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BOUNDEDNESS CRITERIA FOR LINEAR AND MULTILINEAR FRACTIONAL INTEGRAL OPERATORS IN LORENTZ SPACES

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Abstract. In this note we give necessary and sufficient condition on a measure μ guaranteeing the boundedness of the multilinear fractional integral operator $T^m_{\gamma,\mu}$ defined with respect to μ from the product of Lorentz spaces $\prod_{k=1}^m L^{r_k,s_k}(\mu,X)$ to the Lorentz space $L^{p,q}(\mu,X)$. The result is new even for linear fractional integrals $T_{\gamma,\mu}$ (i.e., when $m=1$). From the general results we have a criterion for the validity of Sobolev–type inequality in Lorentz spaces defined for non-doubling measures.

1. Introduction

During the last two decades a considerable attention of researchers was attracted to the study of the mapping properties of integral operators defined on metric measure spaces with non-doubling measure (see e.g. [1, 12–14] and references cited therein). The results regarding the boundedness of such operators in function spaces were mainly obtained under the growth condition on a measure.

The fractional integral (Riesz potential)

$$I_\alpha(f)(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\,dy, \quad x\in\mathbb{R}^n,$$

plays a fundamental role in Harmonic Analysis. It also finds applications in PDEs, such as in the theory of Sobolev embeddings (see, for instance, Maz'ya [10]). The study of multilinear fractional integrals was initiated by L. Grafakos [4]. The author of that paper established the boundedness of the operator

$$B_\alpha(f,g)(x) = \int\limits_{\mathbb{R}^n} \frac{f(x+t)g(x-t)}{|t|^{n-\alpha}}dt, \quad 0<\alpha<n,$$

from $L^{p_1}(\mathbb{R}^n)\times L^{p_2}(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$.

As a tool to understand B_α, the operators

$$\mathcal{I}_\alpha(\overrightarrow{f})(x) = \int\limits_{\mathbb{R}^n} \frac{f_1(y_1)\cdots f_m(y_m)}{(|x-y_1|+\cdots+|x-y_m|)^{mn-\alpha}}d\overrightarrow{y}, \quad x\in\mathbb{R}^n,$$

written in the $m-$ linear form, where $0<\alpha<nm$, $\overrightarrow{f} := (f_1,\dots,f_n)$, $\overrightarrow{y} := (y_1,\dots,y_n)$, were studied as well.

Let (X,d,μ) be a quasi-metric measure space. Our aim is to characterize those measures for which the boundedness of the fractional integral operator

$$T^m_{\gamma,\mu}(\overrightarrow{f})(x) = \int\limits_{X^m} \frac{f_1(y_1)\cdots f_m(y_m)d\mu(\overrightarrow{y})}{\big(d(x,y_1)+\cdots+d(x,y_m)\big)^{m-\gamma}}, \quad 0<\gamma<m, \quad x\in X,$$

where $\overrightarrow{f} = (f_1,\dots,f_m)$, $d\mu(\overrightarrow{y}) = d\mu(y_1)\cdots d\mu(y_m)$ holds from $\prod_{j=1}^m L^{r_j,s_j}(\mu,X)$ to $L^{p,q}(\mu,X)$. Here $L^{r_j,s_j}(\mu,X)$ and $L^{p,q}(\mu,X)$ are Lorentz spaces defined on (X,d,μ).

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This result is new even for linear case $m = 1$. In particular, as a corollary we have a complete characterization of a measure μ guaranteeing the boundedness of the fractional integral operator

$$T_{\gamma,\mu}(g)(x) = \int\limits_X \frac{g(y)}{d(x,y)^{1-\gamma}} d\mu(y), \quad 0 < \gamma < 1, \ \ x \in X,$$

from $L^{r,s}(\mu, X)$ to $L^{p,q}(\mu, X)$. As a corollary, we have also a generalization of the Sobolev-type inequality in Lorentz spaces. In particular, we give necessary and sufficient condition on a measure μ for which the inequality

$$\|T_{\gamma,\mu}(g)\|_{L^{p,q}(\mu,X)} \le C\|g\|_{L^{r,s}(\mu,X)}, \ \ p = \frac{r}{1-\gamma r} \tag{1}$$

holds.

These results for Lebesgue spaces in the multilinear setting were derived in [8] (including weak type estimates), and for the linear case go back to [9] (see also [1], Ch. 6) for quasi-metric measure spaces, and [6,7] for Euclidean spaces. We refer also to [2,3] for the Sobolev–type inequalities in the classical Lebesgue spaces for non-doubling measure (see also [11] for related topics).

2. Preliminaries

Let X be a topological space with a quasi-metric d and a complete measure μ on X. We will assume that the class of compactly supported continuous functions is dense in $L^1(\mu, X)$.

The triple (X, d, μ) is called a quasi-metric measure space.

In the sequel we assume that all the balls $B(x, R)$ with center x and radius R are $\mu-$ measurable with finite measure, and that for every neighborhood V of $x \in X$, there exists $R > 0$ such that $B(x, R) \subset V$.

We say that the measure μ is Ahlfors upper β- regular if there is a positive constant c such that

$$\mu(B(x, R)) \le cR^\beta \tag{2}$$

for all $x \in X$ and $R > 0$.

Let f be a μ-measurable function on X and let $1 \le p, s \le \infty$. We say that f belongs to the Lorentz space $L^{p,s}(\mu, X)$ if

$$\|f\|_{L^{p,s}(\mu,X)} = \begin{cases} \left(s\int\limits_0^\infty \big(\mu\{x\in X : |f(x)| > \tau\}\big)^{s/p} \tau^{s-1} d\tau\right)^{1/s}, & \text{if } \ s < \infty, \\ \sup\limits_{s>0} s\big(\mu(\{x \in X : |f(x)| > s\}\big)^{1/p}, & \text{if } \ s = \infty \end{cases}$$

is finite. It is easy to see that $L^{p,p}(\mu, X)$ coincides with the Lebesgue space $L^p(\mu, X)$ with measure μ.

Denote by f^*_μ a weighted non-increasing rearrangement of f with respect to the measure μ. Then by integration by parts it can be checked that (see also [5]):

$$\|f\|_{L^{p,s}(\mu,X)} \approx \begin{cases} \left(\frac{s}{p}\int\limits_0^\infty \left(t^{1/p} f^*_\mu(t)\right)^s \frac{dt}{t}\right)^{1/s}, & \text{if } \ s < \infty, \\ \sup\limits_{t>0} \left\{t^{1/p} f^*_\mu(t)\right\}, & \text{if } \ s = \infty, \end{cases}$$

3. Main Results

Now we formulate our main statements.

Theorem 1. *Let (X, d, μ) be a quasi-metric measure space, $0 < \gamma < m$, $1 < r_j, s_j < \infty$, $j = 1, \dots, m$. We set $\frac{1}{r} = \sum_{j=1}^m \frac{1}{r_j}$ and $\frac{1}{s} = \sum_{j=1}^m \frac{1}{s_j}$. Suppose that $1 < r < p < \infty$ and q be such that $\frac{r}{p} = \frac{s}{q}$. Then the inequality*

$$\|T^m_{\gamma,\mu}(\overrightarrow{f})\|_{L^{p,q}(\mu,X)} \le C \prod_{k=1}^m \|f_k\|_{L^{r_k,s_k}(\mu,X)},$$

with the positive constant C independent of $\overrightarrow{f} = (f_1, \dots, f_m)$, holds if and only if μ is Ahlfors upper β-regular with $\beta = \frac{rp(m-\gamma)}{rpm+r-p}$ (see (2)).

Theorem 1 implies the following statement.

Theorem 2. *Let (X, d, μ) be a quasi-metric measure space, $0 < \gamma < 1$, and let $1 < r_j, s_j < \infty$, $j = 1, \dots, m$. We set $\frac{1}{r} = \sum_{j=1}^{m} \frac{1}{r_j}$ and $\frac{1}{s} = \sum_{j=1}^{m} \frac{1}{s_j}$. Suppose that $1 < r < \frac{1}{\gamma}$, and $\frac{1}{p} = \frac{1}{r} - \gamma$. Let q be such that $\frac{r}{p} = \frac{s}{q}$. Then the inequality*

$$\|T^m_{\gamma,\mu}(\overrightarrow{f})\|_{L^{p,q}(\mu,X)} \le C \prod_{k=1}^{m} \|f_k\|_{L^{r_k,s_k}(\mu,X)},$$

with the positive constant C independent of $\overrightarrow{f}$, holds if and only if μ is Ahlfors upper 1-regular.

Theorems 1 and 2 yield the following statements for the linear fractional integral operator $T_{\gamma,\mu}$:

Theorem 3. *Let (X, d, μ) be a quasi-metric measure space, $0 < \gamma < 1$, $1 < r < p < \infty$. Let s and q be such that $\frac{r}{p} = \frac{s}{q}$. Then the inequality (1) with the positive constant C independent of g holds if and only if μ is Ahlfors upper β-regular with $\beta = \frac{rp(1-\gamma)}{rp+r-p}$.*

Theorem 4. *Let Let (X, d, μ) be a quasi-metric measure space, $0 < \gamma < 1$. Let $1 < r < \frac{1}{\gamma}$, and $\frac{1}{p} = \frac{1}{r} - \gamma$. Let s and q be such that $\frac{r}{p} = \frac{s}{q}$. Then inequality (1) with the positive constant C independent of g holds if and only if μ is Ahlfors upper 1-regular.*

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