Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 2, 179–196

TWO-WEIGHTED INEQUALITIES FOR MAXIMAL, SINGULAR INTEGRAL OPERATORS AND THEIR COMMUTATORS IN $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$ SPACES

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Abstract. In this paper, we prove the two-weighted boundedness of maximal operator, singular integral operators and their commutators in weighted global Morrey-type spaces $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$. Furthermore, we give weighted global Morrey-type a priori estimates and a priori estimates for non-divergent elliptic equations in $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$ spaces as applications.

1. Introduction

The classical Morrey spaces were introduced by Morrey [32] to study the local behavior of solutions to the second-order elliptic partial differential equations.

Moreover, various Morrey spaces have been defined in the process of study. Guliyev, Mizuhara and Nakai [22, 31, 34] introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$ (see, also [23, 24, 39]).

Recently, Komori and Shirai [29] defined the weighted Morrey spaces $L^{p,\kappa}_{\omega}(\mathbb{R}^n)$ and studied the boundedness of some classical operators such as the Hardy–Littlewood maximal operator and the Calderón–Zygmund operator on these spaces.

Also, Guliyev in [25] first introduced the generalized weighted Morrey spaces $M^{p,\varphi}_{\omega}(\mathbb{R}^n)$ and studied the boundedness of the sublinear operators and their higher order commutators generated by Calderón–Zygmund operators and Riesz potentials in these spaces. Note that Guliyev [25] gave the concept of generalized weighted Morrey space which could be viewed as an extension of both $M^{p,\varphi}_{\omega}(\mathbb{R}^n)$ and $L^{p,\kappa}_{\omega}(\mathbb{R}^n)$ spaces.

Recall that in 1994, in his doctoral thesis [22, pp. 75-76] (see also [23, pp. 123]), Guliyev introduced the local Morrey-type space $L\mathcal{M}_{pq,\omega(\cdot)}(\mathbb{R}^n)$ and complementary local Morrey-type spaces $^{\complement}L\mathcal{M}_{pq,\omega(\cdot)}(\mathbb{R}^n)$ given by

$$\|f\|_{L\mathcal{M}_{pq,\omega(\cdot)}} = \|\omega(r)\,\|f\chi_{B(0,r)}\|_{L_p}\|_{L_q(\mathbb{R}^+)} < \infty$$

and

$$\|f\|_{^{\complement}L\mathcal{M}_{pq,\omega(\cdot)}} = \|\omega(r)\,\|f\chi_{\mathbb{R}^n\setminus B(0,r)}\|_{L_p}\|_{L_q(\mathbb{R}^+)} < \infty,$$

respectively, where ω is a positive measurable function defined on $(0,\infty)$. In [22] (see also [23]), the author found the sufficient conditions for the boundedness of the singular and potential operators in the local Morrey-type spaces $L\mathcal{M}_{pq,\omega(\cdot)}(\mathbb{R}^n)$ and in the complementary local Morrey-type spaces $^{\complement}L\mathcal{M}_{pq,\omega(\cdot)}(\mathbb{R}^n)$.

During the last decades, various classical operators, such as maximal, singular and potential operators were widely investigated both in the classical and in local Morrey-type spaces. In [6, pp. 157], V. I. Burenkov and H. V. Guliyev introduced the space $G\mathcal{M}_{p,\theta,\omega(\cdot)}(\mathbb{R}^n)$. Here and below, we denote $B(x,r) = \{x+y : y \in B(0,r)\}$.

Definition 1.1. Let $0 < p, \theta \le \infty$ and let ω be a non-negative Lebesgue measurable function on $(0,\infty)$.

2020 *Mathematics Subject Classification.* Primary 42B20,42B25, 42B35.

Key words and phrases. Maximal operator; Singular integral operators; Weighted global Morrey-type space; Two-weighted inequality; Hardy inequality; BMO spac.

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1. [22, pp.75-76]. Denote by $LM_{p,\theta,\omega(\cdot)}(\mathbb{R}^n)$ the local Morrey-type space, the space of all functions f Lebesgue measurable on $\mathbb{R}^n$ with a finite quasi-norm

$$\|f\|_{LM_{p,\theta,\omega(\cdot)}} = \|\omega(r)\|f\chi_{B(0,r)}\|_{L_p}\|_{L_\theta(0,\infty)}.$$

2. [6–8]. Denote by $GM_{p,\theta,\omega(\cdot)}(\mathbb{R}^n)$ the global Morrey-type space, the space of all functions f Lebesgue measurable on $\mathbb{R}^n$ with a finite quasi-norm

$$\|f\|_{GM_{p,\theta,\omega(\cdot)}} = \sup_{x\in\mathbb{R}^n} \|f(x+\cdot)\|_{LM_{p,\theta,\omega(\cdot)}} = \sup_{x\in\mathbb{R}^n} \|\omega(r)\|f\chi_{B(x,r)}\|_{L_p}\|_{L_\theta(0,\infty)}.$$

Note that if $\omega(r) = 1$, then $LM_{p,\infty,1}(\mathbb{R}^n) = GM_{p,\infty,1}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$.

If $\|\omega(r)\|_{L_\theta(t,\infty)} = \infty$ for all $t > 0$, then $LM_{p,\theta,\omega} = GM_{p,\theta,\omega} = \emptyset$, where $\emptyset$ is the set of all functions, equivalent to 0 on $\mathbb{R}^n$.

If $\|\omega(r)r^{n/p}\|_{L_\theta(0,t)} = \infty$ for all $t > 0$, then $f(0) = 0$ for all $f \in LM_{p,\theta,\omega}$, continuous at 0, and $GM_{p,\theta,\omega} = \emptyset$ for $0 < p < \infty$.

Furthermore,

$$GM_{p,\infty,r^{-\lambda}}(\mathbb{R}^n) \equiv M^{p,\lambda}(\mathbb{R}^n), \quad 0 < p \le \infty, \quad 0 \le \lambda \le \frac{n}{p}.$$

The spaces $LM_{pq,\omega(\cdot)}(\mathbb{R}^n)$ and $^{\complement}LM_{pq,\omega(\cdot)}(\mathbb{R}^n)$ are denoted, respectively, as local Morrey-type spaces and complementary local Morrey-type spaces, though from the point of view of the role in the development of these spaces they may be also called **local and complementary Morrey–Guliyev spaces**, respectively (see, e.g., [36]).

The local Morrey-type space $LM_{pq,\lambda} = LM_{pq,t^{-\lambda}}(\mathbb{R}^n)$ first appeared in 1981 by D. R. Adams in [1, p. 44] and it also may be called as the **local Morrey–Adams spaces** (see, e.g., [36, 37]).

Let f be a locally integrable function on $\mathbb{R}^n$. The so-called Hardy–Littlewood maximal function is defined by the formula

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int\limits_{B(x,r)} |f(y)|dy,$$

where $|B(x,r)|$ is the Lebesgue measure of the ball $B(x,r)$.

The Calderón–Zygmund type singular operator is defined as

$$Tf(x) = \int\limits_{\mathbb{R}^n} K(x,y)f(y)dy,$$

where $K(x,y)$ is a "standard singular kernel", that is, a continuous function defined on $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : x \ne y\}$ and satisfying the estimates

$$|K(x,y)| \le C|x-y|^{-n} \text{ for all } x \ne y,$$
$$|K(x,y) - K(x,z)| \le C\frac{|y-z|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|y-z|,$$
$$|K(x,y) - K(\xi,y)| \le C\frac{|x-\xi|^\sigma}{|x-y|^{n+\sigma}}, \quad \sigma > 0, \text{ if } |x-y| > 2|x-\xi|.$$

Let

$$T^*f(x) = \sup_{\varepsilon>0} |T_\varepsilon f(x)|$$

be the maximal singular operator, where $T_\varepsilon f(x)$ is the usual truncation

$$T_\varepsilon f(x) = \int\limits_{\{y\in\mathbb{R}^n : |x-y|\ge\varepsilon\}} K(x,y)f(y)dy.$$

It is well known that T^*f exists almost everywhere whenever f is a step function. The almost everywhere existence of the limit (of certain integral averages) was known for a dense subset of L_1 and the result was extended to all of L_1 by establishing control over the corresponding maximal operators.

In this paper our aim is to define weighted global Morrey-type spaces and prove the two-weighted boundedness of a maximal operator, singular integral operators and their commutators in these spaces.

We also aim to give weighted global Morrey-type a priori estimates and a priori estimates for non-divergent elliptic equations as applications.

2. Preliminaries

Let $L_{p,\varphi}(B(x,r))$ denote the weighted L_p-space of measurable functions f for which

$$\|f\|_{L_{p,\varphi}(B(x,r))} \equiv \|f\chi_{B(x,r)}\|_{L_{p,\varphi}(\mathbb{R}^n)} = \Bigg(\int\limits_{B(x,r)} |f(y)|^p \varphi(y) dy \Bigg)^{\frac{1}{p}}.$$

Even though the A_p class is well-known, for completeness, we offer the definition of A_p weight functions.

Definition 2.1. The weight function φ belongs to the class $A_p(\mathbb{R}^n)$ for $1 \le p < \infty$ if the following statement:

$$\sup_{x\in\mathbb{R}^n, r>0} \Bigg(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \varphi^p(y) dy \Bigg)^{\frac{1}{p}} \Bigg(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \varphi^{-p'}(y) dy \Bigg)^{\frac{1}{p'}}$$

is finite and φ belongs to $A_1(\mathbb{R}^n)$, if there exists a positive constant C such that for any $x \in \mathbb{R}^n$ and $r > 0$,

$$|B(x,r)|^{-1} \int\limits_{B(x,r)} \varphi(y) dy \le C \operatorname*{ess\,sup}_{y\in B(x,r)} \frac{1}{\varphi(y)}.$$

The weight function (φ_1, φ_2) belongs to the class $\widetilde{A}_p(\mathbb{R}^n)$ for $1 \le p < \infty$, if the following statement:

$$\sup_{x\in\mathbb{R}^n, r>0} \Bigg(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \varphi_2^p(y) dy \Bigg)^{\frac{1}{p}} \Bigg(\frac{1}{|B(x,r)|} \int\limits_{B(x,r)} \varphi_1^{-p'}(y) dy \Bigg)^{\frac{1}{p'}}$$

is finite.

Lemma 2.2. *Let $1 \le p < \infty$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, then $(\varphi_2^{-1}, \varphi_1^{-1}) \in \widetilde{A}_{p'}(\mathbb{R}^n)$, with $\frac{1}{p} + \frac{1}{p'} = 1$.*

The following theorem has been proved in [33].

Theorem 2.3. *Let $1 \le p < \infty$, then*
1) *$M : L_{p,\varphi}(\mathbb{R}^n) \to L_{p,\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_p(\mathbb{R}^n)$,*
2) *$M : L_{1,\varphi}(\mathbb{R}^n) \to WL_{1,\varphi}(\mathbb{R}^n)$ if and only if $\varphi \in A_1(\mathbb{R}^n)$.*

Theorem 2.4 ([20])**.** *Let $1 < p < \infty$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, then the operator M is bounded from $L_{p,\varphi_1}(\mathbb{R}^n)$ to $L_{p,\varphi_2}(\mathbb{R}^n)$.*

The following theorem has been proved in [19].

Theorem 2.5. *Let $1 < p < \infty$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, then the singular integral operator T is bounded from $L_{p,\varphi_1}(\mathbb{R}^n)$ to $L_{p,\varphi_2}(\mathbb{R}^n)$.*

Corollary 2.6. *Let $1 < p < \infty$ and $\varphi \in A_p(\mathbb{R}^n)$, then the singular integral operator T is bounded in $L_{p,\varphi}(\mathbb{R}^n)$.*

Definition 2.7. Let $0 < p, \theta \le \infty$, $\omega(r)$ be a non-negative measurable function on $(0, \infty)$, $\varphi(r)$ be a measurable function, and $f \in L^{\text{loc}}_{p,\varphi}(\mathbb{R}^n)$. The weighted global Morrey-type spaces $G\mathcal{M}_{p,\theta,\omega,\varphi}$ are defined by the norm

$$\|f\|_{G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n} \Bigg(\int\limits_0^\infty \omega(t) \|f\|^\theta_{L_{p,\varphi}(B(x,t))} dt \Bigg)^{1/\theta}.$$

If $\varphi(r) = 1$, then we obtain global Morrey-type spaces $G\mathcal{M}_{p,\theta,\omega}$ defined in [6].

Let $M^\sharp$ be the sharp maximal function defined by

$$M^\sharp f(x) = \sup_{r>0} |B(x,r)|^{-1} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| dy,$$

where $f_{B(x,r)}(x) = |B(x,r)|^{-1} \int_{B(x,r)} f(y) dy$.

Definition 2.8. We define the $BMO(\mathbb{R}^n)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO} = \sup_{x \in \mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int\limits_{B(x,r)} |f(y) - f_{B(x,r)}| dy < \infty$$

or

$$\|f\|_{BMO} = \inf_C \sup_{x \in \mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int\limits_{B(x,r)} |f(y) - C| dy < \infty.$$

Definition 2.9. Given a measurable function b, the maximal commutator is defined by

$$M_b(f)(x) = \sup_{x \in \mathbb{R}^n,\, r>0} |B(x,r)|^{-1} \int\limits_{B(x,r)} |b(x) - b(y)||f(y)| dy,$$

for all $x \in \mathbb{R}^n$.

This operator plays an important role in the study of commutators of singular integral operators with BMO symbols.

Definition 2.10. Given a measurable function b, the commutator of the Hardy–Littlewood maximal operator M and b are defined by

$$[M,b]f(x) = M(bf)(x) - b(x)Mf(x)$$

for all $x \in \mathbb{R}^n$.

Definition 2.11. We define the $BMO_{p,\varphi}(\mathbb{R}^n)$ $(1 \le p < \infty)$ space as the set of all locally integrable functions f such that

$$\|f\|_{BMO_{p,\varphi}} = \sup_{x \in \mathbb{R}^n,\, r>0} \frac{\|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\varphi}(\mathbb{R}^n)}}{\|\varphi\|_{L_p(B(x,r))}}$$

or

$$\|f\|_{BMO_{p,\varphi}} = \sup_{x \in \mathbb{R}^n,\, r>0} \frac{1}{|B(x,r)|} \|(f(\cdot) - f_{B(x,r)})\chi_{B(x,r)}\|_{L_{p,\varphi}(\mathbb{R}^n)} \|\varphi^{-1}\|_{L_{p'}(B(x,r))} < \infty.$$

Theorem 2.12 ([26])**.** *Let $1 \le p < \infty$ and φ be a Lebesgue measurable function. If $\varphi \in A_p(\mathbb{R}^n)$, then the norms $\|\cdot\|_{BMO_{p,\varphi}}$ and $\|\cdot\|_{BMO}$ are mutually equivalent.*

We will need the following lemma while proving our main theorems.

Lemma 2.13 ([27])**.** *Let $b \in BMO(\mathbb{R}^n)$. Then there is a constant $C > 0$ such that*

$$\left| b_{B(x,r)} - b_{B(x,t)} \right| \le C\|b\|_{BMO} \ln \frac{t}{r} \quad \text{for} \quad 0 < 2r < t,$$

where C is independent of b, x, r and t.

We will use the standard notation for Sobolev spaces and for derivatives, namely, if α is a multi-index, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in \mathbb{Z}^n_+$, we denote $|\alpha| = \sum\limits_{j=1}^{n} \alpha_j$, $D^\alpha = \partial^{\alpha_1}_{x_1} \cdots \partial^{\alpha_n}_{x_n}$ and

$$W^k_p(\Omega) = \{v \in L_p(\Omega) : D^\alpha v \in L_p(\Omega),\ \forall\ |\alpha| \le k\}$$

and the generalized weighted Sobolev-Morrey spaces

$$W^k_{p,\omega}(\Omega, \varphi) = \{v \in \mathcal{M}^{p,\omega}_\varphi(\Omega) : D^\alpha v \in \mathcal{M}^{p,\omega}_\varphi(\Omega),\ \forall\ |\alpha| \le k\}.$$

Let Γ be the standard fundamental solution of the Laplacian operator, namely,

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log |x|^{-1}, & n = 2, \\ \frac{1}{n(n-2)\omega_n} |x|^{2-n}, & n \geq 3, \end{cases}$$

with ω_n the area of the unit sphere in $\mathbb{R}^n$.

Given a function $f \in C_0^\infty(\mathbb{R}^n)$, it is a classic result that the potential ϕ given by

$$\phi(x) = \int \Gamma(x-y) f(y) dy$$

is a solution of $-\triangle\phi = f$ in $\mathbb{R}^n$ and satisfies the estimate

$$\|\phi\|_{W_p^2(\mathbb{R}^n)} \leq C \|f\|_{L_p(\mathbb{R}^n)} \tag{2.1}$$

for $1 < p < \infty$. Indeed, this estimate is a consequence of the Calderón–Zygmund theory of singular integrals (see, e.g., [40]).

On the other hand, a priori estimates like (2.1) for solutions of the Dirichlet problem

$$\begin{cases} -\triangle\phi = f & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{2.2}$$

on smooth bounded domains Ω are also well known (see, e.g., the classic paper by Agmon, Douglis and Nirenberg [4], where a priori estimates for general elliptic problems are proved).

3. Two-weighted Inequalites for the Maximal Operator and its Commutator in the Spaces $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$

In this section we prove the two-weighted boundedness of the maximal operator and maximal commutators in the $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$ weighted global Morrey-type spaces. We need the following two generalized Hardy inequalities which are to be used in the proof of our theorems.

Lemma 3.1. *Let $1 \leq r \leq s \leq \infty$ and let v and w be two functions such that measurable and positive a.e. on $(0,\infty)$. Then there exists a constant C independent of the function h such that*

$$\left(\int_0^\infty \left(\int_0^t h(\tau) d\tau \right)^s w(t) dt \right)^{1/s} \leq C \left(\int_0^\infty h(t)^r v(t) dt \right)^{1/r}, \tag{3.1}$$

if and only if

$$K = \sup_{t>0} \left(\int_t^\infty w(\tau) d\tau \right)^{1/s} \left(\int_0^t v(\tau)^{1-r'} d\tau \right)^{1/r'} < \infty, \tag{3.2}$$

where $r + r' = rr'$. Moreover, if C is the best constant in (3.1) *and K is defined by* (3.2)*, then*

$$K \leq C \leq k(r,s)K. \tag{3.3}$$

Here, the constant $k(r,s)$ in (3.3) *can be written in various forms. For example (see* [35]*):*

$$k(r,s) = r^{1/s}(r'^{1/r'} \quad or \quad k(r,s) = s^{1/s}(s'^{1/r'} \quad or \quad k(r,s) = \left(1 + s/r'\right)^{1/s} \left(1 + r'/s\right)^{1/r'}.$$

Lemma 3.2. *Let $1 \leq r \leq s \leq \infty$ and let v and w be two functions such that measurable and positive a.e. on $(0,\infty)$. Then there exists a constant C independent of the function h such that*

$$\left(\int_0^\infty \left(\int_t^\infty h(\tau) d\tau \right)^s w(t) dt \right)^{1/s} \leq C \left(\int_0^\infty h(t)^r v(t) dt \right)^{1/r} \tag{3.4}$$

if and only if

$$K_1 = \sup_{t>0} \left(\int_0^t w(\tau) d\tau \right)^{1/s} \left(\int_t^\infty v(\tau)^{1-r'} d\tau \right)^{1/r'} < \infty.$$

Moreover, the best constant C in (3.4) *satisfies the inequalities $K_1 \leq C \leq k(r,s)K_1$.*

Note that Lemmas 3.1 and 3.2 were proved by G. Talenti [41], G. Tomaselli [42], B. Muckenhoupt [33] for $1 \le r = s < \infty$, and by J. S. Bradley [5], V. M. Kokilashvili [28], V. G. Maz'ya [30] for $r < s$.

Theorem 3.3. *Let $1 < p < \infty$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$. Then*

$$\|Mf\|_{L_{p,\varphi_2}(B(x,t))} \le C\|\varphi_2\|_{L_p(B(x,t))} \int_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s} \tag{3.5}$$

for every $f \in L_{p,\varphi_1}(\mathbb{R}^n)$, where C does not depend on f, x and t.

Proof. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{B(x,2t)}(y), \quad f_2(y) = f(y)\chi_{\complement_{B(x,2t)}}(y), \quad t > 0, \tag{3.6}$$

and have

$$\|Mf\|_{L_{p,\varphi_2}(B(x,t))} \le \|Mf_1\|_{L_{p,\varphi_2}(B(x,t))} + \|Mf_2\|_{L_{p,\varphi_2}(B(x,t))}.$$

Taking into account that $f_1 \in L_{p,\varphi_1}(\mathbb{R}^n)$, by virtue of Theorem 2.4,

$$\begin{aligned}\|Mf_1\|_{L_{p,\varphi_2}(B(x,t))} &\le \|Mf_1\|_{L_{p,\varphi_2}(\mathbb{R}^n)} \\ &\le C\|f_2\|_{L_{p,\varphi_1}(\mathbb{R}^n)} = C\|f\|_{L_{p,\varphi_1}(B(x,2t))}.\end{aligned}$$

Then

$$\|Mf_1\|_{L_{p,\varphi_2}(B(x,2t))} \le C\|f\|_{L_{p,\varphi_1}(B(x,2t))},$$

where a constant C is independent of f.

Taking into account

$$\|f\|_{L_{p,\varphi_1}(B(x,2t))} \le C\|\varphi_2\|_{L_p(B(x,t))} \int_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s},$$

we get

$$\|Mf_1\|_{L_{p,\varphi_2}(B(x,t))} \le C\|\varphi_2\|_{L_p(B(x,t))} \int_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s}\,. \tag{3.7}$$

Now, observe that for $z \in B(x,t)$, we get

$$\begin{aligned}Mf_2(z) &= \sup_{r>0} |B(z,r)|^{-1} \int_{B(z,r)} |f_2(y)|dy \\ &\le C \sup_{r \ge 2t} \int_{\complement_{B(x,2t)} \cap B(z,r)} |y-z|^{-n}|f(y)|dy \\ &\le C \sup_{r \ge 2t} \int_{\complement_{B(x,2t)} \cap B(z,r)} |x-y|^{-n}|f(y)|dy \\ &\le C \int_{\complement_{B(x,2t)}} |x-y|^{-n}|f(y)|dy.\end{aligned}$$

We prove the following inequality:

$$\int_{\complement_{B(x,t)}} |x-y|^{-n}|f(y)|dy \le C \int_t^\infty s^{-n-1}\|\varphi_1^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\varphi_1}(B(x,s))}ds.$$

Therefore

$$\|Mf_2\|_{L_{p,\varphi_2}(B(x,t))} \le C\Big\| \int\limits_t^\infty s^{-n-1} \|\varphi_1^{-1}\|_{L_{p'}(B(x,s))} \|f\|_{L_{p,\varphi_1}(B(x,s))} ds \Big\|_{L_{p,\varphi_2}(B(x,t))}$$

$$\le C \int\limits_t^\infty s^{-n-1} \|\varphi_1^{-1}\|_{L_{p'}(B(x,s))} \|f\|_{L_{p,\varphi_1}(B(x,s))} ds \, \|\chi_{B(x,t)}\|_{L_{p,\varphi_2}(\mathbb{R}^n)}$$

$$\le C \|\varphi_2\|_{L_p(B(x,t))} \int\limits_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s} . \tag{3.8}$$

From (3.7) and (3.8), we get

$$\|Mf_2\|_{L_{p,\varphi_2}(B(x,t))} \le C \|\varphi_2\|_{L_p(B(x,t))} \int\limits_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s},$$

where a constant C is independent of f. □

In the following theorem we give the necessary condition for the two-weighted boundedness of maximal operator in the spaces $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$.

Theorem 3.4. *Let $1 < p < \infty$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $1 \le \theta_1 \le \theta_2 < \infty$ and the functions ω_1 and ω_2 satisfy conditions*

$$\sup_{t>0} \Big(\int\limits_0^t \omega_2(s) \|\varphi_2\|^{\theta_2}_{L_p(B(x,s))} ds \Big)^{\frac{\theta_1}{\theta_2}} \Big(\int\limits_t^\infty \omega_1(r)^{1-\theta_1'} r^{-\theta_1'} \|\varphi_2\|^{-\theta_1'}_{L_p(B(x,r))} dr \Big)^{\theta_1 - 1} < \infty, \tag{3.9}$$

then the operator M is bounded from $G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\mathbb{R}^n)$ to $G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}(\mathbb{R}^n)$.

Proof. Let $f \in G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\mathbb{R}^n)$. Hence by Theorem 3.3 and Lemma 3.2, we have

$$\|Mf\|_{G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}} = \sup_{x\in\mathbb{R}^n} \Big(\int\limits_0^\infty \omega_2(t) \|Mf\|^{\theta_2}_{L_{p,\varphi_2}(B(x,t))} dt \Big)^{\frac{1}{\theta_2}}$$

$$\le C \sup_{x\in\mathbb{R}^n} \Big(\int\limits_0^\infty \omega_2(t) \|\varphi_2\|^{\theta_2}_{L_p(B(x,t))} \Big(\int\limits_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s} \Big)^{\theta_2} dt \Big)^{\frac{1}{\theta_2}}$$

$$\le C \sup_{x\in\mathbb{R}^n} \Big(\int\limits_0^\infty \frac{\|f\|^{\theta_1}_{L_{p,\varphi_1}(B(x,t))}}{\|\varphi_2\|^{\theta_1}_{L_p(B(x,t))}} t^{-\theta_1} \omega_1(t) t^{\theta_1} \|\varphi_2\|^{\theta_1}_{L_p(B(x,t))} dt \Big)^{\frac{1}{\theta_1}}$$

$$= C \sup_{x\in\mathbb{R}^n} \Big(\int\limits_0^\infty \omega_1(t) \|f\|^{\theta_1}_{L_{p,\varphi_1}(B(x,t))} dt \Big)^{\frac{1}{\theta_1}} = C \|f\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}}. \qquad \square$$

Lemma 3.5 ([2])**.** *Let b be any non-negative locally integrable function. Then*

$$|[M,b]f(x)| \le M_b(f)(x), \ \ x \in \mathbb{R}^n,$$

holds for all $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$.

Theorem 3.6 ([2])**.** *Let $b \in BMO(\mathbb{R}^n)$. Suppose that X is a Banach space of measurable functions defined on $\mathbb{R}^n$. Assume that M is bounded on X. Then the operator M_b is bounded on X and the inequality*

$$\|M_b f\|_X \le C \|b\|_{BMO} \|f\|_X$$

holds with a constant C, independent of f.

Corollary 3.7. *Let $1 \le p < \infty$, $b \in BMO(\mathbb{R}^n)$ and $\varphi \in A_p(\mathbb{R}^n)$, then the operator M_b is bounded in $L_{p,\varphi}(\mathbb{R}^n)$.*

Theorem 3.8. *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1 \in A_p(\mathbb{R}^n)$, then the operator M_b is bounded from $L_{p,\varphi_1}(\mathbb{R}^n)$ to $L_{p,\varphi_2}(\mathbb{R}^n)$.*

Proof. Let $f \in L_{p,\varphi_1}(\mathbb{R}^n)$, $b \in BMO(\mathbb{R}^n)$. The inequality [2, Corollary 1.11],

$$M_b f(x) \leq C\|b\|_{BMO} M^2 f(x)$$

is valid. From this inequality, Theorem 2.4, Corollary 3.7 and the conditions $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1 \in A_p(\mathbb{R}^n)$, we have

$$\begin{aligned}\|M_b f\|_{L_{p,\varphi_2}(B(x,r))} &\leq C\|b\|_{BMO} \left\|M^2 f\right\|_{L_{p,\varphi_2}(B(x,r))}\\ &\leq C\|b\|_{BMO} \left\|M f\right\|_{L_{p,\varphi_1}(B(x,r))} \leq C_1\|b\|_{BMO} \left\|f\right\|_{L_{p,\varphi_1}(B(x,r))},\end{aligned}$$

where $M^2 f(x) = M(Mf(x))$. □

Theorem 3.9. *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1, \varphi_2 \in A_p(\mathbb{R}^n)$, then*

$$\|M_b f\|_{L_{p,\varphi_2}(B(x,t))} \leq C\|b\|_{BMO}\|\varphi_2\|_{L_p(B(x,t))} \int_t^\infty \left(1 + \ln\frac{s}{t}\right) \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s}, \tag{3.10}$$

for every $f \in L_{p,\varphi_1}(\mathbb{R}^n)$, where C does not depend on f, x and t.

Proof. We represent the function f as in (3.6) and have

$$\|M_b f\|_{L_{p,\varphi_2}(B(x,t))} \leq \|M_b f_1\|_{L_{p,\varphi_2}(B(x,t))} + \|M_b f_2\|_{L_{p,\varphi_2}(B(x,t))}.$$

By Theorem 3.8, we obtain

$$\begin{aligned}\|M_b f_1\|_{L_{p,\varphi_2}(B(x,t))} &\leq \|M_b f_1\|_{L_{p,\varphi_2}(\mathbb{R}^n)}\\ &\leq C\|b\|_{BMO}\|f_1\|_{L_{p,\varphi_1}(\mathbb{R}^n)} = C\|b\|_{BMO}\|f\|_{L_{p,\varphi_1}(B(x,2t))},\end{aligned} \tag{3.11}$$

where C does not depend on f. From (3.11), we get

$$\|M_b f_1\|_{L_{p,\varphi_2}(B(x,t))} \leq C\|b\|_{BMO}\|\varphi_2\|_{L_p(B(x,t))} \int_t^\infty \left(1 + \ln\frac{r}{t}\right) \frac{\|f\|_{L_{p,\varphi_1}(B(x,r))}}{\|\varphi_2\|_{L_p(B(x,r))}} \frac{dr}{r}, \tag{3.12}$$

which is easily obtained from the fact that $\|f\|_{L_{p,\varphi_1}(B(x,2t))}$ is non-decreasing in t, therefore $\|f\|_{L_{p,\varphi_1}(B(x,2t))}$ on the right-hand side of (3.11) is dominated by the right-hand side of (3.12).

For $z \in B(x,t)$, we get

$$\begin{aligned}M_b f_2(z) &= \sup_{r>0} |B(z,r)|^{-1} \int_{B(z,r)} |b(z) - b(y)||f_2(y)|dy\\ &\leq C \sup_{r \geq 2t} \int_{\complement B(x,2t) \cap B(z,r)} |y - z|^{-n}|b(z) - b(y)||f(y)|dy\\ &\leq C \sup_{r \geq 2t} \int_{\complement B(x,2t) \cap B(z,r)} |x - y|^{-n}|b(z) - b(y)||f(y)|dy\\ &\leq C \int_t^\infty s^{-n-1}\left(\int_{\{y \in \mathbb{R}^n : 2t \leq |x-y| \leq s\}} |b(y) - b_{B(x,s)}||f(y)|dy\right)ds\\ &+ C \int_t^\infty s^{-n-1}\left(\int_{\{y \in \mathbb{R}^n : 2t \leq |x-y| \leq s\}} |b(z) - b_{B(x,s)}||f(y)|dy\right)ds = I_1 + I_2.\end{aligned}$$

From Hölder's inequality and Theorem 2.12, we obtain

$$I_1 = \int_t^\infty s^{-n-1}\Big(\int_{\{y\in\mathbb{R}^n:2t\le|x-y|\le s\}} |b(y)-b_{B(x,s)}||f(y)|dy\Big)ds$$
$$\le C\|b\|_{BMO}\int_t^\infty s^{-n-1}\|f\|_{L_{p,\varphi_1}(B(x,s))}\|\varphi_1^{-1}\|_{L_{p'}(B(x,s))}ds.$$

To estimate I_2, by Lemma 2.13, we get

$$I_2 = \int_t^\infty s^{-n-1}|b(z)-b_{B(x,s)}|\Big(\int_{\{y\in\mathbb{R}^n:2t\le|x-y|\le s\}} |f(y)|dy\Big)ds$$
$$\le CM_b\chi_{B(x,t)}(z)\int_t^\infty s^{-n-1}\|f\|_{L_{p,\varphi_1}(B(x,s))}\|\varphi_1^{-1}\|_{L_{p'}(B(x,s))}ds$$
$$+C\|b\|_{BMO}\int_t^\infty s^{-n-1}\ln\frac{s}{t}\|f\|_{L_{p,\varphi_1}(B(x,s))}\|\varphi_1^{-1}\|_{L_{p'}(B(x,s))}ds.$$

By Theorem 3.8, we have

$$\|M_bf_2\|_{L_{p,\varphi_2}(B(x,t))} \le \|I_1\|_{L_{p,\varphi_2}(B(x,t))} + \|I_2\|_{L_{p,\varphi_2}(B(x,t))}$$
$$\le C\|b\|_{BMO}\|\varphi_2\|_{L_p(B(x,t))}\int_t^\infty \Big(1+\ln\frac{s}{t}\Big)\frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}}\frac{ds}{s}. \qquad (3.13)$$

Then from (3.12) and (3.13), we obtain (3.10). ☐

In the following theorem we give the necessary condition for the two-weighted boundedness of maximal commutator in the spaces $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$.

Theorem 3.10. *Let $1<p<\infty$, $b\in BMO(\mathbb{R}^n)$, $(\varphi_1,\varphi_2)\in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1,\varphi_2\in A_p(\mathbb{R}^n)$, $1\le\theta_1\le\theta_2<\infty$ and the functions ω_1 and ω_2 satisfy the conditions*

$$\sup_{t>0}\Big(\int_0^t \omega_2(s)\|\varphi_2\|^{\theta_2}_{L_p(B(x,s))}ds\Big)^{\frac{\theta_1}{\theta_2}}$$
$$\times\Big(\int_t^\infty \Big(1+\ln\frac{r}{t}\Big)^{\theta_1'}\omega_1(r)^{1-\theta_1'}r^{-\theta_1'}\|\varphi_2\|^{-\theta_1'}_{L_p(B(x,r))}dr\Big)^{\theta_1-1}<\infty, \qquad (3.14)$$

then the operator M_b is bounded from $G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\mathbb{R}^n)$ to $G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}(\mathbb{R}^n)$.

Proof. Let $f\in G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\mathbb{R}^n)$, $b\in BMO(\mathbb{R}^n)$. Hence by Theorem 3.9 and Lemma 3.2, we obtain

$$\|M_bf\|_{G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}} = \sup_{x\in\mathbb{R}^n}\Big(\int_0^\infty \omega_2(t)\|M_bf\|^{\theta_2}_{L_{p,\varphi_2}(B(x,t))}dt\Big)^{\frac{1}{\theta_2}}$$
$$\le C|b\|_{BMO}\sup_{x\in\mathbb{R}^n}\Big(\int_0^\infty \omega_2(t)\|\varphi_2\|^{\theta_2}_{L_p(B(x,t))}\Big(\int_t^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}}\frac{ds}{s}\Big)^{\theta_2}dt\Big)^{\frac{1}{\theta_2}}$$
$$\le C\ \|b\|_{BMO}\sup_{x\in\mathbb{R}^n}\Big(\int_0^\infty\Big(1+\ln\frac{s}{t}\Big)^{\theta_1}\frac{\|f\|^{\theta_1}_{L_{p,\varphi_1}(B(x,t))}}{\|\varphi_2\|^{\theta_1}_{L_p(B(x,t))}}t^{-\theta_1}\Big(1+\ln\frac{s}{t}\Big)^{-\theta_1}\omega_1(t)t^{\theta_1}\|\varphi_2\|^{\theta_1}_{L_p(B(x,t))}dt\Big)^{\frac{1}{\theta_1}}$$

$$= C\ \|b\|_{BMO} \sup_{x\in\mathbb{R}^n} \left(\int\limits_0^\infty \omega_1(t) \|f\|^{\theta_1}_{L_{p,\varphi_1}(B(x,t))} dt \right)^{\frac{1}{\theta_1}} = C\ \|b\|_{BMO} \|f\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}},$$

which completes the proof. □

4. Two-weighted Inequality for the Singular Operators and their Commutators in the Spaces $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$

Let T be a Calderón–Zygmund singular integral operator and $b \in BMO(\mathbb{R}^n)$. A well known result of Coifman, Rochberg and Weiss [13] states that the commutator operator $[b,T]f = T(bf) - bTf$ is bounded on $L_p(\mathbb{R}^n)$ for $1 < p < \infty$. The commutator of Calderón–Zygmund operators plays an important role in studying the regularity of solutions of elliptic partial differential equations of second order (see, e.g., [10, 12, 16–18]).

In this section, we prove the two-weighted inequalities for singular integral operators and their commutators in the $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$ weighted global Morrey-type spaces. We start with the following lemma.

Lemma 4.1 ([17])**.** *Let $1 < s < \infty$, $b \in BMO(\mathbb{R}^n)$, then there exists $C > 0$ such that for all $x \in \mathbb{R}^n$, the following inequality:*

$$|[b,T]f|(x) \le M(|[b,T]f|(x)) \le C\|b\|_{BMO} \left((M|Tf|^s)^{\frac{1}{s}}(x) + (M|f|^s)^{\frac{1}{s}}(x) \right)$$

holds.

Theorem 4.2. *Let $1 < p < \infty$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$. Then*

$$\|Tf\|_{L_{p,\varphi_2}(B(x,t))} \le C\|\varphi_2\|_{L_p(B(x,t))} \int\limits_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s}$$

for every $f \in L_{p,\omega_1}(\mathbb{R}^n)$, where C does not depend on f, x and t.

Proof. We represent the function f as in (3.6) and have

$$\|Tf\|_{L_{p,\varphi_2}(B(x,t))} \le \|Tf_1\|_{L_{p,\varphi_2}(B(x,t))} + \|Tf_2\|_{L_{p,\varphi_2}(B(x,t))}.$$

From Theorem 2.5, we obtain

$$\|Tf_1\|_{L_{p,\varphi_2}(B(x,t))} \le \|Tf_1\|_{L_{p,\varphi_2}(\mathbb{R}^n)} \le C\|f_1\|_{L_{p,\varphi_1}(\mathbb{R}^n)} = C\|f\|_{L_{p,\varphi_1}(B(x,2t))}, \tag{4.1}$$

where C does not depend on f. From (4.1), we get

$$\|Tf_1\|_{L_{p,\varphi_2}(B(x,t))} \le C\|\varphi_2\|_{L_p(B(x,t))} \int\limits_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s}, \tag{4.2}$$

which is easily obtained from the fact that $\|f\|_{L_{p,\varphi_1}(B(x,2t))}$ is non-decreasing in t, therefore $\|f\|_{L_{p,\varphi_1}(B(x,2t))}$ on the right-hand side of (4.1) is dominated by the right-hand side of (4.2). To estimate $\|Tf_2\|_{L_{p,\varphi_2}(B(x,t))}$, we observe that

$$|Tf_2(z)| \le C \int\limits_{\complement B(x,2t)} \frac{|f(y)|\,dy}{|y-z|^n},$$

where $z \in B(x,t)$ and the inequalities $|x-z| \le t$, $|z-y| \ge 2t$ imply $\frac{1}{2}|z-y| \le |x-y| \le \frac{3}{2}|z-y|$, finally, we get

$$|Tf_2(z)| \le C \int\limits_{\complement B(x,2t)} |x-y|^{-n}|f(y)|dy.$$

To estimate $Tf_2(z)$, for $z \in B(x,t)$, choosing $\delta > 0$ from Theorem 2.12, we have

$$\int\limits_{\complement_{B(x,t)}} |x-y|^{-n}|f(y)|dy$$
$$\leq C\int\limits_t^\infty s^{-n-1} \int\limits_{\{y\in\mathbb{R}^n:2t\leq|x-y|\leq s\}} |f(y)|dyds$$
$$\leq C\int\limits_t^\infty s^{-n-1}\|\varphi_1^{-1}\chi_{B(x,s)}\|_{L_{p'}(\mathbb{R}^n)}\|f\|_{L_{p,\varphi_1}(B(x,s))}ds.$$

We prove the following inequality:

$$\int\limits_{\complement_{B(x,t)}} |x-y|^{-n}|f(y)|dy \leq C\int\limits_t^\infty s^{-n-1}\|\varphi_1^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\varphi_1}(B(x,s))}ds. \tag{4.3}$$

Hence by inequality (4.3), we get

$$\|Tf_2\|_{L_{p,\varphi_2}(B(x,t))} \leq C\|\chi_{B(x,t)}\|_{L_{p,\varphi_2}(\mathbb{R}^n)}\int\limits_t^\infty s^{-n-1}\|\varphi_1^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\varphi_1}(B(x,s))}ds$$
$$\leq C\|\varphi_2\|_{L_p(B(x,t))}\int\limits_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}}\,\frac{ds}{s}. \tag{4.4}$$

From (4.2) and (4.4), we arrive at (3.5). □

In the following theorem, we give the necessary condition for the two-weighted boundedness of singular integral operators in the spaces $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$.

Theorem 4.3. *Let $1 < p < \infty$ and $(\varphi_1,\varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $1 \leq \theta_1 \leq \theta_2 < \infty$ and the functions ω_1 and ω_2 satisfy condition* (3.9).

Then the operator T is bounded from $G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\mathbb{R}^n)$ to $G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}(\mathbb{R}^n)$.

Proof. Let $f \in G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\mathbb{R}^n)$. From Theorem 4.2 and Lemma 3.2, we get

$$\|Tf\|_{G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}}$$
$$\leq C\sup_{x\in\mathbb{R}^n}\left(\int\limits_0^\infty \omega_2(t)\|\varphi_2\|^{\theta_2}_{L_p(B(x,t))}\left(\int\limits_t^\infty \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}}\,\frac{ds}{s}\right)^{\theta_2}dt\right)^{\frac{1}{\theta_2}}$$
$$\leq C\sup_{x\in\mathbb{R}^n}\left(\int\limits_0^\infty \frac{\|f\|^{\theta_1}_{L_{p,\varphi_1}(B(x,t))}}{\|\varphi_2\|^{\theta_1}_{L_p(B(x,t))}}\,t^{-\theta_1}\omega_1(t)t^{\theta_1}\|\varphi_2\|^{\theta_1}_{L_p(B(x,t))}dt\right)^{\frac{1}{\theta_1}}$$
$$= C\sup_{x\in\mathbb{R}^n}\left(\int\limits_0^\infty \omega_1(t)\|f\|^{\theta_1}_{L_{p,\varphi_1}(B(x,t))}dt\right)^{\frac{1}{\theta_1}} = C\|f\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}}.$$ □

The following theorem gives the two-weighted boundedness of the operator $[b,T]$ in the $L_{p,\varphi}(\mathbb{R}^n)$ spaces.

Theorem 4.4. *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and $(\varphi_1,\varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1 \in A_p(\mathbb{R}^n)$. Then the operator $[b,T]$ is bounded from $L_{p,\varphi_1}(\mathbb{R}^n)$ to $L_{p,\varphi_2}(\mathbb{R}^n)$.*

Proof. Let $f \in L_{p,\varphi}(\mathbb{R}^n)$, $b \in BMO(\mathbb{R}^n)$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1 \in A_p(\mathbb{R}^n)$. From Lemma 4.1, Theorem 2.3, Theorem 2.4 and Corollary 2.6, we have

$$\begin{aligned} \|[b,T]f\|_{L_{p,\varphi_2}(\mathbb{R}^n)} &\le \|M([b,T]f)\|_{L_{p,\varphi_2}(\mathbb{R}^n)} \le C\|b\|_{BMO} \left\| (M|Tf|^s)^{\frac{1}{s}} + (M|f|^s)^{\frac{1}{s}} \right\|_{L_{p,\varphi_2}(\mathbb{R}^n)} \\ &\le C\|b\|_{BMO} \left[\left\| (M|Tf|^s)^{\frac{1}{s}} \right\|_{L_{p,\varphi_2}(\mathbb{R}^n)} + \left\| (M|f|^s)^{\frac{1}{s}} \right\|_{L_{p,\varphi_2}(\mathbb{R}^n)} \right] \\ &\le C\|b\|_{BMO} \left[\left\| (|Tf|^s)^{\frac{1}{s}} \right\|_{L_{p,\varphi_1}(\mathbb{R}^n)} + \left\| (|f|^s)^{\frac{1}{s}} \right\|_{L_{p,\varphi_1}(\mathbb{R}^n)} \right] \le C\|b\|_{BMO} \|f\|_{L_{p,\varphi_1}(\mathbb{R}^n)}. \end{aligned}$$ □

We can easily get the following

Theorem 4.5. *Let $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$ and $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1, \varphi_2 \in A_p(\mathbb{R}^n)$. Then*

$$\|[b,T]f\|_{L_{p,\varphi_2}(B(x,t))} \le C\|b\|_{BMO}\|\varphi_2\|_{L_p(B(x,t))} \int_t^\infty \left(1 + \ln \frac{r}{t}\right) \frac{\|f\|_{L_{p,\varphi_1}(B(x,r))}}{\|\varphi_2\|_{L_p(B(x,r))}} \frac{dr}{r} \tag{4.5}$$

for every $f \in L_{p,\varphi_1}(\mathbb{R}^n)$, where C does not depend on f, x and t.

Proof. We represent the function f as in (3.6) and have

$$\|[b,T]f\|_{L_{p,\varphi_2}(B(x,t))} \le \|[b,T]f_1\|_{L_{p,\varphi_2}(B(x,t))} + \|[b,T]f_2\|_{L_{p,\varphi_2}(B(x,t))}.$$

By Theorem 4.4, we obtain

$$\begin{aligned} &\|[b,T]f_1\|_{L_{p,\varphi_2}(B(x,t))} \le \|[b,T]f_1\|_{L_{p,\varphi_2}(\mathbb{R}^n)} \\ &\le C\|b\|_{BMO}\|f_1\|_{L_{p,\varphi_1}(\mathbb{R}^n)} = C\|b\|_{BMO}\|f\|_{L_{p,\varphi_1}(B(x,2t))}, \end{aligned} \tag{4.6}$$

where C does not depend on f. From (4.6), we obtain

$$\|[b,T]f_1\|_{L_{p,\varphi_2}(B(x,t))} \le C\|b\|_{BMO}\|\varphi_2\|_{L_p(B(x,t))} \int_t^\infty \left(1 + \ln \frac{s}{t}\right) \frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}} \frac{ds}{s} \tag{4.7}$$

which is easily obtained from the fact that $\|f\|_{L_{p,\omega_1}(B(x,2t))}$ is non-decreasing in t, therefore $\|f\|_{L_{p,\omega_1}(B(x,2t))}$ on the right-hand side of (4.6) is dominated by the right-hand side of (4.7). To estimate $\|[b,T]f_2\|_{L_{p,\omega_2}(B(x,t))}$, we observe that

$$|[b,T]f_2(z)| \le C \int_{\mathbb{R}^n \setminus B(x,2t)} |b(z) - b(y)| \frac{|f(y)|}{|y-z|^n} \, dy,$$

where $z \in B(x,t)$ and the inequalities $|x - z| \le t$, $|z - y| \ge 2t$ imply $\frac{1}{2}|z - y| \le |x - y| \le \frac{3}{2}|z - y|$, and therefore

$$|[b,T]f_2(z)| \le C \int_{\mathbb{R}^n \setminus B(x,2t)} |x-y|^{-n} |b(z) - b(y)| \, |f(y)| dy.$$

To estimate $[b,T]f_2$, we first prove the following auxiliary inequality:

$$\begin{aligned} &\int_{\mathbb{R}^n \setminus B(x,t)} |x-y|^{-n} |b(z) - b(y)||f(y)| dy \\ &\le C\|b\|_{BMO} \int_t^\infty s^{-n} \left(1 + \ln \frac{s}{t}\right) \|\omega_1^{-1}\|_{L_{p'}(B(x,s))} \|f\|_{L_{p,\omega_1}(B(x,s))} \frac{ds}{s}. \end{aligned} \tag{4.8}$$

To estimate $[b,T]f_2(z)$, we observe that for $z \in B(x,t)$, we have

$$\int\limits_{\mathbb{R}^n\setminus B(x,t)} |x-y|^{-n}|b(z)-b(y)||f(y)|dy$$

$$\le \int\limits_{\mathbb{R}^n\setminus B(x,t)} |x-y|^{-n}|b(y)-b_{B(x,t)}||f(y)|dy$$

$$+ \int\limits_{\mathbb{R}^n\setminus B(x,t)} |x-y|^{-n}|b(z)-b_{B(x,t)}||f(y)|dy = J_1 + J_2.$$

Now, we choose $\delta > 0$ and by Theorem 2.12 and Lemma 2.13, we obtain

$$J_1 = \int\limits_{\mathbb{R}^n\setminus B(x,t)} |x-y|^{-n}|b(y)-b_{B(x,t)}||f(y)|dy$$

$$\le C\|b\|_{BMO}\int\limits_t^\infty s^{-n-1}\|\omega_1^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}ds$$

$$+ C\|b\|_{BMO}\int\limits_t^\infty s^{-n-1}\ln\frac{s}{t}\|\omega_1^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}ds.$$

To estimate J_2, we have

$$J_2 = |b(z)-b_{B(x,t)}| \int\limits_{\mathbb{R}^n\setminus B(x,t)} |x-y|^{-n}|f(y)|dy$$

$$\le CM_b\chi_{B(x,t)}(z)\int\limits_t^\infty s^{-n}\|\omega_1^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}\frac{ds}{s},$$

where C does not depend on x, t.

Hence by inequality (4.8), we get

$$\|[b,T]f_2\|_{L_{p,\omega_2}(B(x,t))} \le C\|\chi_{B(x,t)}\|_{L_{p,\omega_2}(\mathbb{R}^n)}\|b\|_{BMO}$$

$$\times \int\limits_t^\infty s^{-n}\left(1+\ln\frac{s}{t}\right)\|\omega_1^{-1}\|_{L_{p'}(B(x,s))}\|f\|_{L_{p,\omega_1}(B(x,s))}\frac{ds}{s}$$

$$\le C\|b\|_{BMO}\|\omega_2\|_{L_p(B(x,t))}\int\limits_t^\infty \left(1+\ln\frac{s}{t}\right)\frac{\|f\|_{L_{p,\omega_1}(B(x,s))}}{\|\omega_2\|_{L_p(B(x,s))}}\frac{ds}{s}\,. \tag{4.9}$$

From (4.7) and (4.9), we arrive at (4.5). □

In the following theorem we prove the two-weighted boundedness of commutators of singular integral operators in the spaces $G\mathcal{M}_{p,\theta,\omega,\varphi}(\mathbb{R}^n)$.

Theorem 4.6. *Let* $1 < p < \infty$, $b \in BMO(\mathbb{R}^n)$, $(\varphi_1,\varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1,\varphi_2 \in A_p(\mathbb{R}^n)$, $1 \le \theta_1 \le \theta_2 < \infty$ *and the functions* ω_1 *and* ω_2 *satisfy condition* (3.14).

Then the operator $[b,T]$ *is bounded from* $G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\mathbb{R}^n)$ *to* $G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}(\mathbb{R}^n)$.

Proof. Let $f \in G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\mathbb{R}^n)$, $b \in BMO(\mathbb{R}^n)$. Hence by Theorem 4.5 and Lemma 3.2, we obtain

$$\|[b,T]f\|_{G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}} = \sup_{x\in\mathbb{R}^n}\left(\int\limits_0^\infty \omega_2(t)\|[b,T]f\|^{\theta_2}_{L_{p,\varphi_2}(B(x,t))}dt\right)^{\frac{1}{\theta_2}}$$

$$\leq C\|b\|_{BMO}\sup_{x\in\mathbb{R}^n}\left(\int\limits_0^\infty \omega_2(t)\|\varphi_2\|^{\theta_2}_{L_p(B(x,t))}\left(\int\limits_t^\infty\Big(1+\ln\frac{s}{t}\Big)\frac{\|f\|_{L_{p,\varphi_1}(B(x,s))}}{\|\varphi_2\|_{L_p(B(x,s))}}\frac{ds}{s}\right)^{\theta_2}dt\right)^{\frac{1}{\theta_2}}$$

$$\leq C\sup_{x\in\mathbb{R}^n}\left(\int\limits_0^\infty\Big(1+\ln\frac{s}{t}\Big)^{\theta_1}\frac{\|f\|^{\theta_1}_{L_{p,\varphi_1}(B(x,t))}}{\|\varphi_2\|^{\theta_1}_{L_p(B(x,t))}}t^{-\theta_1}\Big(1+\ln\frac{s}{t}\Big)^{-\theta_1}\omega_1(t)t^{\theta_1}\|\varphi_2\|^{\theta_1}_{L_p(B(x,t))}dt\right)^{\frac{1}{\theta_1}}$$

$$= C\sup_{x\in\mathbb{R}^n}\left(\int\limits_0^\infty\omega_1(t)\|f\|^{\theta_1}_{L_{p,\varphi_1}(B(x,t))}dt\right)^{\frac{1}{\theta_1}} = C\|f\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}},$$

which completes the proof. □

5. Weighted Global Morrey-type a Priori Estimates

In this section, we consider the Dirichlet problem (2.2) in the bounded domains Ω. We assume that $\partial\Omega$ is of the class C^2.

$$\phi(x)=\int\limits_\Omega G(x,y)f(y)dy \tag{5.1}$$

is the solution of this problem, where $G(x,y)$ is the Green function that can be written as

$$G(x,y)=\Gamma(x-y)+h(x,y)$$

with $h(x,y)$ satisfying, for each fixed $y\in\Omega$,

$$\begin{cases}\triangle_x h(x,y)=0 & x\in\Omega,\\ h(x,y)=-\Gamma(x-y) & x\in\partial\Omega.\end{cases}$$

If $P(y,Q)$ is the Poisson kernel, then $h(x,y)$ is given by

$$h(x,y)=-\frac{1}{(n-2)\omega_n}\int\limits_{\partial\Omega}\frac{1}{|x-Q|^{n-2}}P(y,Q)dS(Q),$$

where dS denotes the surface measure on $\partial\Omega$.

The inequalities

$$G(x,y)\leq\begin{cases}C\log|x-y| & \text{if } n=2,\\ C|x-y|^{2-n} & \text{if } n\geq 3,\end{cases}$$

and

$$|D_{x_i}G(x,y)|\leq C|x-y|^{1-n}$$

are satisfied by the Green function (see [43]). Thus

$$D_{x_i}\phi(x)=\int\limits_\Omega D_{x_i}G(x,y)f(y)dy.$$

We need the following lemma to get the second derivatives of ϕ from the representation (5.1). We denote by $d(x)$ the distance to the boundary, $d(x)=\inf\limits_{Q\in\partial\Omega}|x-Q|$.

Lemma 5.1. *Given $\alpha\in\mathbb{Z}^n_+$ ($|\alpha|>0$ if $n=2$), there exists a constant C depending only on n and α such that*

$$|D^\alpha h(x,y)|\leq Cd(x)^{2-n-|\alpha|}.$$

We find that for each $x\in\Omega$, $D_{x_ix_j}h(x,y)$ is bounded uniformly in a neighborhood of x and therefore

$$D_{x_ix_j}\int\limits_\Omega h(x,y)f(y)dy=\int\limits_\Omega D_{x_ix_j}h(x,y)f(y)dy.$$

Moreover, since $|D_{x_j}\Gamma(x)| \leq C|x|^{1-n}$, we obtain

$$D_{x_j}\int\limits_{\Omega}\Gamma(x-y)f(y)dy = \int\limits_{\Omega} D_{x_j}\Gamma(x-y)f(y)dy.$$

$D_{x_ix_j}\Gamma$ is not an integrable function, so we cannot interchange the order between integration and second derivatives. The known standard argument shows that

$$D_{x_i}\int\limits_{\Omega} D_{x_j}\Gamma(x-y)f(y)dy = Kf(x) + c(x)f(x),$$

where c is a bounded function and

$$Kf(x) = \lim_{\varepsilon\to 0}\int\limits_{|x-y|>\varepsilon} D_{x_ix_j}\Gamma(x-y)f(y)dy.$$

Here and in the sequel, we assume hat f is defined in $\mathbb{R}^n$ extending the original f by zero.

Since $D_{x_j}\Gamma \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and is a homogeneous function of degree $1-n$, the operator K is a Calderón–Zygmund singular integral operator and $D_{x_ix_j}\Gamma(x-y)$ is homogeneous of degree $-n$ and has vanishing average on the unit sphere (see Lemma 11.1 in [3, page 152]). It follows from the general theory given in [9] that K is a bounded operator in $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

Furthermore, the maximal operator

$$\widetilde{K}f(x) = \sup_{\varepsilon>0}\left|\int\limits_{|x-y|>\varepsilon} D_{x_ix_j}\Gamma(x-y)f(y)dy\right|$$

is also bounded in $L_p(\mathbb{R}^n)$ for $1 < p < \infty$.

We can now give and prove our main result.

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain and $1 < p < \infty$, $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n), 1 \leq \theta_1 \leq \theta_2 < \infty$. Let the functions $\omega_1(r)$ and $\omega_2(r)$ satisfy condition (3.9), $f \in G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\Omega)$ and ϕ be the solution of problem (2.2), then there exists a constant C depending only on n and Ω such that*

$$\|\phi\|_{W^2_{p,\omega_2,\theta_2}(\Omega,\varphi_2)} \leq C\|f\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\Omega)}. \tag{5.2}$$

Proof. We will need the following estimate for the Green function. This estimate has been proved by A. Dall'Acqua and G. Sweers in [14], however, they assume that the domain is more regular than C^2.

Let Ω be a bounded C^2 domain and $G(x,y)$ be the Green function of problem (2.2) in Ω. There exists a constant C depending only on n and Ω such that for $(x,y) \in \Omega\times\Omega$,

$$|D_{x_ix_j}G(x,y)| \leq C\frac{d(x)}{|x-y|^{n+1}}.$$

Our result follows from the following inequalities (see [15]).

There exists a constant C depending only on n and Ω such that for any $x \in \Omega$,

$$|\phi(x)| + |D_{x_i}\phi(x)| \leq CMf(x), \tag{5.3}$$

$$|D_{x_ix_j}\phi(x)| \leq C\left(\widetilde{K}f(x) + Mf(x) + |f(x)|\right). \tag{5.4}$$

Theorems 3.4 and 4.3 imply that the operators M and $\widetilde{K}$ are bounded from $G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\Omega)$ to $G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}(\Omega)$. Therefore (5.2) follows immediately from inequalities (5.3) and (5.4). □

Now, we get the following

Theorem 5.3. *Let $\Omega \subset \mathbb{R}^n$ be a bounded C^2 domain and $1 < p < \infty$, $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n), 1 \leq \theta_1 \leq \theta_2 < \infty$. Let the functions $\omega_1(r)$ and $\omega_2(r)$ satisfy condition (3.9), $f \in G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\Omega)$ and ϕ be the solution of problem (2.2), then there exists a constant C depending only on n and Ω such that*

$$\|\phi\|_{W^2_{p,\omega_2,\theta_2}(\Omega,\varphi_2)} \leq C\|f\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\Omega)}.$$

6. A Priori Estimates for Non-divergent Elliptic Equations in Weighted Global Morrey-type Spaces

Definition 6.1. Suppose that Ω is an open set in $\mathbb{R}^n$. For any $f \in BMO(\Omega)$ and $r > 0$, we define

$$\eta(r) = \sup_{x\in\Omega,\rho\le r} \frac{1}{|\Omega(x,\rho)|} \int\limits_{\Omega(x,\rho)} |a(y) - a_{\Omega(x,\rho)}| dy < \infty,$$

where $a_\Omega = 1/|\Omega| \int_\Omega a(y)dy$. If $\eta(r) \to 0$ for $r \to 0^+$, we say that any $a \in BMO(\Omega)$ is from the space $VMO(\Omega)$.

Let $1 < p < \infty$, $f \in G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\Omega)$ and the functions φ satisfy the condition

$$\sup_{r>0} \left(\int\limits_0^r \omega_2(s) \|\varphi_2\|^{\theta_2}_{L_p(B(x,s))} ds \right)^{\frac{\theta_1}{\theta_2}}$$
$$\times \left(\int\limits_r^d \left(1 + \ln\frac{t}{r}\right)^{\theta_1'} \omega_1(t)^{1-\theta_1'} t^{-\theta_1'} \|\varphi_2\|^{-\theta_1'}_{L_p(B(x,t))} dt \right)^{\theta_1 - 1} < \infty, \tag{6.1}$$

where C is independent of x and r.

We get the following result from Theorem 4.6.

Corollary 6.2. *Let $1 < p < \infty$, $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1, \varphi_2 \in A_p(\mathbb{R}^n)$, $1 \le \theta_1 \le \theta_2 < \infty$ and the functions ω_1 and ω_2 satisfy condition (6.1). Suppose Ω is an open set in $\mathbb{R}^n$ and $a \in VMO(\Omega)$. If the kernel K is a constant or variable Calderón–Zygmund kernel on $\mathbb{R}^n$ and T is the corresponding Calderón–Zygmund determinant, then for any $\varepsilon > 0$, there exists a positive number $\rho_0 = \rho_0(\varepsilon, \eta)$ such that, for any ball $B(0,r)$ with radius $r \in (0, \rho_0)$, $\Omega(0,r) \ne \emptyset$ for all $f \in G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\Omega(0,r))$,*

$$\|[a,T]f\|_{G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}(\Omega(0,r))} \le C\varepsilon \|f\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\Omega(0,r))},$$

where $C = C(n,p,\varphi,K,M)$ is independent of ε, f and r.

Suppose that Ω is a bounded domain in $\mathbb{R}^n$, $n \ge 3$ and $\partial\Omega \in C^{1,1}$ and the coefficients $a_{ij}(x)$, $i,j = 1,\dots,n$, are symmetric and uniformly elliptic in Ω, that is, for some $\Lambda > 0$ and any $\xi \in \mathbb{R}^n$,

$$a_{ij}(x) = a_{ji}(x), \Lambda^{-1}|\xi|^2 \le \sum_{i,j=1}^n a_{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2$$

a.e. $x \in \Omega$. Furthermore, let $a_{ij}(x) \in VMO(\Omega)$, the spaces of mean oscillation functions vanish to zero according to D. Sarason [38].

Take into account the Dirichlet problem

$$\begin{cases} Lu = f & \text{almost everywhere in } \ \Omega, \\ u = 0 & \text{on the } \ \partial\Omega. \end{cases} \tag{6.2}$$

Let

$$\Gamma(x,t) = \frac{1}{(n-2)\omega_n (\det a_{i,j})^{1/2}} \left(\sum_{i,j=1}^n A_{ij}(x) t_i t_j \right)^{(2-n)/2},$$
$$\Gamma_i(x,t) = \frac{\partial}{\partial t_i} \Gamma(x,t),$$
$$\Gamma_{ij}(x,t) = \frac{\partial^2}{\partial t_i \partial t_j} \Gamma(x,t),$$

for a.e. $x \in B$ and $\forall t \in \mathbb{R}^n \setminus \{0\}$, where $(A_{ij})_{n\times n}$ is the inverse of the matrix $(a_{ij})_{n\times n}$. We have for $u \in W_0^{2,p}$ (see [11, 18, 21]) the following formulas:

$$u_{x_ix_j}(x) = P \cdot V \cdot \int_B \Gamma_{ij}(x, x-y)\Big[\sum_{k,l=1}(a_{kl}(x) - a_{kl}(y))u_{x_kx_l}(y) + Lu(y)\Big]dy$$
$$+ Lu(x) \int_{|y|=1} \Gamma_i(x,y) y_j d\delta_y,$$

a.e. $x \in B \subset \Omega$, where B is a ball in Ω.

Theorem 6.3. *Let Ω be a bounded domain in $\mathbb{R}^n$, $1 < p < \infty$, $(\varphi_1, \varphi_2) \in \widetilde{A}_p(\mathbb{R}^n)$, $\varphi_1, \varphi_2 \in A_p(\mathbb{R}^n)$, $1 \le \theta_1 \le \theta_2 < \infty$ and the functions ω_1 and ω_2 satisfy condition* (6.1). *Suppose that $a_{ij} \in VMO(\Omega)$ for $i, j = 1, 2, \dots, n$,*

$$M \equiv \max_{i,j=1,\dots,n} \max_{|\beta|\le 2n} \Big\| \frac{\partial^\beta}{\partial t^\beta} \Gamma_{ij}(x, T) \Big\|_{L_\infty} < \infty,$$

$f \in G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(\Omega)$ and u is the solution of problem (6.2), *then there exists a positive constant C such that for any ball $B \subset \Omega$:*

$$\|u_{x_ix_j}\|_{G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}(B)} \le C\|Lu\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(B)}.$$

Proof. One can easily check that Γ_{ij} is a variable Calderón–Zygmund kernel. From Corollary 6.2 and representation $u_{x_ix_j}$, for any $\varepsilon > 0$, we get

$$\|u_{x_ix_j}\|_{G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}(B)} \le C\varepsilon\|u_{x_ix_j}\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(B)} + C\|Lu\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(B)}.$$

Choosing ε small enough (for example, $C\varepsilon < 1$), we obtain

$$\|u_{x_ix_j}\|_{G\mathcal{M}_{p,\theta_2,\omega_2,\varphi_2}(B)} \le (C/(1 - C\varepsilon))\|Lu\|_{G\mathcal{M}_{p,\theta_1,\omega_1,\varphi_1}(B)}.$$

Therefore the proof is completed. □

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(Received 15.06.2022)

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