the surface S_2, the Dirichlet boundary condition

$$\{U^{(2)}\}^+ = p^{(2)} \quad \text{on } S_2 \times [0, \infty),$$

and the initial conditions

$$u_j^{(1)}(x,0)=0,\quad \partial_t\, u_j^{(1)}(x,0)=0,\quad x\in\Omega_1,\ \ j=\overline{1,4},$$
$$u_j^{(2)}(x,0)=0,\quad \partial_t\, u_j^{(2)}(x,0)=0,\quad x\in\Omega_2,\ \ j=\overline{1,8}.$$

3.2. **Formulation of the Neumann boundary-transmission dynamical problem $(TN)_t$.** We are looking for a solution

$$U^{(1)}=(u^{(1)},\vartheta^{(1)})^\top=(u^{(1)},u_4^{(1)})^\top,$$
$$U^{(2)}=(u^{(2)},\vartheta^{(2)},\phi^{(2)},\varphi^{(2)},\psi^{(2)})^\top=(u^{(2)},u_4^{(2)},u_5^{(2)},\dots,u_9^{(2)})^\top$$

of the dynamical equations

$$A^{(1)}(\partial_x,\partial_t)U^{(1)}=\Phi_1\quad\text{in }\ \Omega_1\times[0,\infty),$$
$$A^{(2)}(\partial_x,\partial_t)U^{(2)}=\Phi_2\quad\text{in }\ \Omega_2\times[0,\infty),$$

which satisfy on the surface S_1 the following boundary-transmission conditions:

$$\{u_j^{(1)}\}^+-\{u_j^{(2)}\}^+=f_j^{(1)}\qquad\text{on }\ S_1\times[0,\infty),\ \ j=\overline{1,4},$$
$$\{T^{(1)}(\partial_x,n,\partial_t)U^{(1)}\}_j^+ +\{T^{(2)}(\partial_x,\nu,\partial_t)U^{(2)}\}_j^+=f_j^{(2)}\quad\text{on }\ S_1\times[0,\infty),\ \ j=\overline{1,4},\ \ \nu=-n,$$

and the boundary conditions

$$\{u_j^{(2)}\}^+=Q_j^{(2)},\quad S_1\times[0,\infty),\ \ j=\overline{5,9},$$

while on the surface S_2, the Neumann boundary condition

$$\{T^{(2)}(\partial_x,\nu,\partial_t)U^{(2)}\}^+=q^{(2)}\quad\text{on }\ S_2\times[0,\infty),$$

and the initial conditions

$$u_j^{(1)}(x,0)=0,\quad \partial_t\, u_j^{(1)}(x,0)=0,\quad x\in\Omega_1,\ \ j=\overline{1,4},$$
$$u_j^{(2)}(x,0)=0,\quad \partial_t\, u_j^{(2)}(x,0)=0,\quad x\in\Omega_2,\ \ j=\overline{1,8}.$$

3.3. **Formulation of the mixed boundary-transmission dynamical problem $(TM)_t$.** We are looking for a solution

$$U^{(1)}=(u^{(1)},\vartheta^{(1)})^\top=(u^{(1)},u_4^{(1)})^\top,$$
$$U^{(2)}=(u^{(2)},\vartheta^{(2)},\phi^{(2)},\varphi^{(2)},\psi^{(2)})^\top=(u^{(2)},u_4^{(2)},u_5^{(2)},\dots,u_9^{(2)})^\top$$

of the dynamical equations

$$A^{(1)}(\partial_x,\partial_t)U^{(1)}=\Phi_1\quad\text{in }\ \Omega_1\times[0,\infty),$$
$$A^{(2)}(\partial_x,\partial_t)U^{(2)}=\Phi_2\quad\text{in }\ \Omega_2\times[0,\infty),$$

which satisfy on the surface S_1 the following boundary-transmission conditions:

$$\{u_j^{(1)}\}^+-\{u_j^{(2)}\}^+=f_j^{(1)}\qquad\text{on }\ S_1\times[0,\infty),\ \ j=\overline{1,4},$$
$$\{T^{(1)}(\partial_x,n,\partial_t)U^{(1)}\}_j^+ +\{T^{(2)}(\partial_x,\nu,\partial_t)U^{(2)}\}_j^+=f_j^{(2)}\quad\text{on }\ S_1\times[0,\infty),\ \ j=\overline{1,4},\ \ \nu=-n,$$

and the boundary conditions

$$\{u_j^{(2)}\}^+=Q_j^{(2)}\quad\text{on }\ S_1\times[0,\infty),\ \ j=\overline{5,9},$$

while on the surface S_2, the mixed boundary conditions

$$\{U^{(2)}\}^+=p_2^{(D)}\quad\text{on }\ S_2^{(D)}\times[0,\infty),$$
$$\{T^{(2)}(\partial_x,\nu,\partial_t)U^{(2)}\}^+=q_2^{(N)}\quad\text{on }\ S_2^{(N)}\times[0,\infty),$$

and the initial conditions

$$u_j^{(1)}(x,0)=0,\quad \partial_t\, u_j^{(1)}(x,0)=0,\quad x\in\Omega_1,\ \ j=\overline{1,4},$$

$$u_j^{(2)}(x,0)=0, \quad \partial_t\, u_j^{(2)}(x,0)=0, \quad x\in\Omega_2, \;\; j=\overline{1,8},$$

where

$$S_2=\overline{S_{\,2}^{(D)}}\cup\overline{S_{\,2}^{(N)}}, \quad S_2^{(D)}\cap S_2^{(N)}=\varnothing, \quad \ell=\partial S_2^{(D)}=\partial S_2^{(N)}\in C^\infty.$$

Remark 3.1. Taking into account the homogeneous initial conditions of the boundary-transmission dynamical problems $(DT)_t$, $(NT)_t$, $(MT)_t$, from the 9-th equation of the basic dynamical system of equations, when $t=0$, and the corresponding boundary condition, we can find the function $\psi^{(2)}(x,0)$ for $x\in\Omega_2$. Note that in formulating the boundary-transmission dynamical problems $(DT)_t$, $(NT)_t$, $(MT)_t$, we can consider the homogeneous initial conditions without loss of generality (see [4,16]).

By H^s with $s\in\mathbb{R}$, we denote the Sobolev-Slobodetsky space. Let $\mathscr{M}_0$ be a smooth surface without boundary. For a proper sub-manifold $\mathscr{M}\subset\mathscr{M}_0$, we denote by $\widetilde{H}^s(\mathscr{M})$ the subspace of $H^s(\mathscr{M}_0)$,

$$\widetilde{H}^s(\mathscr{M})=\big\{g\,:\,g\in H^s(\mathscr{M}_0),\,\operatorname{supp} g\subset\overline{\mathscr{M}}\big\},$$

while $H^s(\mathscr{M})$ stands for the space of restriction on $\mathscr{M}$ of functions from $H^s(\mathscr{M}_0)$.

Let $\mathbb{B}$ be some Banach space and let $a>0$ and $m\in\mathbb{N}\cup 0$.

Definition 3.2. By $C_a^m([0,\infty),\mathbb{B})$ we denote the set of all $\mathbb{B}$–valued functions, which are m-times continuously differentiable on $[0,\infty)$ and satisfying the conditions

$$\frac{\partial^l u(0)}{\partial t^l}=0, \quad l=0,\dots,m, \quad \Big\|\frac{\partial^l u(t)}{\partial t^l}\Big\|_{\mathbb{B}}=O(e^{\alpha t}), \quad \forall\alpha>a>0, \quad l=0,\dots,m.$$

Definition 3.3. By $C_{0,a}^m([0,\infty),\mathbb{B})$ we denote the set of all $\mathbb{B}$–valued functions, which are m-times continuously differentiable on $[0,\infty)$ and satisfying the conditions

$$\frac{\partial^l u(0)}{\partial t^l}=0, \;\; l=0,\dots,m-2, \quad \Big\|\frac{\partial^l u(t)}{\partial t^l}\Big\|_{\mathbb{B}}=O(e^{at}), \;\; l=0,\dots,m.$$

We will study the solvability of the above-formulated dynamical boundary-transmission problems in the spaces

$$C_a^m([0,\infty),[H^1(\Omega_1)]^4)\times C_a^m([0,\infty),[H^1(\Omega_2)]^9) \quad \text{with} \;\; m\geq 2 \;\; \text{and} \;\; a>0,$$

assuming that

$$\begin{gathered}
\Phi_1\in C_{0,a}^M([0,\infty),[L_2(\Omega_1)]^4), \quad \Phi_2\in C_{0,a}^M([0,\infty),[L_2(\Omega_2)]^9), \quad f_j^{(1)}\in C_{0,a}^{M+2}([0,\infty),H^{\frac12}(S_1)),\\
f_j^{(2)}\in C_{0,a}^{M+2}([0,\infty),H^{-\frac12}(S_1)), \;\; j=\overline{1,4}, \quad Q_j^{(2)}\in C_{0,a}^{M+2}([0,\infty),H^{\frac12}(S_1)), \;\; j=\overline{5,9},\\
p^{(2)}\in C_{0,a}^{M+2}([0,\infty),[H^{\frac12}(S_1)]^9), \quad q^{(2)}\in C_{0,a}^{M+2}([0,\infty),[H^{-\frac12}(S_1)]^9),\\
p_2^{(D)}\in C_{0,a}^{M+2}([0,\infty),[H^{\frac12}(S_2^{(D)})]^9), \quad q_2^{(N)}\in C_{0,a}^{M+2}([0,\infty),[H^{-\frac12}(S_2^{(N)}]^9),
\end{gathered}$$

where M is an appropriately chosen natural number. Further, note that the initial conditions are satisfied automatically.

4. Boundary-transmission Problems of Pseudo-oscillations

Using the Laplace transform

$$\widetilde{f}(\tau)=\int_0^\infty e^{-\tau t}f(t)dt, \;\; \tau=\sigma+i\omega, \quad \sigma=\operatorname{Re}\tau>a>0,\;\omega\in\mathbb{R},$$

the Dirichlet, Neumann and mixed boundary-transmission dynamical problems can be reduced to the following boundary-transmission problems of pseudo-oscillation equations $(TD)_\tau$, $(TN)_\tau$ and $(TM)_\tau$, depending on the complex parameter τ.

We are looking for a solution

$$\begin{aligned}
&\widetilde{U}^{(1)}=(\widetilde{u}^{(1)},\widetilde{\vartheta}^{(1)})^\top=(\widetilde{u}^{(1)},\widetilde{u}_4^{(1)})^\top\in[H^1(\Omega_1)]^4,\\
&\widetilde{U}^{(2)}=(\widetilde{u}^{(2)},\widetilde{\vartheta}^{(2)},\widetilde{\phi}^{(2)},\widetilde{\varphi}^{(2)},\widetilde{\psi}^{(2)})^\top=(\widetilde{u}^{(2)},\widetilde{u}_4^{(2)},\widetilde{u}_5^{(2)},\dots,\widetilde{u}_9^{(2)})^\top\in[H^1(\Omega_2)]^9
\end{aligned}$$

of the pseudo-oscillation equations

$$A^{(1)}(\partial_x,\tau)\widetilde{U}^{(1)}=\widetilde{\Phi}_1 \quad \text{in } \ \Omega_1,$$
$$A^{(2)}(\partial_x,\tau)\widetilde{U}^{(2)}=\widetilde{\Phi}_2 \quad \text{in } \ \Omega_2,$$

which satisfy on the surface S_1 the following boundary-transmission conditions:

$$\{\widetilde{u}_j^{(1)}\}^+-\{\widetilde{u}_j^{(2)}\}^+=\widetilde{f}_j^{(1)} \qquad \text{on } \ S_1, \ \ j=\overline{1,4},$$
$$\{T^{(1)}(\partial_x,n,\tau)\widetilde{U}^{(1)}\}_j^+ + \{T^{(2)}(\partial_x,\nu,\tau)\widetilde{U}^{(2)}\}_j^+=\widetilde{f}_j^{(2)} \quad \text{on } \ S_1, \quad j=\overline{1,4}, \ \ \nu=-n,$$

and the boundary conditions

$$\{\widetilde{u}_j^{(2)}\}^+=\widetilde{Q}_j^{(2)} \ \text{ on } \ \ S_1, \ \ j=\overline{5,9},$$

while on the surface S_2, the Dirichlet boundary condition for the Dirichlet boundary-transmission problem $(TD)_\tau$,

$$\{\widetilde{U}^{(2)}\}^+=\widetilde{p}^{(2)} \ \text{ on } \ S_2,$$

or the Neumann boundary condition for the Neumann boundary-transmission problem $(TN)_\tau$,

$$\{T^{(2)}(\partial_x,\nu,\tau)\widetilde{U}^{(2)}\}^+=\widetilde{q}^{(2)} \ \text{ on } \ S_2,$$

or the mixed boundary conditions for the mixed boundary-transmission problem $(TM)_\tau$,

$$\{\widetilde{U}^{(2)}\}^+=\widetilde{p}_2^{(D)} \quad \text{on } \ S_2^{(D)}, \quad \{T^{(2)}(\partial_x,\nu,\tau)\widetilde{U}^{(2)}\}^+=\widetilde{q}_2^{(N)} \ \text{ on } \ S_2^{(N)},$$

where $\operatorname{Re}\tau>a>0$,

$$\widetilde{\Phi}_1\in[L_2(\Omega_1)]^4, \quad \widetilde{\Phi}_2\in[L_2(\Omega_2)]^9,$$
$$\widetilde{f}_j^{(1)}\in H^{\frac12}(S_1), \quad \widetilde{f}_j^{(2)}\in H^{-\frac12}(S_1), \ \ j=\overline{1,4},$$
$$\widetilde{Q}_j^{(2)}\in H^{\frac12}(S_1), \ \ j=\overline{5,9}, \quad \widetilde{p}^{(2)}\in[H^{\frac12}(S_2)]^9,$$
$$\widetilde{q}^{(2)}\in[H^{-\frac12}(S_2)]^9, \quad \widetilde{p}_2^{(D)}\in[H^{\frac12}(S_2^{(D)})]^9, \quad \widetilde{q}_2^{(N)}\in[H^{-\frac12}(S_2^{(N)})]^9,$$

and

$$A^{(1)}(\partial_x,\tau)=\big[A_{ij}^{(1)}(\partial_x,\tau)\big]_{4\times4}$$
$$:=\begin{bmatrix}[\delta_{ij}(\lambda^{(1)}+\mu^{(1)})\Delta+(\lambda^{(1)}+\varkappa^{(1)})\partial_i\partial_j-\tau^2\rho_1\delta_{ij}]_{3\times3}, & -\tau\beta_0^{(1)}[\partial_i]_{3\times1}\\ -\tau\beta_0^{(1)}[\partial_j]_{1\times3} & -\tau^2a^{(1)}+k^{(1)}\Delta\end{bmatrix}_{4\times4},$$
$$T^{(1)}=T^{(1)}(\partial_x,n,\tau)=\big[T_{ij}^{(1)}(\partial_x,n,\tau)\big]_{4\times4}$$
$$:=\begin{bmatrix}[\lambda^{(1)}n_i\partial_j+\mu^{(1)}n_j\partial_i+\delta_{ij}(\mu^{(1)}+\varkappa^{(1)})n_k\partial_k]_{3\times3}, & [-\tau\beta_0^{(1)}n]_{3\times1}\\ [0]_{1\times3} & k^{(1)}n_l\partial_l\end{bmatrix}_{4\times4},$$

the matrix differential pseudo-oscillation operator of thermo-electro-elasticity is defined as follows:

$$A^{(2)}(\partial_x,\tau)=[A_{ij}^{(2)}(\partial_x,\tau)]_{9\times9},$$
$$A_{ij}^{(2)}(\partial_x,\tau)=\delta_{ij}(\mu^{(2)}+\varkappa^{(2)})\partial_l\partial_l+(\lambda^{(2)}+\mu^{(2)})\partial_i\partial_j-\tau^2\rho_2\delta_{ij},$$
$$A_{i4}^{(2)}(\partial_x,\tau))=-\tau\beta_0^{(2)}\partial_i, \quad A_{i,j+4}^{(2)}(\partial_x,\tau)=-\varkappa^{(2)}\varepsilon_{ijl}\partial_l,$$
$$A_{i8}^{(2)}(\partial_x,\tau)=\lambda_0^{(2)}\partial_i, \quad A_{i9}^{(2)}(\partial_x,\tau)=0,$$
$$A_{4j}^{(2)}(\partial_x,\tau)=-\tau\beta_0^{(2)}\partial_j, \quad A_{44}^{(2)}(\partial_x,\tau)=k^{(2)}\partial_l\partial_l-\tau^2a^{(2)},$$
$$A_{4,j+4}^{(2)}(\partial_x,\tau)=0, \quad A_{48}^{(2)}(\partial_x,\tau)=\nu_1^{(2)}\partial_l\partial_j-\tau c_0^{(2)}, \quad A_{49}^{(2)}(\partial_x,\tau)=-\nu_3^{(2)}\partial_l\partial_l,$$
$$A_{i+4,j}^{(2)}(\partial_x,\tau)=-\varkappa^{(2)}\varepsilon_{ijl}\partial_l, \ \ A_{i+4,4}^{(2)}(\partial_x,\tau)=0,$$
$$A_{i+4,j+4}^{(2)}(\partial_x,\tau)=\delta_{ij}\gamma^{(2)}\partial_l\partial_l+(\alpha^{(2)}+\beta^{(2)})\partial_i\partial_j-(2\varkappa^{(2)}+\tau^2I_0^{(2)})\delta_{ij},$$
$$A_{i+4,8}^{(2)}(\partial_x,\tau)=0, \quad A_{i+4,9}^{(2)}(\partial_x,\tau)=0,$$

$$A^{(2)}_{8j}(\partial_x,\tau) = -\lambda^{(2)}_0\partial_j, \quad A^{(2)}_{84}(\partial_x,\tau) = \nu^{(2)}_1\partial_l\partial_l + \tau c^{(2)}_0,$$
$$A^{(2)}_{8,j+4}(\partial_x,\tau) = 0, \quad A^{(2)}_{88}(\partial_x,\tau) = a^{(2)}_0\partial_l\partial_l - (\xi^{(2)}_0 + \tau^2 j^{(2)}_0),$$
$$A^{(2)}_{89}(\partial_x,\tau) = -\lambda^{(2)}_2\partial_l\partial_l, \quad A^{(2)}_{9j}(\partial_x,\tau) = 0, \quad A^{(2)}_{94}(\partial_x,\tau) = \nu^{(2)}_3\partial_l\partial_l,$$
$$A^{(2)}_{9,j+4}(\partial_x,\tau) = 0 \quad A^{(2)}_{98}(\partial_x,\tau) = \lambda^{(2)}_2\partial_l\partial_l, \quad A^{(2)}_{99}(\partial_x,\tau) = \chi^{(2)}\partial_l\partial_l, \;\; i,j = 1,2,3,$$

the corresponding stress differential operator of thermo-electro-elasticity is defined as follows:

$$T^{(2)} = T^{(2)}(\partial_x,\nu,\tau) := [T^{(2)}_{ij}(\partial_x,\nu,\tau)]_{9\times 9},$$

where

$$T^{(2)}_{ij}(\partial_x,\nu,\tau) = \lambda^{(2)}\nu_i\partial_j + \mu^{(2)}\nu_j\partial_i + \delta_{ij}(\mu^{(2)} + \varkappa^{(2)})\nu_k\partial_k, \quad T^{(2)}_{i4}(\partial_x,\nu,\tau) = -\tau\beta^{(2)}_0\nu_i,$$
$$T^{(2)}_{i,j+4}(\partial_x,\nu,\tau) = -\varkappa^{(2)}\varepsilon_{ijk}\nu_k, \quad T^{(2)}_{i8}(\partial_x,\nu,\tau) = \lambda^{(2)}_0\nu_i, \quad T^{(2)}_{i9}(\partial_x,\nu,\tau) = 0,$$
$$T^{(2)}_{4,j}(\partial_x,\nu,\tau) = 0, \quad T^{(2)}_{44}(\partial_x,\nu,\tau) = k^{(2)}\nu_l\partial_l,$$
$$T^{(2)}_{4,j+4}(\partial_x,\nu,\tau) = -\nu^{(2)}_2\varepsilon_{ljk}\nu_l\partial_k, \quad T^{(2)}_{48}(\partial_x,\nu,\tau) = \nu^{(2)}_1\nu_k\partial_k,$$
$$T^{(2)}_{49}(\partial_x,\nu,\tau) = -\nu^{(2)}_3\nu_k\partial_k, \quad T^{(2)}_{i+4,j}(\partial_x,\nu,\tau) = 0,$$
$$T^{(2)}_{i+4,4}(\partial_x,\nu,\tau) = \nu^{(2)}_2\varepsilon_{lik}\nu_l\partial_k, \quad T^{(2)}_{i+4,j+4}(\partial_x,\nu,\tau) = \alpha^{(2)}\nu_i\partial_j + \beta^{(2)}\nu_j\partial_i + \delta_{ij}\gamma^{(2)}\nu_k\partial_k,$$
$$T^{(2)}_{i+4,8}(\partial_x,\nu,\tau) = b^{(2)}_0\varepsilon_{lik}\nu_l\partial_k, \quad T^{(2)}_{i+4,9}(\partial_x,\nu,\tau) = \lambda^{(2)}_1\varepsilon_{lik}\nu_l\partial_k,$$
$$T^{(2)}_{8j}(\partial x,\nu,\tau) = 0, \quad T^{(2)}_{84}(\partial_x,\nu,\tau) = \nu^{(2)}_1\nu_k\partial_k,$$
$$T^{(2)}_{8,j+4}(\partial_x,\nu,\tau) = -b^{(2)}_0\varepsilon_{lik}\nu_l\partial_k, \;\; T^{(2)}_{88}(\partial_x,\nu,\tau) = a^{(2)}_0\nu_k\partial_k, \quad T^{(2)}_{89}(\partial_x,\nu,\tau) = -\lambda^{(2)}_2\nu_k\partial_k,$$
$$T^{(2)}_{9j}(\partial x,\nu,\tau) = 0, \quad T^{(2)}_{94}(\partial_x,\nu,\tau) = \nu^{(2)}_3\nu_k\partial_k,$$
$$T^{(2)}_{9,j+4}(\partial_x,\nu,\tau) = -\lambda^{(2)}_1\varepsilon_{ljk}\nu_l\partial_k, \quad T^{(2)}_{98}(\partial_x,\nu,\tau) = \lambda^{(2)}_2\nu_k\partial_k,$$
$$T^{(2)}_{99}(\partial_x,\nu,\tau) = \chi^{(2)}\nu_k\partial_k, \;\; i,j = 1,2,3.$$

Now, let us formulate the existence and uniqueness, the regularity theorems of boundary-transmission pseudo-oscillation problems $(TD)_\tau, (TN)_\tau$ and $(TM)_\tau$, proven in [17].

Theorem 4.1. *Let $S_1, S_2 \in C^\infty$, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 > 0$, $\omega \in \mathbb{R}$, and $\widetilde{\Phi}_1 \in [L_2(\Omega_1)]^4$, $\widetilde{\Phi}_2 \in [L_2(\Omega_2)]^9$, $\widetilde{f}^{(1)}_j \in H^{\frac12}(S_1)$, $\widetilde{f}^{(2)}_j \in H^{-\frac12}(S_1)$, $j = \overline{1,4}$, $\widetilde{Q}^{(2)}_j \in H^{\frac12}(S_1)$, $j = \overline{5,9}$, $\widetilde{p}^{(2)} \in [H^{\frac12}(S_2)]^9$. Then the Dirichlet boundary-transmission problem $(TD)_\tau$ has a unique solution*

$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9.$$

Theorem 4.2. *Let $S_1, S_2 \in C^{m,a}$, $0 < \beta < \alpha \le 1$, $m \ge 2$, $m \in \mathbb{N}$, and*

$$\widetilde{\Phi}_1 \in [C^{k-2,\beta}(\overline{\Omega}_1)]^4, \quad \widetilde{\Phi}_2 \in [C^{k-2,\beta}(\overline{\Omega}_2)]^9,$$
$$\widetilde{f}^{(1)}_j \in C^{k,\beta}(S_1), \quad \widetilde{f}^{(2)}_j \in C^{k-1,\beta}(S_1), \;\; j = \overline{1,4},$$
$$\widetilde{Q}^{(2)}_j \in C^{k,\beta}(S_1), \;\; j = \overline{5,9}, \quad \widetilde{p}^{(2)} \in [C^{k,\beta}(S_2)]^9, \;\; k = \overline{2,m}.$$

Then the Dirichlet boundary-transmission problem $(TD)_\tau$ has a unique solution

$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [C^{k,\beta}(\overline{\Omega}_1)]^4 \times [C^{k,\beta}(\overline{\Omega}_2)]^9.$$

Corollary 4.3. *Let $S_1, S_2 \in C^\infty$ and $\widetilde{\Phi}_1 \in [C^\infty(\overline{\Omega}_1)]^4$, $\widetilde{\Phi}_2 \in [C^\infty(\overline{\Omega}_2)]^9$, $\widetilde{f}^{(1)}_j \in C^\infty(S_1)$, $\widetilde{f}^{(2)}_j \in C^\infty(S_1)$, $j = \overline{1,4}$, $\widetilde{Q}^{(2)}_j \in C^\infty(S_1)$, $j = \overline{5,9}$, $\widetilde{p}^{(2)} \in [C^\infty(S_2)]^9$. Then the unique solution $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ of the Dirichlet problem $(TD)_\tau$ belongs to the class $[C^\infty(\overline{\Omega}_1)]^4 \times [C^\infty(\overline{\Omega}_2)]^9$, i.e.,*

$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [C^\infty(\overline{\Omega}_1)]^4 \times [C^\infty(\overline{\Omega}_2)]^9.$$

Theorem 4.4. *Let $S_1, S_2 \in C^\infty$, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 > 0$, $\omega \in \mathbb{R}$, and*

$$\widetilde{\Phi}_1 \in [L_2(\Omega_1)]^4, \quad \widetilde{\Phi}_2 \in [L_2(\Omega_2)]^9,$$
$$\widetilde{f}_j^{(1)} \in H^{\frac{1}{2}}(S_1), \quad \widetilde{f}_j^{(2)} \in H^{-\frac{1}{2}}(S_1), \;\; j = \overline{1,4},$$
$$\widetilde{Q}_j^{(2)} \in H^{\frac{1}{2}}(S_1), \;\; j = \overline{5,9}, \quad \widetilde{q}^{(2)} \in [H^{-\frac{1}{2}}(S_2)]^9.$$

Then the Neumann boundary-transmission problem $(TN)_\tau$ has a unique solution

$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9.$$

Theorem 4.5. *Let $S_1, S_2 \in C^{m,\alpha}$, $0 < \beta < \alpha \leq 1$, $m \geq 2$, $m \in \mathbb{N}$, and*

$$\widetilde{\Phi}_1 \in [C^{k-2,\beta}(\overline{\Omega}_1)]^4, \quad \widetilde{\Phi}_2 \in [C^{k-2,\beta}(\overline{\Omega}_2)]^9,$$
$$\widetilde{f}_j^{(1)} \in C^{k,\beta}(S_1), \quad \widetilde{f}_j^{(2)} \in C^{k-1,\beta}(S_1), \;\; j = \overline{1,4},$$
$$\widetilde{Q}_j^{(2)} \in C^{k,\beta}(S_1), \;\; j = \overline{5,9}, \quad \widetilde{q}^{(2)} \in [C^{k-1,\beta}(S_2))]^9, \;\; k = \overline{2,m}.$$

Then problem $(TN)_\tau$ has a unique solution

$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [C^{k,\beta}(\overline{\Omega}_1)]^4 \times [C^{k,\beta}(\overline{\Omega}_2)]^9.$$

Corollary 4.6. *Let the following conditions:*

$$S_1, S_2 \in C^\infty, \quad \widetilde{\Phi}_1 \in [C^\infty(\overline{\Omega}_1)]^4, \quad \widetilde{\Phi}_2 \in [C^\infty(\overline{\Omega}_2)]^9,$$
$$\widetilde{f}_j^{(1)} \in C^\infty(S_1), \quad \widetilde{f}_j^{(2)} \in C^\infty(S_1), \;\; j = \overline{1,4},$$
$$\widetilde{Q}_j^{(2)} \in C^\infty(S_1), \;\; j = \overline{5,9}, \quad \widetilde{q}^{(2)} \in [C^\infty(S_2)]^9,$$

be fulfilled, then the unique solution to problem $(TN)_\tau$ is infinitly differentiable, i.e.,

$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [C^\infty(\overline{\Omega}_1)]^4 \times [C^\infty(\overline{\Omega}_2)]^9.$$

Theorem 4.7. *Let $S_1, S_2 \in C^\infty$, $\tau = \sigma + i\omega$, $\sigma > \sigma_0 > 0$, $\omega \in \mathbb{R}$, and*

$$\widetilde{\Phi}_1 \in [L_2(\Omega_1)]^4, \;\; \widetilde{\Phi}_2 \in [L_2(\Omega_2)]^9,$$
$$\widetilde{f}_j^{(1)} \in H^{\frac{1}{2}}(S_1), \quad \widetilde{f}_j^{(2)} \in H^{-\frac{1}{2}}(S_2), \;\; j = \overline{1,4},$$
$$\widetilde{Q}_j^{(2)} \in H^{\frac{1}{2}}(S_1), \;\; j = \overline{5,9}, \quad \widetilde{p}_2^{(D)} \in [H^{\frac{1}{2}}(S_2^{(D)})]^9, \quad \widetilde{q}_2^{(N)} \in [H^{-\frac{1}{2}}(S_2^{(N)}]^9.$$

Then the mixed boundary-transmission problem $(TM)_\tau$ has a unique solution

$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9.$$

Let us introduce the notation

$$d := \frac{c\, b_0^{(2)} + p\, \lambda_1^{(2)} + q\, \nu_2^{(2)}}{2\gamma^{(2)}},$$

where

$$c := \frac{1}{2}(b_0^{(2)} b_{11} + \lambda_1^{(2)} b_{21} + \nu_2^{(2)} b_{31}), \quad p := \frac{1}{2}(b_0^{(2)} b_{12} + \lambda_1^{(2)} b_{22} + \nu_2^{(2)} b_{32}),$$
$$q := \frac{1}{2}(b_0^{(2)} b_{13} + \lambda_1^{(2)} b_{23} + \nu_2^{(2)} b_{33}),$$
$$[b_{jk}]_{3\times 3} := \begin{bmatrix} a_0^{(2)} & -\lambda_2^{(2)} & \nu_1^{(2)} \\ \lambda_2^{(2)} & \chi^{(2)} & \nu_3^{(2)} \\ \nu_1^{(2)} & -\nu_3^{(2)} & k^{(2)} \end{bmatrix}^{-1}.$$

The following regularity theorem holds:

Theorem 4.8. *Suppose $S_1, S_2 \in C^\infty$ and*

$$\widetilde{\Phi}_1 \in [C^\infty(\overline{\Omega}_1)]^4, \quad \widetilde{\Phi}_2 \in [C^\infty(\overline{\Omega}_2)]^9,$$
$$\widetilde{f}_j^{(1)} \in C^\infty(S_1), \quad \widetilde{f}_j^{(2)} \in C^\infty(S_1), \ \ j=\overline{1,4},$$
$$\widetilde{Q}_j^{(2)} \in C^\infty(S_1), \ \ j=\overline{5,9}, \quad \widetilde{p}_2^{(D)} \in [C^\infty(\overline{S}_2^{(D)})]^9, \quad \widetilde{q}_2^{(N)} \in [C^\infty(\overline{S}_2^{(N)})]^9.$$

Then:

1) *If $d<0$, then the unique solution $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ to the mixed boundary-transmission problem $(TM)_\tau$ belongs to $[C^\infty(\overline{\Omega}_1)]^4 \times [C^{\gamma_1}(\overline{\Omega}_2)]^9$, i.e.,*
$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [C^\infty(\overline{\Omega}_1)]^4 \times [C^{\gamma_1}(\overline{\Omega}_2)]^9,$$
where $\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\operatorname{arctg} 2\sqrt{-d}$, γ_1 depends on the material constants, does not depend on the geometry of the exceptional line $\ell = \partial S_2^{(D)} = \partial S_2^{(N)}$ and may take any values from the interval $(0, \frac{1}{2})$.
2) *If $d \geq 0$, then the unique solution to the corresponding boundary-transmission problem $(TM)_\tau$*
$$(\widetilde{U}^{(1)}, \widetilde{U}^{(2)}) \in [C^\infty(\overline{\Omega_1})]^4 \times [C^{\frac{1}{2}}(\overline{\Omega_2})]^9.$$

In order to perform the inverse Laplace transform of the solution $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ of boundary-transmission problems $(TD)_\tau$, $(TN)_\tau$ and $(TM)_\tau$, i.e.,

$$U^{(q)}(\cdot,t) = \frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+\infty} e^{\tau t}\widetilde{U}^{(q)}(\cdot,\tau)d\tau, \ \ q=1,2, \ \ \alpha > a,$$

we need the estimates in τ of

$$\|\widetilde{U}^{(1)}(\cdot,\tau)\|_{[H^1(\Omega_1)]^4}, \qquad \|\widetilde{U}^{(2)}(\cdot,\tau)\|_{[H^1(\Omega_2)]^9},$$

when $|\tau| \to \infty$ $(\operatorname{Re}\tau > a)$.

Using the integration by parts formula for the data of the boundary-transmission pseudo-oscillation problems $(TD)_\tau, (TN)_\tau$ and $(TM)_\tau$, for $\operatorname{Re}\tau > a > 0$, we deduce the following inequalities:

$$\|\widetilde{\Phi}^{(1)}(\cdot,\tau)\|_{[L_2(\Omega_1)]^4} \leq C|\tau|^{-M}, \qquad \|\widetilde{\Phi}^{(2)}(\cdot,\tau)\|_{[L_2(\Omega_2)]^9} \leq C|\tau|^{-M},$$
$$\|\widetilde{f}_j^{(1)}(\cdot,\tau)\|_{H^{\frac{1}{2}}(S_1)} \leq C|\tau|^{-M-2}, \qquad \|\widetilde{f}_j^{(2)}(\cdot,\tau)\|_{H^{-\frac{1}{2}}(S_1)} \leq C|\tau|^{-M-2}, \ \ j=\overline{1,4},$$
$$\|\widetilde{Q}_j^{(2)}(\cdot,\tau)\|_{H^{\frac{1}{2}}(S_1)} \leq C|\tau|^{-M-2}, \ \ j=\overline{5,9}, \qquad \|\widetilde{p}^{(2)}(\cdot,\tau)\|_{[H^{\frac{1}{2}}(S_2)]^9} \leq C|\tau|^{-M-2},$$
$$\|\widetilde{q}^{(2)}(\cdot,\tau)\|_{[H^{-\frac{1}{2}}(S_2)]^9} \leq C|\tau|^{-M-2}, \qquad \|\widetilde{p}_2^{(D)}(\cdot,\tau)\|_{[H^{\frac{1}{2}}(S_2^{(D)})]^9} \leq C|\tau|^{-M-2},$$
$$\|\widetilde{q}_2^{(N)}(\cdot,\tau)\|_{[H^{-\frac{1}{2}}(S_2^{(N)})]^9} \leq C|\tau|^{-M-2}, \tag{4.1}$$

where C is a constant, independent of τ.

Let us consider the Dirichlet boundary-transmission problem $(TD)_\tau$.

To obtain the similar estimates (see (4.1)) for the corresponding solution $(\widetilde{U}^{(1)}, \widetilde{U}^{(2)})$ of the Dirichlet boundary-transmission problem $(TD)_\tau$ we use the representation

$$\widetilde{U}^{(1)} = V_1^{(1)} + V_2^{(1)}, \quad V_q^{(1)} = (v_{q,1}^{(1)}, \dots, v_{q,4}^{(1)})^\top = (v_q^{(1)}, v_{q,4}^{(1)})^\top, \ \ q=1,2,$$
$$\widetilde{U}^{(2)} = W_1^{(2)} + W_2^{(2)}, \quad W_q^{(2)} = (w_{q,1}^{(2)},, w_{q,9}^{(2)})^\top = (w_q^{(2)}, w_{q,4}^{(2)}, \dots, w_{q,9}^{(2)})^\top, \ \ q=1,2,$$

where $(V_1^{(1)}, W_1^{(2)})$ and $(V_2^{(1)}, W_2^{(2)})$ are the solutions of the following boundary-transmission Problem 4.1 and Problem 4.2, respectively.

Problem 4.1. Find a vector function $(V_1^{(1)}, W_1^{(2)}) \in [H^1(\Omega_1)]^4 \times [H^1(\Omega_2)]^9$ satisfying the pseudo-oscillation differential equations

$$A^{(1)}(\partial_x, \tau_0)V_1^{(1)} = \widetilde{\Phi}_1 \quad \text{in} \ \ \Omega_1,$$

$$A^{(2)}(\partial_x,\tau_0)W_1^{(2)}=\widetilde{\Phi}_2 \quad \text{in } \ \Omega_2,$$

the boundary-transmission conditions on the surface S_1 :

$$\{v_{1,j}^{(1)}\}^+-\{w_{1,j}^{(1)}\}^+=\widetilde{f}_j^{(1)} \qquad \text{on } \ S_1, \quad j=\overline{1,4},$$
$$\{T^{(1)}(\partial_x,\nu,\tau_0)V_1^{(1)}\}_j^+ + \{T^{(2)}(\partial_x,\nu,\tau_0)W_1^{(2)}\}_j^+=\widetilde{f}_j^{(2)} \quad \text{on } \ S_1, \quad j=\overline{1,4}, \ \ \nu=-n,$$

and the boundary conditions

$$\{w_{1,j}^{(2)}\}^+=\widetilde{Q}_j^{(2)}, \quad S_1, \ \ j=\overline{5,9},$$

while on the surface S_2, the Dirichlet boundary condition for the Dirichlet boundary-transmission problem $(TD)_\tau$,

$$\{W_1^{(2)}\}^+=\widetilde{p}^{(2)} \quad \text{on } \ S_2,$$

where τ_0 is a fixed complex number such that $\operatorname{Re}\tau_0>0$.

Problem 4.2. Find a vector function $(V_2^{(1)},W_2^{(2)})\in[H^1(\Omega_1)]^4\times[H^1(\Omega_2)]^9$ satisfying the pseudo-oscillation differential equations

$$A^{(1)}(\partial_x,\tau)V_2^{(1)}=\Psi^{(1)} \quad \text{in } \ \Omega_1,$$
$$A^{(2)}(\partial_x,\tau)W_2^{(2)}=\Psi^{(2)} \quad \text{in } \ \Omega_2,$$

the boundary-transmission conditions on the surface S_1 :

$$\{v_{2,j}^{(1)}\}^+-\{w_{2,j}^{(1)}\}^+=0 \qquad \text{on } \ S_1, \quad j=\overline{1,4},$$
$$\{T^{(1)}(\partial_x,\nu,\tau)V_2^{(1)}\}_j^+ + \{T^{(2)}(\partial_x,\nu,\tau)W_2^{(2)}\}_j^+=G_j \quad \text{on } \ S_1, \quad j=\overline{1,4}, \ \ \nu=-n,$$

and the boundary conditions

$$\{w_{2,j}^{(2)}\}^+=0, \quad S_1, \ \ j=\overline{5,9},$$

while on the surface S_2, the Dirichlet boundary condition for the Dirichlet boundary-transmission problem $(TD)_\tau$

$$\{W_2^{(2)}\}^+=0 \quad \text{on } \ S_2,$$

where

$$\Psi^{(1)}:=[A^{(1)}(\partial_x,\tau_0)-A^{(1)}(\partial_x,\tau)]V_1^{(1)}, \quad \Psi^{(2)}:=[A^{(2)}(\partial_x,\tau_0)-A^{(2)}(\partial_x,\tau)]W_1^{(2)},$$
$$G_j:=\{[T^{1)}(\partial_x,n,\tau_0)-T^{(1)}(\partial_x,n,\tau)]V_1^{(1)}\}_j^+ + \{[T^{(2)}(\partial_x,\nu,\tau_0)-T^{(2)}(\partial_x,\nu,\tau)]W_1^{(2)}\}_j^+, \ \ j=\overline{1,4},$$

i.e.,

$$\Psi^{(1)}=\begin{bmatrix}\rho_1(\tau^2-\tau_0^2)v_1^{(1)}+(\tau-\tau_0)\beta_0^{(1)}\big[\partial_i\big]_{3\times1}v_{1,4}^{(1)}\\ (\tau-\tau_0)\beta_0^{(1)}\partial_i v_{1,i}^{(1)}+(\tau^2-\tau_0^2)a^{(1)}v_{1,4}^{(1)}\end{bmatrix},$$

$$\Psi^{(2)}=\begin{bmatrix}\rho_2(\tau^2-\tau_0^2)w_1^{(2)}+(\tau-\tau_0)\beta_0^{(2)}\big[\partial_i\big]_{3\times1}w_{1,4}^{(2)}\\ (\tau-\tau_0)\beta_0^{(2)}\partial_i w_{1,i}^{(2)}+(\tau^2-\tau_0^2)a^{(2)}w_{1,4}^{(2)}+(\tau-\tau_0)c_0^{(2)}w_{1,8}^{(2)}\\ I_0^{(2)}(\tau^2-\tau_0^2)\big[w_{1,j+4}^{(2)}\big]_{3\times1}\\ (\tau-\tau_0)c_0^{(2)}w_{1,4}^{(2)}+(\tau^2-\tau_0^2)j_0^{(2)}w_{1,8}^{(2)}\\ 0\end{bmatrix},$$

$$G_j=(\tau-\tau_0)\beta_0^{(1)}n_j v_{1,4}^{(1)}+(\tau-\tau_0)\beta_0^{(2)}\nu_j w_{1,4}^{(2)}, \ \ j=1,2,3, \ \ G_4=0.$$

By Theorem 4.1, Problem 4.1 is uniquely solvable in $[H^1(\Omega_1)]^4\times[H^1(\Omega_2)]^9$, then the following estimates

$$\|V_1^{(1)}\|_{[H^1(\Omega_1)]^4}+\|W_1^{(2)}\|_{[H^1(\Omega_2)]^9}\le C'\Big(\|\widetilde{\Phi}_1\|_{[L_2(\Omega_1)]^4}+\|\widetilde{\Phi}_2\|_{[L_2(\Omega_2)]^9}$$
$$+\sum_{j=1}^4\big(\|\widetilde{f}_j^{(1)}\|_{H^{\frac12}(S_1)}+\|\widetilde{f}_j^{(2)}\|_{H^{-\frac12}(S_1)}\big)+\sum_{j=5}^9\|\widetilde{Q}_j^{(2)}\|_{H^{\frac12}(S_1)}+\|\widetilde{p}^{(2)}\|_{[H^{\frac12}(S_2)]^9}\Big),$$

hold, where the constant C' does not depend on τ. Taking into account estimates (4.1), we obtain

$$\|V_1^{(1)}\|_{[H^1(\Omega_1)]^4} + \|W_1^{(2)}\|_{[H^1(\Omega_2)]^9} \le C_1|\tau|^{-M}, \quad \operatorname{Re}\tau > a > 0. \tag{4.2}$$

where the constant C_1 does not depend on τ.

It is clear that the estimates of the data of Problem 4.2 follow from (4.2) with respect to τ:

$$\|\Psi^{(1)}\|_{[L_2(\Omega_1)]^4} \le C|\tau|^{-M+2}, \qquad \|\Psi^{(2)}\|_{[L_2(\Omega_2)]^9} \le C|\tau|^{-M+2},$$
$$\|G_j\|_{H^{-\frac{1}{2}}(S_1)} \le C|\tau|^{-M+1}, \ \ j=\overline{1,4}. \tag{4.3}$$

Suppose $(V_2^{(1)}, W_2^{(2)})$ is a solution to Problem 4.2. Let us write Green's formulas for the vector functions $V_2^{(1)}$ and $W_2^{(2)}$ in the domains Ω_1 and Ω_2, respectively:

$$\int\limits_{\Omega_1} A^{(1)}(\partial_x,\tau)V_2^{(1)}\cdot V_2{}^{(1)}dx + \int\limits_{\Omega_1} E_\tau^{(1)}(V_2^{(1)},\overline{V_2}^{(1)})dx = \langle\{T^{(1)}V_2^{(1)}\}^+,\{V_2{}^{(1)}\}^+\rangle_{S_1}, \tag{4.4}$$

$$\int\limits_{\Omega_2} A^{(2)}(\partial_x,\tau)W_2^{(2)}\cdot W_2{}^{(2)}dx + \int\limits_{\Omega_2} E_\tau^{(2)}(W_2^{(2)},\overline{W_2}^{(2)})dx = \langle\{T^{(2)}W_2^{(2)}\}^+,\{W_2{}^{(2)}\}^+\rangle_{S_1\cup S_2}, \tag{4.5}$$

where

$$V_2^{(1)} = (v_{2,1}^{(1)},\dots,v_{2,4}^{(1)})^\top = (v_2^{(1)},v_{2,4}^{(1)})^\top, \quad v_2^{(1)} = (v_{2,1}^{(1)},v_{2,2}^{(1)},v_{2,3}^{(1)})^\top,$$
$$W_2^{(2)} = (w_{2,1}^{(2)},\dots,w_{2,9}^{(2)})^\top = (w_2^{(2)},w_{2,4}^{(2)},\phi_2^{(2)},w_{2,8}^2,w_{2,9}^{(2)})^\top,$$
$$w_2^{(2)} = (w_{2,1}^{(2)},w_{2,2}^{(2)},w_{2,3}^{(2)})^\top, \quad \phi_2^{(2)} = (w_{2,5}^{(2)},w_{2,6}^{(2)},w_{2,7}^{(2)})^\top,$$

$$E_\tau^{(1)}(V_2^{(1)},\overline{V_2}^{(1)}) = \mathscr{E}(v_2^{(1)},\overline{v_2}^{(1)}) + \rho_1\tau^2|v_2^{(1)}|^2 - \tau\beta_0^{(1)}v_{2,4}^{(1)}\operatorname{div}\overline{v_2}^{(1)} + k^{(1)}|\operatorname{grad}v_{2,4}^{(1)}|^2$$
$$+\tau\beta_0^{(1)}\operatorname{div}v_2^{(1)}\overline{v}_{2,4}^{(1)} + \tau^2a^{(1)}|v_{2,4}^{(1)}|^2,$$
$$\mathscr{E}(v_2^{(1)},\overline{v_2}^{(1)}) = (\mu^{(1)}+\varkappa^{(1)})|\operatorname{grad}v_2^{(1)}|^2 + (\lambda^{(1)}+\mu^{(1)})|\operatorname{div}v_2^{(1)}|^2.$$

Here and in what follows, $a\cdot b$ denotes the scalar product of two, in general, complex-valued vectors

$$a\cdot b = \sum_{k=1}^N a_k\overline{b}_k, \quad a,b\in\mathbb{C}^N.$$

Obviously, $\mathscr{E}(v_2^{(1)},\overline{v}_2^{(1)}) > 0$,

$$E_\tau^{(2)}(W_2^{(2)},\overline{W}_2^{(2)}) = B(v^{(2)},\overline{v}^{(2)}) + 2i\lambda_1^{(2)}\varepsilon_{ijk}\operatorname{Im}(\partial_k w_{2,9}^{(2)}\,\partial_i\overline{w}_{2,j+4}^{(2)}) + 2i\lambda_2^{(2)}\operatorname{Im}(\partial_j w_{2,8}^{(2)}\,\partial_j\overline{w}_{2,9}^{(2)})$$
$$+2i\nu_3^{(2)}\operatorname{Im}(\partial_j w_{2,4}^{(2)}\,\partial_j\overline{w}_{2,9}^{(2)}) + 2i\tau\beta_0^{(2)}\operatorname{Im}(\partial_j w_{2,j}^{(2)}\,\overline{w}_{2,4}^{(2)}) + 2i\tau c_0^{(2)}\operatorname{Im}(w_{2,8}^{(2)}\,\overline{w}_{2,4}^{(2)})$$
$$+\tau^2\big(\rho_2|w_2^{(2)}|^2 + I_0^{(2)}|\phi_2^{(2)}|^2 + j_0^{(2)}|w_{2,8}^{(2)}|^2 + a^{(2)}|w_{2,4}^{(2)}|^2\big);$$

here, we assume that $B(v^{(2)},\overline{v}^{(2)})$ is positive definite with respect to the vector $v^{(2)} = (e_{ij},\varkappa_{ij},\zeta_j,\varphi,$ $T,\vartheta_i,E_i)$, $B(v^{(2)},\overline{v}^{(2)}) > 0$ $\forall v^{(2)}\neq 0$, where $e_{ij} = \partial_i w_{2,j}^{(2)} + \varepsilon_{jik}w_{2,k+4}^{(2)}$, $\varkappa_{ij} = \partial_i w_{2,j+4}^{(2)}$, $\zeta_j = \partial_j w_{2,8}^{(2)}$, $\varphi = w_{2,8}^{(2)}$, $T = \tau w_{2,4}^{(2)}$, $\vartheta_i^{(2)} = \partial_i w_{2,4}^{(2)}$, $E_i = -\partial_i w_{2,9}^{(2)}$ (for the definition of this form see [5] formula (2.19)).

Adding Green's formulas (4.4) and (4.5) and taking into account the fact that $(V_2^{(1)},W_2^{(2)})\in$ $[H^1(\Omega_1)]^4\times[H^1(\Omega_2)]^9$ is a solution to the boundary-transmission Problem 4.2, we get

$$\int\limits_{\Omega_1}\Psi^{(1)}\cdot V_2^{(1)}dx + \int\limits_{\Omega_2}\Psi^{(2)}\cdot W_2{}^{(2)}dx + \int\limits_{\Omega_1}E_\tau^{(1)}(V_2^{(1)},\overline{V}_2^{(1)})dx + \int\limits_{\Omega_2}E_\tau^{(2)}(W_2^{(2)},W_2^{(2)})dx$$
$$= \langle\sum_{j=1}^4\{T^{(1)}V_2^{(1)}\}_j^+,\{V_2^{(1)}\}_j^+\rangle_{S_1} + \langle\sum_{j=1}^4\{T^{(2)}W_2^{(2)}\}_j^+,\{W_2{}^{(2)}\}_j^+\rangle_{S_1}$$

$$= \sum_{j=1}^{4} \langle \big(\{T^{(1)} V_2^{(1)}\}_j^+ + \{T^{(2)} W_2^{(2)}\}_j^+ \big), \{W_2{}^{(2)}\}_j^+ \rangle_{S_1} = \sum_{j=1}^{4} \langle G_j, \{W_2^{(2)}\}_j^+ \rangle_{S_1},$$

where

$$\{W_2^{(2)}\}_j^+ = \{w_{2,j}^{(2)}\}^+, \ \ j = \overline{1,9}, \quad \{V_2^{(1)}\}_j^+ = \{v_{2,j}^{(1)}\}^+, \ \ j = \overline{1,4}.$$

Therefore we obtain

$$\int\limits_{\Omega_1} E_\tau^{(1)}(V_2^{(1)}, \overline{V}_2^{(1)}) dx + \int\limits_{\Omega_2} E_\tau^{(2)}(W_2^{(2)}, \overline{W}_2^{(2)}) dx$$

$$= \sum_{j=1}^{4} \langle G_j, \{W_2^{(2)}\}_j^+ \rangle_{S_1} - \int\limits_{\Omega_1} \Psi^{(1)} \cdot V_2^{(1)} dx - \int\limits_{\Omega_2} \Psi^{(2)} \cdot W_2{}^{(2)} dx.$$

Similarly, we get (see [5])

$$\int\limits_{\Omega_1} E_\tau^{(1)}(V_2^{(1)}, \overline{V_2}^{(1)}) dx + \int\limits_{\Omega_2} \widetilde{E}_\tau^{(2)}(W_2^{(2)}, \overline{W_2}^{(2)}) dx$$

$$= \sum_{j=1}^{4} \langle G_j \, , \{W_2^{(2)}\}_j^+ \rangle_{S_1} - \int\limits_{\Omega_1} \Psi^{(1)} \cdot V_2^{(1)} dx - \int\limits_{\Omega_2} \Psi^{(2)} \cdot W_2{}^{(2)} dx, \tag{4.6}$$

where

$$\widetilde{E}_\tau^{(2)}(W_2^{(2)}, \overline{W}_2^{(2)}) := B^{(2)}(v^{(2)}, \overline{v}^{(2)}) + 2i\tau\beta_0^{(2)} \operatorname{Im}(\partial_j w_{2,j}^{(2)} \, \overline{w}_{2,4}^{(2)}) + 2i\tau c_0^{(2)} \operatorname{Im}(w_{2,8}^{(2)} \, \overline{w}_{2,4}^{(2)})$$
$$+\tau^2 \big(\rho_2 |w_2^{(2)}|^2 + I_0^{(2)} |\phi_2^{(2)}|^2 + j_0^{(2)} |w_{2,8}^{(2)}|^2 + a^{(2)} |w_{2,4}^{(2)}|^2 \big),$$

Now, let us first take the real part of equality (4.6) and then the imaginary part, where

$$\tau = \sigma + i\omega, \quad \tau^2 = (\sigma^2 - \omega^2) + 2i\sigma\omega, \quad \sigma > \sigma_0 > 0, \ \ \omega \in \mathbb{R}.$$

Thus we obtain the following integral equalities:

$$\int\limits_{\Omega_1} \Big[\mathscr{E}(v_2^{(1)}, \overline{v}_2^{(1)}) + (\sigma^2 - \omega^2) \rho_1 |v_2^{(1)}|^2 - 2\beta_0^{(1)} \omega \operatorname{Im}(\overline{v}_{2,4}^{(1)} \operatorname{div} v_2^{(1)}) + k^{(1)} |\operatorname{grad} v_{2,4}^{(1)}|^2$$
$$+ a^{(1)} (\sigma^2 - \omega^2) |v_{2,4}^{(1)}|^2 \Big] dx,$$
$$+ \int\limits_{\Omega_2} \Big[B(v^{(2)}, \overline{v}^{(2)}) - 2\omega\beta_0^{(2)} \operatorname{Im}(\overline{w}_{2,4}^{(2)} \operatorname{div} w_2^{(2)}) - 2\omega c_0^{(2)} \operatorname{Im}(w_{2,8}^{(2)} \, \overline{w}_{2,4}^{(2)}) + (\sigma^2 - \omega^2) \big(\rho_2 |w_2^{(2)}|^2$$
$$+ I_0^{(2)} |\phi_2^{(2)}|^2 + j_0^{(2)} |w_{2,8}^{(2)}|^2 + a^{(2)} |w_{2,4}^{(2)}|^2 \big) \Big] dx$$
$$= \operatorname{Re} \sum_{j=1}^{4} \langle G_j, \{W_2{}^{(2)}\}_j^+ \rangle_{S_1} - \operatorname{Re} \int\limits_{\Omega_1} \Psi^{(1)} \cdot V_2^{(1)} dx - \operatorname{Re} \int\limits_{\Omega_2} \Psi^{(2)} \cdot W_2{}^{(2)} dx, \tag{4.7}$$

$$\int\limits_{\Omega_1} \Big[2\sigma\omega\rho_1 |v_2^{(1)}|^2 + 2a^{(1)} \sigma\omega |v_{2,4}^{(1)}|^2 + 2\beta_0^{(1)} \sigma \operatorname{Im}(\overline{v}_{2,4}^{(1)} \operatorname{div} v_2^{(1)}) \Big] dx$$
$$+ \int\limits_{\Omega_2} \Big[2\sigma\beta_0^{(2)} \operatorname{Im}(\overline{w}_{2,4}^{(2)} \operatorname{div} w_2^{(2)}) + 2\sigma c_0^{(2)} \operatorname{Im}(w_{2,8}^{(2)} \, \overline{w}_{2,4}^{(2)}) + 2\sigma\omega \big(\rho_2 |w_2^{(2)}|^2$$
$$+ I_0^{(2)} |\phi_2^{(2)}|^2 + j_0^{(2)} |w_{2,8}^{(2)}|^2 + a^{(2)} |w_{2,4}^{(2)}|^2 \big) \Big] dx$$
$$= \operatorname{Im} \sum_{j=1}^{4} \langle G_j, \{W_2^{(2)}\}_j^+ \rangle_{S_1} - \operatorname{Im} \int\limits_{\Omega_1} \Psi^{(1)} \cdot V_2^{(1)} dx - \operatorname{Im} \int\limits_{\Omega_2} \Psi^{(2)} \cdot W_2{}^{(2)} dx. \tag{4.8}$$

Multiplying (4.8) by $\frac{\omega}{\sigma}$ and adding equality (4.7), we get

$$\int\limits_{\Omega_1}\Big[\mathscr{E}(v_2^{(1)},\overline{v}_2^{(1)})+(\sigma^2+\omega^2)\rho_1|v_2^{(1)}|^2+k^{(1)}|\operatorname{grad}v_{2,4}^{(1)}|^2+a^{(1)}(\sigma^2+\omega^2)|v_{2,4}^{(1)}|^2\Big]dx$$
$$+\int\limits_{\Omega_2}\Big[B(v^{(2)},\overline{v}^{(2)})+(\sigma^2+\omega^2)\big(\rho_2|w_2^{(2)}|^2+I_0^{(2)}|\phi_2^{(2)}|^2+j_0^{(2)}|w_{2,8}^{(2)}|^2+a^{(2)}|w_{2,4}^{(2)}|^2\big)\Big]dx$$
$$=\operatorname{Re}\sum_{j=1}^{4}\langle G_j,\{W_2{}^{(2)}\}_j^+\rangle_{S_1}-\operatorname{Re}\int\limits_{\Omega_1}\Psi^{(1)}\cdot V_2^{(1)}dx-\operatorname{Re}\int\limits_{\Omega_2}\Psi^{(2)}\cdot W_2{}^{(2)}dx$$
$$+\frac{\omega}{\sigma}\operatorname{Im}\sum_{j=1}^{4}\langle G_j,\{W_2{}^{(2)}\}_j^+\rangle_{S_1}-\frac{\omega}{\sigma}\operatorname{Im}\int\limits_{\Omega_1}\Psi^{(1)}\cdot V_2^{(1)}dx-\frac{\omega}{\sigma}\operatorname{Im}\int\limits_{\Omega_2}\Psi^{(2)}\cdot W_2{}^{(2)}dx. \tag{4.9}$$

From equlity (4.9), we obtain for $|\tau|\to\infty$ the following estimates:

$$c_1\bigg(\int\limits_{\Omega_1}\mathscr{E}(v_2^{(1)},\overline{v}_2^{(1)})dx+\|v_2^{(1)}\|^2_{[L_2(\Omega_1)]^3}+\|v_{2,4}^{(1)}\|^2_{H^1(\Omega_1)}\bigg)$$
$$+c_2\bigg(\int\limits_{\Omega_2}B(v^{(2)},\overline{v}^{(2)})dx+\|w_2^{(2)}\|^2_{[L_2(\Omega_2)]^3}+\|w_{2,4}^{(2)}\|^2_{L_2(\Omega_2)}+\|\phi_2^{(2)}\|^2_{[L_2(\Omega_2)]^3}+\|w_{2,8}^{(2)}\|^2_{L_2(\Omega_2)}\bigg)$$
$$\le\sum_{j=1}^{4}\|G_j\|_{H^{-\frac12}(S_1)}\|\{W_2{}^{(2)}\}_j^+\|_{H^{\frac12}(S_1)}+\|\Psi^{(1)}\|_{[L_2(\Omega_1)]^4}\|V_2^{(1)}\|_{[L_2(\Omega_1)]^4}$$
$$+\|\Psi^{(2)}\|_{[L_2(\Omega_1)]^9}\|W_2{}^{(2)}\|_{[L_2(\Omega_2)]^9}+\frac{|\omega|}{\sigma}\|G_j\|_{H^{-\frac12}(S_1)}\|\{W_2{}^{(2)}\}_j^+\|_{H^{\frac12}(S_1)}$$
$$+\frac{|\omega|}{\sigma}\|\Psi^{(1)}\|_{[L_2(\Omega_1)]^4}\|V_2^{(1)}\|_{[L_2(\Omega_1)]^4}+\frac{|\omega|}{\sigma}\|\Psi^{(2)}\|_{[L_2(\Omega_1)]^9}\|W_2{}^{(2)}\|_{[L_2(\Omega_2)]^9}, \tag{4.10}$$

where $c_1:=\min\{\rho_1,k^{(1)},a^{(1)},1\}$ and $c_2:=\min\{\rho_2,I_0^{(2)},j_0^{(2)},a^{(2)},1\}$.

Now, in the left part of inequality (4.10) we use the positive-definiteness of the form $B(v^{(2)},v^{(2)})$ and Poincaré's inequality, and in the right part of the same inequality we use the trace theorem and estimates (4.3). Thus we get

$$\Big(\|V_2^{(1)}\|_{[H^1[(\Omega_1)]^4}+\|W_2^{(2)}\|_{H^1[(\Omega_2)]^9}\Big)^2\le c|\tau|^{-M+3}\Big(\|V_2^{(1)}\|_{[H^1[(\Omega_1)]^4}+\|W_2^{(2)}\|_{[H^1[(\Omega_2)]^9}\Big)$$

and therefore, we obtain

$$|V_2^{(1)}\|_{[H^1[(\Omega_1)]^4}+\|W_2^{(2)}\|_{[H^1[(\Omega_2)]^9}\le c|\tau|^{-M+3}\ \text{ for }\ |\tau|\to\infty, \tag{4.11}$$

where c is a positive number, not depending on the complex parameter τ.

Thus, in view of (4.2) and (4.11), we conclude

$$\|\widetilde{U}^{(1)}\|_{[H^1[(\Omega_1)]^4}\le c|\tau|^{-M+3},\quad \|\widetilde{U}^{(2)}\|_{[H^1[(\Omega_2)]^9}\le c|\tau|^{-M+3}\ \text{ for }\ |\tau|\to\infty. \tag{4.12}$$

Similarly, we can obtain the same estimates of (4.12) for the solutions of the $(TN)_\tau$ and $(TM)_\tau$ pseudo-oscillation boundary-transmission problems. In the case of the $(TN)_\tau$ pseudo-oscillation boundary-transmission problem, the Dirichlet conditions in Problem 4.1 and Problem 4.2 on the surface S_2 should be replaced by the Neumann boundary conditions, and in the case of the $(TM)_\tau$ pseudo-oscillation boundary-transmission problem, the Dirichlet conditions in Problem 4.1 and Problem 4.2 on the surface S_2 should be replaced by the mixed boundary conditions.

In turn, from estimates (4.12) with $M>m+4$ and the inverse Laplace transform

$$U^{(1)}(\cdot,t)=\frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+\infty}e^{\tau t}\widetilde{U}^{(1)}(\cdot,\tau)d\tau,\quad \alpha>a>0,$$

$$U^{(2)}(\cdot,t)=\frac{1}{2\pi i}\int\limits_{\alpha-i\infty}^{\alpha+\infty} e^{\tau t}\widetilde{U}^{(2)}(\cdot,\tau)d\tau,\quad \alpha>a>0$$

of the solution $(\widetilde{U}^{(1)},\widetilde{U}^{(2)})$ of the boundary-transmission pseudo-oscillation problems, we have found that

$$U^{(1)}\in C_a^m\big([0,+\infty),\big[H^1(\Omega_1)\big]^4\big),$$
$$U^{(2)}\in C_a^m\big([0,+\infty),\big[H^1(\Omega_2)\big]^9\big)\ \text{ with }\ m\geq 2.$$

Therefore we arrive at the following existence and uniqueness results for the original boundary-transmission dynamical problems $(TD)_t$, $(TN)_t$, $(TM)_t$.

Theorem 4.9. *Let* $S_1,S_2\in C^\infty$ *and*

$$\Phi_1\in C_{a,0}^{m+5}\big([0,+\infty),\big[L_2(\Omega_1)\big]^4\big),\quad \Phi_2\in C_{a,0}^{m+5}\big([0,+\infty),\big[L_2(\Omega_2)\big]^9\big),$$
$$f_j^{(1)}\in C_{a,0}^{m+7}\big([0,+\infty),H^{\frac12}(S_1)\big),\quad f_j^{(2}\in C_{a,0}^{m+7}\big([0,+\infty),H^{\frac12}(S_1)\big)\ \ j=\overline{1,4},$$
$$Q_j^{(2)}\in C_{a,0}^{m+7}\big([0,+\infty),H^{\frac12}(S_1)\big),\ \ j=\overline{5,9},\ \ p^{(2)}\in C_{a,0}^{m+7}\big([0,+\infty),\big[H^{\frac12}(S_2)\big]^9\big).$$

Then the boundary-transmission dynamical problem $(TD)_t$ *has a unique solution* $(U^{(1)},U^{(2)})$ *in the space*

$$C_a^m\big([0,+\infty),\big[H^1(\Omega_1)\big]^4\big)\times\in C_a^m\big([0,+\infty),\big[H^1(\Omega_2)\big]^9\big)\quad \textit{for}\quad m\geq 2.$$

Theorem 4.10. *Let* $S_1,S_2\in C^\infty$ *and*

$$\Phi_1\in C_{a,0}^{m+5}\big([0,+\infty),\big[L_2(\Omega_1)\big]^4\big),\quad \Phi_2\in C_{a,0}^{m+5}\big([0,+\infty),\big[L_2(\Omega_2)\big]^9\big),$$
$$f_j^{(1)}\in C_{a,0}^{m+7}\big([0,+\infty),H^{\frac12}(S_1)\big),\quad f_j^{(2}\in C_{a,0}^{m+7}\big([0,+\infty),H^{\frac12}(S_1)\big)\ \ j=\overline{1,4},$$
$$Q_j^{(2)}\in C_{a,0}^{m+7}\big([0,+\infty),H^{\frac12}(S_1)\big)\ \ j=\overline{5,9},\ \ q^{(2)}\in C_{a,0}^{m+7}\big([0,+\infty),\big[H^{-\frac12}(S_2)\big]^9\big).$$

Then the boundary-transmission dynamical problem $(TN)_t$ *has a unique solution* $(U^{(1)},U^{(2)})$ *in the space*

$$C_a^m\big([0,+\infty),\big[H^1(\Omega_1)\big]^4\big)\times\in C_a^m\big([0,+\infty),\big[H^1(\Omega_2)\big]^9\big)\quad \textit{for}\quad m\geq 2.$$

Theorem 4.11. *Let* $S_1,S_2\in C^\infty$ *and*

$$\Phi_1\in C_{a,0}^{m+5}\big([0,+\infty),\big[L_2(\Omega_1)\big]^4\big),\quad \Phi_2\in C_{a,0}^{m+5}\big([0,+\infty),\big[L_2(\Omega_2)\big]^9\big),$$
$$f_j^{(1)}\in C_{a,0}^{m+7}\big([0,+\infty),H^{\frac12}(S_1)\big),\quad f_j^{(2}\in C_{a,0}^{m+7}\big([0,+\infty),H^{\frac12}(S_1)\big)\ \ j=\overline{1,4},$$
$$Q_j^{(2)}\in C_{a,0}^{m+7}\big([0,+\infty),H^{\frac12}(S_1)\big)\ \ j=\overline{5,9},\ \ p_2^{(D)}\in C_{a,0}^{m+7}\big([0,+\infty),\big[H^{\frac12}(S_2^{(D)})\big]^9\big),$$
$$q_2^{(N)}\in C_{a,0}^{m+7}([0,\infty),[H^{-\frac12}(S_N)]^9).$$

Then the boundary-transmission dynamical problem $(TM)_t$ *has a unique solution* $(U^{(1)},U^{(2)})$ *in the space*

$$C_a^m\big([0,+\infty),\big[H^1(\Omega_1)\big]^4\big)\times\in C_a^m\big([0,+\infty),\big[H^1(\Omega_2)\big]^9\big)\quad \textit{for}\quad m\geq 2.$$

Let us introduce the notation

$$C_{a,0}^m\big([0,+\infty),C^\infty(\overline{\Omega}_q)\big):=\bigcap_{k=1}^{\infty}C_{a,0}^m\big([0,+\infty),C^k(\overline{\Omega}_q)\big),\quad q=1,2,$$
$$C_{a,0}^m\big([0,+\infty),C^\infty(S_q)\big):=\bigcap_{k=1}^{\infty}C_{a,0}^m\big([0,+\infty),C^k(S_q)\big),\quad q=1,2,$$
$$C_a^m\big([0,+\infty),C^\infty(\overline{\Omega}_q)\big):=\bigcap_{k=1}^{\infty}C_a^m\big([0,+\infty),C^k(\overline{\Omega}_q)\big).$$

The following regularity results for the solutions of the dynamical boundary-transmission problems $(TD)_t$ and $(TN)_t$ follow directly from Theorems 4.2, 4.5, Corollaries 4.3, 4.6 and Theorems 4.9, 4.10.

Theorem 4.12. *Let $S_1, S_2 \in C^{r,\alpha}$ with $0 < \beta < \alpha \le 1$, and let*

$$(U^{(1)}, U^{(2)}) \in C_a^m\big([0,+\infty), \big[H^1(\Omega_1)\big]^4\big) \times \in C_a^m\big([0,+\infty), \big[H^1(\Omega_2)\big]^9\big)$$

for $m \ge 2$, $a > 0$ be a solution of the dynamical problem $(TD)_t$ for the data $\Phi_1 \in C_{a,0}^{m+5}\big([0,+\infty), [C^{k-2,\beta}(\overline{\Omega}_1)]^4\big)$, $\Phi_2 \in C_{a,0}^{m+5}\big([0,+\infty), [C^{k-2,\beta}(\overline{\Omega}_2)]^9\big)$, $k \ge 2$, $k \in \mathbb{N}$, $m \ge 2$,

$$f_j^{(1)} \in C_{a,0}^{m+7}\big([0,+\infty), C^{k,\beta}(S_1)\big), \quad f_j^{(2} \in C_{a,0}^{m+7}\big([0,+\infty), C^{k-1,\beta}(S_1)\big), \quad j = \overline{1,4},$$

$$Q_j^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty), C^{k,\beta}(S_1)\big), \ \ j = \overline{5,9}, \quad p^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty), [C^{k,\beta}(S_2)]^9\big), \ \ k = \overline{2,r}, \ r \in \mathbb{N}.$$

Then the Dirichlet boundary-transmission dynamical problem $(TD)_t$ has a unique solution

$$(U^{(1)}, U^{(2)}) \in C_a^m\big([0,+\infty), [C^{k,\beta}(\overline{\Omega}_1)]^4\big) \times C_a^m\big([0,+\infty), [C^{k,\beta}(\overline{\Omega}_2)]^9\big).$$

Corollary 4.13. *Let $S_1, S_2 \in C^\infty$ and*

$$\Phi_1 \in C_{a,0}^{m+5}\big([0,+\infty), [C^\infty(\overline{\Omega}_1)]^4\big), \quad \Phi_2 \in C_{a,0}^{m+5}\big([0,+\infty), [C^\infty(\overline{\Omega}_2)]^9\big),$$

$$f_j^{(1)} \in C_{a,0}^{m+7}\big([0,+\infty), C^\infty(S_1)\big), \quad f_j^{(2)} C_{a,0}^{m+7}\big([0,+\infty), C^\infty(S_2)\big), \quad j = \overline{1,4},$$

$$Q_j^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty), C^\infty(S_1)\big), \quad j = \overline{5,9}, \quad p^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty), [C^\infty(S_2)]^9\big)$$

for all $k \ge 2$, $k \in \mathbb{N}$, and $m \ge 2$, $a > 0$.

Then the unique solution $(U^{(1)}, U^{(2)})$ of the Dirichlet boundary-transmission dynamical problem $(TD)_t$ belongs to the class $C_a^m\big([0,+\infty), [C^\infty(\overline{\Omega}_1)]^4\big) \times C_a^m\big([0,+\infty), [C^\infty(\overline{\Omega}_2)]^9\big)$ i.e.,

$$(U^{(1)}, U^{(2)}) \in C_a^m\big([0,+\infty), [C^\infty(\overline{\Omega}_1)]^4\big) \times C_a^m\big([0,+\infty), [C^\infty(\overline{\Omega}_2)]^9\big).$$

Theorem 4.14. *Let $S_1, S_2 \in C^{r,\alpha}$ with $0 < \beta < \alpha \le 1$, and let*

$$(U^{(1)}, U^{(2)}) \in C_a^m\big([0,+\infty), \big[H^1(\Omega_1)\big]^4\big) \times \in C_a^m\big([0,+\infty), \big[H^1(\Omega_2)\big]^9\big)$$

for $m \ge 2$, $a > 0$ be a solution of the dynamical problem $(TN)_t$ for the data

$$\Phi_1 \in C_{a,0}^{m+5}\big([0,+\infty), [C^{k-2,\beta}(\overline{\Omega}_1)]^4\big), \quad \Phi_2 \in C_{a,0}^{m+5}\big([0,+\infty), [C^{k-2,\beta}(\overline{\Omega}_2)]^9\big), \ \ k \ge 2, \ \ k \in \mathbb{N}, \ \ m \ge 2,$$

$$f_j^{(1)} \in C_{a,0}^{m+7}\big([0,+\infty), C^{k,\beta}(S_1)\big), \quad f_j^{(2} \in C_{a,0}^{m+7}\big([0,+\infty), C^{k-1,\beta}(S_1)\big), \ \ j = \overline{1,4},$$

$$Q_j^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty), C^{k,\beta}(S_1)\big), \ \ j = \overline{5,9}, \quad q^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty), [C^{k-1,\beta}(S_2)]^9\big), \ \ k = \overline{2,r}, \ r \in \mathbb{N}.$$

Then the Neumann boundary-transmission dynamical problem $(TN)_t$ has a unique solution

$$(U^{(1)}, U^{(2)}) \in C_a^m\big([0,+\infty), [C^{k,\beta}(\overline{\Omega}_1)]^4\big) \times C_a^m\big([0,+\infty), [C^{k,\beta}(\overline{\Omega}_2)]^9\big).$$

Corollary 4.15. *Let $S_1, S_2 \in C^\infty$ and*

$$\Phi_1 \in C_{a,0}^{m+5}\big([0,+\infty), [C^\infty(\overline{\Omega}_1)]^4\big), \quad \Phi_2 \in C_{a,0}^{m+5}\big([0,+\infty), [C^\infty(\overline{\Omega}_2)]^9\big),$$

$$f_j^{(1)} \in C_{a,0}^{m+7}\big([0,+\infty), C^\infty(S_1)\big), \quad f_j^{(2)} C_{a,0}^{m+7}\big([0,+\infty), C^\infty(S_2)\big), \ \ j = \overline{1,4},$$

$$Q_j^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty), C^\infty(S_1)\big), \ \ j = \overline{5,9}, \quad q^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty), [C^\infty(S_2)]^9\big)$$

for all $k \ge 2$, $k \in \mathbb{N}$, and $m \ge 2$, $a > 0$.

Then the unique solution $(U^{(1)}, U^{(2)})$ of the Neumann boundary-transmission dynamical problem $(TN)_t$ belongs to the class $C_a^m\big([0,+\infty), [C^\infty(\overline{\Omega}_1)]^4\big) \times C_a^m\big([0,+\infty), [C^\infty(\overline{\Omega}_2)]^9\big)$ i.e.,

$$(U^{(1)}, U^{(2)}) \in C_a^m\big([0,+\infty), [C^\infty(\overline{\Omega}_1)]^4\big) \times C_a^m\big([0,+\infty), [C^\infty(\overline{\Omega}_2)]^9\big).$$

From the asymptotic formula of the solution of the pseudo-oscillation problem (cf. formula (8.9) in [5] and also (5.26) in [6]) and the approach developed in paper [6], we can obtain the estimates of the first coefficients and the remainder term of the asymptotic expansion with respect to the complex parameter τ. Then, using the inverse Laplace transform in the asymptotic expansion of a solution, we can obtain the following optimal regularity result for the boundary-transmission dynamical problem $(TM)_t$ near the line $\ell = \partial S_2^{(D)} = \partial S_2^{(N)}$.

Theorem 4.16. *Suppose $S_1, S_2 \in C^\infty$ and*

$$\Phi_1 \in C_{a,0}^{m+5}\big([0,+\infty),[C^\infty(\overline{\Omega}_1)]^4\big), \quad \Phi_2 \in C_{a,0}^{m+5}\big([0,+\infty),[C^\infty(\overline{\Omega}_2)]^9\big),$$
$$f_j^{(1)} \in C_{a,0}^{m+7}\big([0,+\infty),C^\infty(S_1)\big), \quad f_j^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty),\in C^\infty(S_1)\big), \;\; j=\overline{1,4},$$
$$Q_j^{(2)} \in C_{a,0}^{m+7}\big([0,+\infty),C^\infty(S_1)\big), \;\; j=\overline{5,9}, \quad p_2^{(D)} \in C_{a,0}^{m+7}\big([0,+\infty),[C^\infty(\overline{S}_2^{(D)})]^9\big),$$
$$q_2^{(N)} \in C_{a,0}^{m+7}\big([0,+\infty),[C^\infty(\overline{S}_2^{(N)})]^9\big),$$

for $m \geq 2$, $a > 0$. Then:

1) *If $d < 0$, then the unique solution $(U^{(1)}, U^{(2)})$ to the mixed boundary-transmission dynamical problem $(TM)_t$ belongs to $C_a^m\big([0,+\infty),[C^\infty(\overline{\Omega}_1)]^4\big) \times C_a^m\big([0,+\infty),[C^{\gamma_1}(\overline{\Omega}_2)]^9\big)$, i.e.,*

$$(U^{(1)}, U^{(2)}) \in C_a^m\big([0,+\infty),[C^\infty(\overline{\Omega}_1)]^4\big) \times C_a^m\big([0,+\infty),[C^{\gamma_1}(\overline{\Omega}_2)]^9\big),$$

where $\gamma_1 = \frac{1}{2} - \frac{1}{\pi}\operatorname{arctg} 2\sqrt{-d}$, γ_1 depends on the material constants, does not depend on the geometry of the exceptional line $\ell = \partial S_2^{(D)} = \partial S_2^{(N)}$ and may take any values from the interval $(0, \frac{1}{2})$.

2) *If $d \geq 0$, then the unique solution to the corresponding mixed boundary-transmission dynamical problem $(TM)_t$*

$$(U^{(1)}, U^{(2)}) \in C_a^m\big([0,+\infty),[C^\infty(\overline{\Omega}_1)]^4\big) \times C_a^m\big([0,+\infty),[C^{\frac{1}{2}}(\overline{\Omega}_2)]^9\big).$$

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Sokhumi State University, 9 A. Politkovskaia Str., Tbilisi 0186, Georgia
Email address: anikatoloraia@gmail.com