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ON UNIVERSAL FUNCTIONS REPRESENTING CERTAIN CLASSES OF FUNCTIONS

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Abstract. In the paper, a class A of real-valued functions defined on the interval $(0, 1)$ is considered. The notion of a universal representing function for the class A is introduced. In particular, for the class of continuous functions $C(0, 1)$, the existence of a universal representing function F is established. It is shown that for any continuous function $f \in C(0, 1)$, there exists a subset of the interval $(0, 1)$ such that if the values of the universal function F are known on this subset, the values of the function f can be determined at each point of the interval $(0, 1)$.

Let A be a certain class of real-valued functions defined on the interval $(0, 1)$ and containing at least two functions that differ from each other at least at one point in the interval $(0, 1)$.

Definition 1. A set $E \subset (0, 1)$ will be called a uniqueness set for the class A if for any functions $f(x)$ and $g(x)$ of the class A such that

$$f(x) = g(x), \quad \text{when } \; x \in E,$$

it follows that $f(x) = g(x)$ for any $x \in (0, 1)$.

Let $\Omega(A)$ be the family of all uniqueness sets for the class A. Thus, for any $E \in \Omega(A)$, we have: if for any functions $f(x)$ and $g(x)$ of the class A such that

$$f(x) = g(x), \quad \text{when } \; x \in E,$$

it follows that $f(x) = g(x)$, for any $x \in (0, 1)$. It is evident that for any class A, the set $E = (0, 1)$ belongs to $\Omega(A)$.

Note that each function mentioned below in this paper is a real-valued function and each class of functions is a class of real-valued functions.

Definition 2. We say that a function $F(x)$ defined on $(0, 1)$ is a universal representing function for the class A if for any function $f(x) \in A$, there exists a set $E \in \Omega(A)$ such that

$$F(x) = f(x), \quad \text{for any } \; x \in E.$$

For such functions $F(x)$, for brevity, we will henceforth say that $F(x)$ is a representing function of the class A. Here, we state some properties of representing functions.

Property 1. If the function $F(x)$ is a representing function of the class A, then $F(x) \notin A$.

Proof. Suppose the opposite. That is, let $F(x)$ be a representing function for the class A and $F(x) \in A$. Then, since $F(x)$ is a representing function for the class A, for any $f(x) \in A$, there exists a set $E \in \Omega(A)$ such that $F(x) = f(x)$, for any $x \in E$.

Furthermore, due to the assumption that $F \in A$, we have

$$F \in A, \quad f \in A, \quad \text{and there exists a set } \; E \in \Omega(A) \; \text{ such that}$$
$$F(x) = f(x), \quad \text{when } \; x \in E.$$

Therefore, since E is a uniqueness set for the class A, we have $F(x) \equiv f(x)$, for any $x \in (0, 1)$.

That is, for any $f(x) \in A$, we have $F(x) \equiv f(x)$ for $x \in (0, 1)$. In other words, is a class of functions containing only one function. Property 1 is proved. □

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Property 2. There exists a class of functions A for which there is no representing function for the class A.

Indeed, for example, if A is the class of all functions defined on $(0,1)$, then there exists no representing function for the class A. This is due to the fact that if we assume that some $F(x)$ is a representing function for the class A, then according to Property 1, we have $F \notin A$, which is impossible since A is the class of all functions defined on $(0,1)$.

Property 3. If A and B are the classes of functions defined on $(0,1)$ such that $A \subset B$, and the function $F(x)$ is a representing function for the class B, then $F(x)$ is likewise a representing function for the class A.

The validity of this property follows directly from Definition 2 and the fact that if $A \subset B$, then $\Omega(B) \subset \Omega(A)$.

In this work, we consider $A = C(0,1)$, the class of all continuous functions on $(0,1)$. Notice that for E to be a uniqueness set for the class $C(0,1)$, it is necessary and sufficient for E to be a dense set in $(0,1)$.

Given Property 2, a natural question arises: does there exist a representing function for the class $C(0,1)$?

The answer to this question is positive. Namely, the following statement holds.

Theorem 1. *The values of any function defined on* $(0,1)$ *can be altered on a subset of Lebesgue linear measure zero and the first category from the interval* $(0,1)$ *so that the resulting function* $F(x)$ *will be a universal representing function for the class* $C(0,1)$.

The validity of this theorem follows from the following, more general, statement.

Theorem 2. *There exist an* H*-subset of Lebesgue linear measure zero and the first category in the interval* $(0,1)$ *and a function* $h(x)$ *defined on the set* H *such that the following holds: if any function* $\varphi(x)$ *defined on* $(0,1)$ *is replaced by* $h(x)$ *on the set* H*, then the resulting function* $F(x)$ *will be a universal representing function for the class* $C(0,1)$.

The following statement follows directly from the above theorems.

Corollary. *For any function* $\varphi(x)$ *defined on* $(0,1)$*, there exists a function* $F(x)$ *such that* $F(x) = \varphi(x)$ *almost everywhere on* $(0,1)$ *and at the same time* $F(x)$ *is a universal representing function of the class* $C(0,1)$.

Note that by virtue of Property 3, each function $F(x)$ from the above statements (Theorem 1, Theorem 2 and Corollary) is a universal representing function for any class of functions A, where $A \subset C(0,1)$.

Let a function $\Phi(x)$ be defined on $(0,1)$, E be a subset of $(0,1)$ and let x_0 be a limit point of the set E.

The notation $\lim\limits_{\substack{x\to x_0 \\ x\in E}} \Phi(x) = a$ means that if $x \to x_0$ and $x \in E$, then $\Phi(x) \to a$.

The following statement shows that by using any universal representing function for the class $C(0,1)$, it is possible to determine the value of any continuous function $f(x) \in C(0,1)$ at each point of the interval $(0,1)$.

Namely, the following statement holds.

Theorem 3. *Let* $F(x)$ *be an arbitrary universal representing function for the class* $C(0,1)$*. Then for any continuous function* $f(x) \in C(0,1)$*, there exists a dense subset E in* $(0,1)$ *such that for each point* $x_0 \in (0,1)$*, the equality*

$$\lim_{\substack{x\to x_0 \\ x\in E}} F(x) = f(x_0)$$

holds.

At the end of the paper, note that analogous definitions and statements to those presented above can be formulated and proved, respectively, for more general classes of mappings rather than just for the class $C(0,1)$.

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