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STRUCTURAL EQUATIONS OF THE TANGENT SPACE $T(T(Vn))$

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Abstract. In this paper we consider the manifold tangent space $T(T(Vn))$ of the tangent space $T(Vn)$. Invariant I-forms of the space $T(T(Vn))$ are defined and their structural equations are obtained. Structural equations of the space $T(T(Vn))$ are obtained.

It is well known that various geometrical structures on a given space are defined by concrete fields of differential-geometric objects. If new fibers join the given fiber spaces, then for these new fibers there sometimes appear new geometric structures generated by the original structures. In that case the new geometrical structures are regarded as peculiar analogs of the facts of internal geometry of the equipped surface.

Let us consider the manifold tangent space $T(T(Vn))$ with local coordinates $(x^i, y^{\bar{i}}, y^i, z^{\bar{i}})$, $i,j,k = \overline{1,n}$, $\bar{i},\bar{j},\bar{k} = \overline{1,n}$, where $x^i, y^{\bar{i}}$ are the coordinates of the basis $T(Vn)$, and y^i, $z^{\bar{i}}$ are the coordinates of the fiber T_z, $z \in T(Vn)$. In other words, the vector fields $\mathfrak{X}$

$$\mathfrak{X} = y^i \frac{\partial}{\partial x^i} + z^{\bar{i}} \frac{\partial}{\partial y^{\bar{i}}}$$

generate the fiber space $T(T(Vn))$. Then it is obvious that the local coordinates $(x^i, y^{\bar{i}}, y^i, z^{\bar{i}})$ of a point of the fiber space $T(T(Vn))$ are transformed as follows:

$$\overline{x}^i = \overline{x}^i(x^k), \quad \overline{y}^i = x^i_k y^k, \quad \overline{y}^{\bar{i}} = x^{\bar{i}}_{\bar{k}} y^{\bar{k}}, \quad \overline{z}^{\bar{i}} = x^{\bar{i}}_{\bar{k}} z^{\bar{k}} + x^{\bar{i}}_{\bar{k}j} y^{\bar{k}} y^j. \tag{1}$$

On the space $T(T(Vn))$ we can define the following forms:

$$\theta^i = dy^i + \omega^i_k y^k, \quad \theta^{\bar{i}} = dy^{\bar{i}} + \omega^{\bar{i}}_{\bar{k}} y^{\bar{k}}, \quad \vartheta^{\bar{i}} = dz^{\bar{i}} + \omega^{\bar{i}}_{\bar{k}} z^{\bar{k}} + \omega^{\bar{i}}_{\bar{k}j} y^{\bar{k}} y^j. \tag{2}$$

The full equipment of the tangent space $TT(T(Vn))$ can be determined by using differential-geometric objects Γ^i_j, $G^{\bar{i}}_{\bar{k}}$, $G^{\bar{i}}_j$, $L^{\bar{i}}_k$, $C^{\bar{k}}_i$, the transformation law of which has the form:

$$x^{\bar{i}}_{\bar{k}} G^{\bar{k}}_j = x^{\bar{k}}_j \overline{G}^{\bar{i}}_{\bar{k}} + x^{\bar{i}}_{\bar{k}j} y^{\bar{k}}, \tag{3}$$

$$x^p_i \Gamma^i_k = x^i_k \overline{\Gamma}^p_i + x^p_{ki} y^i, \tag{4}$$

$$x^{\bar{i}}_{\bar{j}} L^{\bar{j}}_k = x^j_k \overline{L}^{\bar{i}}_j + \overline{G}^{\bar{i}}_{\bar{j}} x^{\bar{j}}_{\bar{p}k} y^{\bar{p}} + \overline{G}^{\bar{i}}_p x^p_{jk} y^j + x^{\bar{i}}_{\bar{j}k} z^{\bar{j}} + x^{\bar{i}}_{\bar{j}pk} y^{\bar{j}} y^p, \tag{5}$$

$$x^{\bar{i}}_{\bar{j}} G^{\bar{j}}_{\bar{k}} = x^{\bar{j}}_{\bar{k}} \overline{G}^{\bar{i}}_{\bar{j}} + x^{\bar{i}}_{\bar{k}j} y^j, \tag{6}$$

$$x^i_k C^{\bar{j}}_i = x^{\bar{j}}_{\bar{i}} \overline{C}^{\bar{i}}_k + \overline{Q}^{\bar{p}}_k x^{\bar{j}}_{\bar{p}i} y^i + \overline{E}^i_k x^{\bar{j}}_{\bar{p}i} y^{\bar{p}} - x^{\bar{j}}_{\bar{i}k} z^{\bar{i}} - x^{\bar{j}}_{\bar{i}pk} y^{\bar{i}} y^p. \tag{7}$$

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Differentiating the law of transformation (3)–(7) in the usual way, after appropriate calculations we obtain:

$$\begin{gathered}
\nabla G_k^{\bar i} - \theta_k^{\bar i} = \nabla_j G_k^{\bar i}\omega^j + \nabla_{\bar j} G_k^{\bar i}\theta^{\bar j},\\
\nabla G_{\bar k}^{\bar j} - \theta_{\bar k}^{\bar j} = \nabla_i G_{\bar k}^{\bar j}\omega^i + \nabla_{\bar i} G_{\bar k}^{\bar j}\theta^{\bar i} + \nabla_{\bar i} G_{\bar k}^{\bar j}\theta^i,\\
\nabla \Gamma_j^i - \theta_j^i = \nabla_p \Gamma_j^i \omega^p + \nabla_{\bar k}\Gamma_j^i\theta^{\bar k} + \nabla_{\bar p}\Gamma_j^i\theta^p,\\
\nabla L_k^{\bar i} - G_{\bar j}^{\bar i}\theta_k^{\bar j} - G_p^{\bar i}\theta_k^p - \vartheta_k^{\bar i} = \nabla_p L_k^{\bar i}\omega^p + \nabla_{\bar j} L_k^{\bar i}\theta^{\bar j} + \nabla_{\bar p} L_k^{\bar i}\theta^p + \nabla_{\bar p} L_k^{\bar i}\vartheta^{\bar p},\\
\nabla L_k^{\bar i} + G_k^{\bar j}\theta_{\bar j}^{\bar i} + \Gamma_k^p\theta_p^{\bar i} - \vartheta_k^{\bar i} = \nabla_p C_k^{\bar i}\omega^p + \nabla_{\bar j} C_k^{\bar i}\theta^{\bar j} + \nabla_{\bar p} C_k^{\bar i}\theta^p + \nabla_{\bar p} C_k^{\bar i}\vartheta^{\bar p},
\end{gathered} \tag{8}$$

where, the quantities $\nabla_p C_k^{\bar i}$, $\nabla_{\bar j} C_k^{\bar i}$, $\nabla_{\bar p} C_k^{\bar i}$, $\nabla_{\bar p} C_k^{\bar i}$ are the expansion coefficients of I-forms $\nabla C_k^{\bar i} + G_k^{\bar j}\theta_{\bar j}^{\bar i} + \Gamma_k^p\theta_p^{\bar i} - \vartheta_k^{\bar i}$ by I-forms ω^j, $\theta^{\bar j}$, θ^i, $\vartheta^{\bar p}$ and they are called Pfaffian derivatives of the first, second, third and fourth rolls respectively for $C_k^{\bar i}$. In a holonomic frame, i.e. when

$$\omega^j = dx^j, \quad \theta^{\bar j} = dy^{\bar j}, \quad \theta^i = dy^i, \quad \vartheta^{\bar p} = dz^{\bar p},$$

$$\nabla C_k^{\bar i} + G_k^{\bar j}\theta_{\bar j}^{\bar i} + \Gamma_k^p\theta_p^{\bar i} - \vartheta_k^{\bar i} = \nabla_p C_k^{\bar i}dx^p + \nabla_{\bar j} C_k^{\bar i}dy^{\bar j} + \nabla_{\bar p} C_k^{\bar i}dy^p + \nabla_{\bar p} C_k^{\bar i}dz^{\bar p},$$

they coincide with ordinary partial derivatives, i.e.

$$\nabla_k C_j^{\bar i} = \frac{\partial C_j^{\bar i}}{\partial x^k}, \qquad \nabla_{\bar j} C_k^{\bar i} = \frac{\partial C_k^{\bar i}}{\partial y^{\bar j}},$$

$$\nabla_{\bar p} C_k^{\bar i} = \frac{\partial C_k^{\bar i}}{\partial y^k}, \qquad \nabla_{\bar p} C_k^{\bar i} = \frac{\partial C_k^{\bar i}}{\partial z^{\bar p}}\,.$$

Differential equations (8) are called differential equations of the field of a linearly connected object

$$\Gamma = (G_k^{\bar i}, G_{\bar k}^{\bar j}, \Gamma_j^i, L_k^{\bar i}, C_k^{\bar i}).$$

The I-forms of these connections have the form:

$$\widetilde{\theta}^{\,\bar i} = \theta^{\bar i} + G_k^{\bar i}\omega^k, \tag{9}$$

$$\widetilde{\theta}^{i} = \theta^i + \Gamma_k^i\omega^k, \tag{10}$$

$$\widetilde{\vartheta}^{\,\bar i} = \vartheta^{\bar i} + L_k^{\bar i}\omega^k + G_{\bar k}^{\bar i}\theta^{\bar k} + G_k^{\bar i}\theta^k. \tag{11}$$

Differentiating externally equalities (9), we obtain:

$$D\widetilde{\theta}^{\,\bar i} + dG_k^{\bar i}\wedge\omega^k + G_k^{\bar i}D\omega^k.$$

From here, due to (1) and (8), we obtain:

$$D\widetilde{\theta}^{\,\bar i} - \widetilde{\theta}^{\,\bar k}\wedge\left(\omega_{\bar k}^{\bar i} + G_{\bar k j}^{\bar i}\omega^j\right) = R_{pq}^{\bar i}\omega^p\wedge\omega^q,$$

where

$$R_{pq}^{\bar i} = \overset{\Gamma}{\nabla}_{[p} G_{q]}^{\bar i}.$$

Differentiating externally equalities (10), we obtain:

$$D\widetilde{\theta}^i = D\theta^i + d\,\Gamma_k^i\wedge\omega^k + \Gamma_k^i D\omega^k.$$

From here, due to (2) and (8), we obtain:

$$\begin{gathered}
D\widetilde{\theta}^i = \widetilde{\theta}^k\wedge\widetilde{\omega}_k^i + (\nabla_p\Gamma_k^i - \nabla_{\bar j}\Gamma_k^i G_p^{\bar j} - \nabla_{\bar q}\Gamma_k^i\Gamma_p^q)\omega^p\wedge\omega^k + \nabla_{\bar j}\Gamma_k^i\widetilde{\theta}^{\,\bar j}\wedge\omega^k\\
+(\nabla_{\bar p}\Gamma_k^i - \Gamma_{pk}^i)\widetilde{\theta}^p\wedge\omega^k
\end{gathered}$$

or

$$D\widetilde{\theta}^i = \widetilde{\theta}^k\wedge\widetilde{\omega}_k^i + \mathbb{R}_{pk}^i\omega^p\wedge\omega^k + \mathbb{R}_{\bar j k}^i\widetilde{\theta}^{\,\bar j}\wedge\omega^k + \mathbb{S}_{pk}^i\widetilde{\theta}^p\wedge\omega^k,$$

where

$$\mathbb{R}^{i}_{pk} = \overset{\Gamma}{\nabla}_{[p}\Gamma^{i}_{k]} - \nabla_{\overline{q}}\Gamma_{[k}\Gamma^{q}_{p]}, \qquad \mathbb{R}^{i}_{\overline{j}k} = \nabla_{\overline{j}}\Gamma^{i}_{k},$$
$$\mathbb{S}^{i}_{pk} = \nabla_{[\overline{p}}\Gamma^{i}_{k]} - \Gamma^{i}_{[pk]}.$$

Differentiating externally equalities (11), we have:

$$D\widetilde{\vartheta}^{\,\overline{i}} = D\theta^{\overline{i}} + dL^{\overline{i}}_{k} \wedge \omega^{k} + L^{\overline{i}}_{k} D\omega^{k} + dG^{\overline{i}}_{\overline{k}} \wedge \theta^{\overline{k}} + G^{\overline{i}}_{\overline{k}} D\theta^{\overline{k}} + dG^{\overline{i}}_{k} \wedge \theta^{k} + G^{\overline{i}}_{k} D\theta^{k}.$$

From here

$$D\widetilde{\vartheta}^{\,\overline{i}} - \widetilde{\vartheta}^{\,\overline{j}} \wedge \widetilde{\omega}^{\overline{i}}_{\overline{j}} = \nabla_{j} L^{\overline{i}}_{k}\omega^{j} \wedge \omega^{k} + \nabla_{\overline{j}} L^{\overline{i}}_{k}\,\theta^{\overline{j}} \wedge \omega^{k} + \nabla_{\overline{j}} L^{\overline{i}}_{k}\,\theta^{j} \wedge \omega^{k}$$
$$+\nabla_{\overline{j}} L^{\overline{i}}_{k}\vartheta^{\overline{j}} \wedge \omega^{k} + \nabla_{j} G^{\overline{i}}_{\overline{k}}\omega^{j} \wedge \theta^{\overline{k}} + \nabla_{\overline{j}}\, G^{\overline{i}}_{\overline{k}}\,\theta^{\overline{j}} \wedge \theta^{\overline{k}} + \nabla_{\overline{j}}\, G^{\overline{i}}_{\overline{k}}\,\theta^{j} \wedge \theta^{\overline{k}}$$
$$+\nabla_{j} G^{\overline{i}}_{k}\omega^{j} \wedge \theta^{k} + \nabla_{\overline{j}}\, G^{\overline{i}}_{k}\theta^{\overline{j}} \wedge \theta^{k}.$$

From here, by virtue of (9), (10) and (11), we obtain:

$$D\widetilde{\vartheta}^{\,\overline{i}} - \widetilde{\vartheta}^{\,\overline{j}} \wedge \widetilde{\omega}^{\overline{i}}_{\overline{j}} = \mathbb{A}^{\overline{i}}_{pk}\omega^{p} \wedge \omega^{k} + \mathbb{B}^{\,\overline{i}}_{\,\overline{j}k}\widetilde{\vartheta}^{\,\overline{j}} \wedge \omega^{k} + \mathbb{M}^{\,\overline{i}}_{\,\overline{j}\,\overline{k}}\theta^{\overline{j}} \wedge \theta^{\overline{k}}$$
$$+\mathbb{C}^{\overline{i}}_{jk}\widetilde{\vartheta}^{\,\overline{j}} \wedge \omega^{k} + \mathbb{D}^{\,\overline{i}}_{\,\overline{j}k}\widetilde{\theta}^{\,i} \wedge \theta^{\overline{k}} + \mathbb{E}^{\,\overline{i}}_{\,\overline{j}k}\widetilde{\vartheta}^{\,\overline{j}} \wedge \omega^{k},$$

where

$$\mathbb{A}^{\overline{i}}_{pk} = \nabla_{p} L^{\overline{i}}_{k} - \nabla_{\overline{j}} L^{\overline{i}}_{k} G^{\overline{j}}_{p} - \nabla_{\overline{j}} L^{\overline{i}}_{k}\Gamma^{j}_{p} - \nabla_{\overline{j}} L^{\overline{i}}_{k} C^{\overline{j}}_{p} + \nabla_{k} G^{\,\overline{i}}_{\,\overline{j}}\Gamma^{j}_{p}$$
$$+\nabla_{\overline{q}}\, G^{\,\overline{i}}_{\,\overline{j}}\, G^{\overline{q}}_{p}\, G^{\overline{j}}_{k} + \nabla_{\overline{q}}\, G^{\,\overline{i}}_{\,\overline{j}}\,\Gamma^{q}_{p}\, G^{\overline{j}}_{k} + \nabla_{k} G^{\overline{i}}_{j}\Gamma^{j}_{p} + \nabla_{\overline{j}}\, G^{\overline{i}}_{q}\, G^{\overline{j}}_{p}\,\Gamma^{i}_{k},$$
$$\mathbb{B}^{\overline{i}}_{\overline{j}k} = \nabla_{\overline{j}} L^{\overline{i}}_{k} - \nabla_{\overline{p}} L^{\overline{i}}_{k} G^{\overline{p}}_{\overline{j}} - \nabla_{k} G^{\,\overline{i}}_{\,\overline{j}} + \nabla_{\overline{q}}\, G^{\overline{i}}_{\overline{j}}\, G^{\overline{q}}_{k} - \nabla_{\overline{j}}\, G^{\overline{i}}_{\overline{q}}\, G^{\overline{q}}_{k} + \nabla_{\overline{p}}\, G^{\overline{i}}_{\overline{j}}\,\Gamma^{p}_{k} - \nabla_{\overline{j}}\, G^{\overline{i}}_{p}\,\Gamma^{p}_{k},$$
$$\mathbb{C}^{\overline{i}}_{jk} = \nabla_{\overline{j}} L^{\overline{i}}_{k} - \nabla_{\overline{q}} L^{\overline{i}}_{k} G^{\overline{q}}_{j} - \nabla_{\overline{j}}\, G^{\overline{i}}_{\overline{q}}\, G^{\overline{q}}_{k} - \nabla_{k} G^{\overline{i}}_{j} + \nabla_{\overline{q}}\, G^{\overline{i}}_{j}\, G^{\overline{q}}_{k},$$
$$\mathbb{D}^{\overline{i}}_{\overline{j}k} = \nabla_{\overline{k}}\, G^{\overline{i}}_{\overline{j}} - \nabla_{\overline{j}}\, G^{\overline{i}}_{k},$$
$$\mathbb{E}^{\overline{i}}_{\overline{j}k} = \nabla_{\overline{j}} L^{\overline{i}}_{k} - G^{\overline{i}}_{\overline{j}k},$$
$$\mathbb{M}^{\overline{i}}_{\overline{j}k} = \nabla_{\overline{k}}\, G^{\overline{i}}_{\overline{j}}.$$

So, the external differential equations

$$D\omega^{i} = \omega^{k} \wedge \widetilde{\omega}^{i}_{k},$$
$$D\widetilde{\theta}^{\,\overline{i}} = \widetilde{\vartheta}^{\,\overline{k}} \wedge \omega^{\overline{i}}_{\overline{k}} + R^{\overline{i}}_{pq}\omega^{p} \wedge \omega^{q},$$
$$D\widetilde{\theta}^{i} = \widetilde{\theta}^{k} \wedge \widetilde{\omega}^{i}_{k} + \mathbb{R}^{i}_{pk}\omega^{p} \wedge \omega^{k} + \mathbb{R}^{i}_{\overline{j}k}\widetilde{\theta}^{\,\overline{j}} \wedge \omega^{k} + \mathbb{S}^{i}_{pk}\widetilde{\theta}^{p} \wedge \omega^{k},$$
$$D\widetilde{\vartheta}^{\,\overline{i}} = \widetilde{\vartheta}^{\,\overline{j}} \wedge \widetilde{\omega}^{\,\overline{i}}_{\,\overline{j}} + \mathbb{A}^{\overline{i}}_{pk}\omega^{p} \wedge \omega^{k} + \mathbb{B}^{\overline{i}}_{\overline{j}k}\widetilde{\vartheta}^{\,\overline{j}} \wedge \omega^{k} + \mathbb{M}^{\overline{i}}_{\overline{j}\,\overline{k}}\theta^{\overline{j}} \wedge \theta^{\overline{k}}$$
$$+\mathbb{C}^{\overline{i}}_{jk}\widetilde{\vartheta}^{\,\overline{j}} \wedge \omega^{k} + \mathbb{D}^{\overline{i}}_{\overline{j}k}\widetilde{\theta}^{i} \wedge \theta^{\overline{k}} + \mathbb{E}^{\overline{i}}_{\overline{j}k}\widetilde{\vartheta}^{\,\overline{j}} \wedge \omega^{k}$$

are structural equations of the space $T(T(Vn))$. See [1–7] for related topics.

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