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ON SOME FINITE SYSTEMS OF VECTORS IN THE EUCLIDEAN PLANE

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Abstract. In this note, an algorithmic version of Minkowski's well-known theorem is considered for the special case of dimension $m = 2$.

The following classical and well-known result on convex polyhedra was obtained by Minkowski many years ago (see, for example, [1, 5]).

Theorem 1. *Let $m \geq 2$ be a natural number, $\{e_1, e_2, \dots e_k\}$ be a finite system of distinct unit vectors in the space $\mathbf{R}^m$, not all of which belong to a vector hyperplane of $\mathbf{R}^m$, and let $\{s_1, s_2, \dots, s_k\}$ be a family of strictly positive real numbers satisfying the relation*

$$\sum \{s_i e_i : 1 \leq i \leq k\} = 0.$$

Then there exists a unique (up to translations) m-dimensional convex polyhedron P in $\mathbf{R}^m$ with all facets $\{F_1, F_2, \dots, F_k\}$ such that for every natural index $j \in [1, k]$, the $(m-1)$-dimensional volume of F_j is equal to s_j, and e_j is an exterior normal vector of F_j.

Some further interesting extensions of Minkowski's theorem can be found, e.g., in [2] and [7].

For $m \geq 3$, all the known proofs of this theorem are non-elementary, because they use the standard methods of mathematical analysis. So, it makes no sense to speak on finding an algorithmic construction of such a polyhedron P. However, in the case $m = 2$, the same question is meaningful and we will consider it below.

In the sequel, we will need one simple auxiliary proposition.

Lemma 1. *If $m = 2$, then Minkowski's theorem is equivalent to the following statement:*

Let $k \geq 3$ be a natural number, $\{e_1, e_2, \dots, e_k\}$ be a finite system of nonzero vectors in the plane $\mathbf{R}^2$, no two of which are of the same direction, and let

$$e_1 + e_2 + \cdots + e_k = 0.$$

Then there exists a unique (up to translations) non-degenerate oriented convex k-gon P in $\mathbf{R}^2$ such that the set of all oriented sides of P coincides with the given system.

In other words, Lemma 1 says that there exists a permutation ϕ of the set $\{1, 2, \dots, k\}$ such that the re-enumerated system of vectors

$$\{e_{\phi(1)}, e_{\phi(2)}, \dots, e_{\phi(k)}\}$$

gives us all successive sides of some oriented convex k-gon P in $\mathbf{R}^2$.

Since there are exactly $k!$ permutations of $\{1, 2, \dots, k\}$, there arises a natural question: find a maximally simple construction of the above-mentioned convex polygon P. Clearly, from the algorithmic point of view, such a construction should be of minimal complexity with respect to an initial natural parameter k.

Lemma 2. *Let $\{e_1, e_2, \dots, e_k\}$ be a nonempty finite system of distinct unit vectors in the plane $\mathbf{R}^2$.*

The following two assertions are valid:

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(1) *if* $k = 2n$, *then there exists a straight line* l *passing through the origin of* $\mathbf{R}^2$ *and having the property that exactly* n *many vectors of this system belong to one of the open half-planes determined by* l *and the rest* n *vectors of this system belong to the other open half-plane determined by* l*;*

(2) *if* $k = 2n+1$, *then there exists a straight line* l *passing through the origin of* $\mathbf{R}^2$ *and having the property that exactly* $n+1$ *many vectors of this system belong to one of the open half-planes determined by* l *and the rest* n *vectors of this system belong to the other open half-plane determined by* l.

Moreover, there is an algorithm of complexity $O(k\ln(k))$ *for finding the required separating line* l *in both cases* (1) *and* (2).

In the process of proving Lemma 2, the two possible situations should be taken into account:

(i) the origin of $\mathbf{R}^2$ does not belong to $\mathrm{conv}\{e_1, e_2, \dots, e_k\}$;

(ii) the origin of $\mathbf{R}^2$ belongs to $\mathrm{conv}\{e_1, e_2, \dots, e_k\}$.

Remark 1. Obviously, using appropriate projections and induction on m, Lemma 2 can be generalized to finite systems of unit vectors in the Euclidean space $\mathbf{R}^m$, where $m \geq 2$. Also, we would like to mention one interesting fact closely connected with the separating hyperplanes for such systems of vectors.

Let $\{e_1, e_2, \dots, e_{2k}\}$ be a nonempty finite system of distinct unit vectors in $\mathbf{R}^m$, where $m \geq 2$, and suppose that the following condition is satisfied:

If H is any hyperplane in $\mathbf{R}^m$ passing through the origin of $\mathbf{R}^m$ and containing no vector from $\{e_1, e_2, \dots, e_{2k}\}$, then there are exactly k many vectors of this system which belong to one of the open half-spaces determined by H and the rest k vectors of this system belong to the other open half-space determined by H.

Then there exists a partition of $\{e_1, e_2, \dots, e_{2k}\}$ into two-element subsets such that the vectors belonging to any member of the partition are opposite to each other.

Keeping in mind Lemma 2, the proof of the last statement can be obtained by using induction on m (cf. [6], where this fact was applied to some characteristic properties of m-dimensional parallelepipeds in $\mathbf{R}^m$).

Remark 2. The following statement is analogous to Lemma 2 and proved similarly to the proof of this lemma.

Let $k \geq 3$ be a natural number, $\{z_1, z_2, \dots, z_k\}$ be a finite system of pairwise different points in the plane $\mathbf{R}^2$, and let z be a point of $\mathbf{R}^2$ which is in a general position with respect to $\{z_1, z_2, \dots, z_k\}$ (i.e., z does not belong to the straight line passing through any two distinct points of the system).

For such a point z, these two assertions are always valid:

(a) if $k = 2n$, then there exists a straight line $l \subset \mathbf{R}^2$ passing through z and having the property that exactly n many points of the system belong to one of the open half-planes determined by l and the rest n points of this system belong to the other open half-plane determined by l;

(b) if $k = 2n+1$, then there exists a straight line $l \subset \mathbf{R}^2$ passing through z and having the property that exactly $n+1$ many points of the system belong to one of the open half-planes determined by l and the rest n points of this system belong to the other open half-plane determined by l.

Moreover, there is an algorithm of complexity $O(k\ln(k))$ for finding the required separating line l in both cases (a) and (b).

In addition, it should be mentioned that if there is at least one straight line passing through two distinct points of the system and containing z, then it may happen that both assertions (a) and (b) become false.

Lemma 3. *Let* $\mathbf{N}$ *denote the set of all natural numbers and let*

$$f : \mathbf{N} \to \mathbf{N}$$

be an increasing (*in general, not strictly increasing*) *function satisfying the relation*

$$f(2n) \leq 2f(n+1) + q \ (n \in \mathbf{N}),$$

where q *is some real constant* (*one may assume, without loss of generality, that* $q \geq f(3)$).

Then the inequality $f(n) \leq 4qn - q$ *for* $n \geq 3$ *takes place. Consequently,* $f(n) = O(n)$.

Actually, Lemma 3 is a very special version of the well-known Master Theorem (see, for instance, [3, 4]). It makes sense to emphasize that in the formulation of this lemma the condition that f is a monotone function is essential and cannot be omitted.

Theorem 2. *Let $k \geq 3$ be a natural number, $\{e_1, e_2, \dots, e_k\}$ be a finite system of nonzero vectors in the plane $\mathbf{R}^2$, no two of which are of the same direction, and let*

$$e_1 + e_2 + \cdots + e_k = 0.$$

Then there exists an algorithm of the complexity $O(k\ln(k))$ for finding a permutation ϕ of $\{1, 2, \dots, k\}$ such that the system of vectors

$$\{e_{\phi(1)}, e_{\phi(2)}, \dots, e_{\phi(k)}\}$$

gives all successive sides of some oriented convex k-gon P in $\mathbf{R}^2$.

Remark 3. Theorem 2 implies that if $k \geq 3$ and $\{e_1, e_2, \dots, e_k\}$ is a finite system of nonzero vectors in the plane $\mathbf{R}^2$ which are not collinear and for which the equality $e_1 + e_2 + \cdots + e_k = 0$ holds true, then there exists a permutation ϕ of $\{1, 2, \dots, k\}$ such that the system $\{e_{\phi(1)}, e_{\phi(2)}, \dots, e_{\phi(k)}\}$ gives all successive sides of an oriented convex k-gon P in $\mathbf{R}^2$ (however, in this k-gon P, some adjacent sides may be collinear).

Remark 4. Preserving the notation and assumptions of Theorem 2, suppose additionally that l is a straight line passing through the origin of $\mathbf{R}^2$ and separating the system of unit vectors

$$V = \{e_1/||e_1||, e_2/||e_2||, \dots, e_k/||e_k||\},$$

as is indicated in Lemma 2. Then, using this line l and two linear arcwise orderings produced canonically by l on the two respective parts of V, the required convex k-gon P for the system $\{e_1, e_2, \dots, e_k\}$ can be constructed by an algorithm of complexity $O(k)$. Note that a similar situation occurs when studying the problem of finding all affine diameters of a given k-element set E in the plane $\mathbf{R}^2$. This problem is easily reduced to finding all affine diameters of the set of vertices of $\mathrm{conv}(E)$. As is well known, all successive vertices of the convex polygon $\mathrm{conv}(E)$ are obtained by applying to E an algorithm of complexity $O(k\ln(k))$. At the same time, it can be demonstrated that if E itself is the set of all successive vertices of a convex k-gon, then all affine diameters of E can be found with the aid of an algorithm of complexity $O(k)$. Note, by the way, that in this case we also have an upper estimate $3k/2$ for the total number $af(E)$ of affine diameters of E. The equality $af(E) = 3k/2$ is valid if and only if the set of all sides of $\mathrm{conv}(E)$ admits a partition into two-element subsets so that the sides belonging to any member of the partition are parallel. On the other hand, if E is an arbitrary k-element subset of $\mathbf{R}^2$, then it may happen that $af(E)$ is of order k^2 (as simple examples show). Also, if E is the set of all vertices of a convex k-gonal prism in the space $\mathbf{R}^3$, then $af(E)$ is again of order k^2.

It follows from Minkowski's theorem formulated at the beginning of this communication that the above-mentioned convex k-gon P is unique up to the translations of the plane $\mathbf{R}^2$. For dimension $m \geq 3$, the uniqueness part in Minkowski's theorem is usually justified by referring to the Brunn-Minkowski inequality (see, for instance, [1]). In the case $m = 2$, the uniqueness of P can be established by a direct recursive construction of P.

Lemma 4. *Preserving the assumptions and notation of Theorem 2, let (e_i, e_j) be a pair of distinct vectors from the system $\{e_1, e_2, \dots, e_k\}$ such that the angle between e_i and e_j takes minimum value.*

Then these two vectors are necessarily the adjacent sides of the convex k-gon P.

Replacing e_i and e_j by their sum $e_i + e_j$ and using the inductive argument, one can construct a unique (up to translations) oriented convex $(k-1)$-gon P' for the system of vectors

$$(\{e_1, e_2, \dots, e_k\} \setminus \{e_i, e_j\}) \cup \{e_i + e_j\}.$$

Afterwards, replacing in P' the side $e_i + e_j$ by the two adjacent sides e_i and e_j, it becomes possible to obtain the required oriented convex k-gon P. As a by-product, this construction also yields the uniqueness of P (up to translations of $\mathbf{R}^2$).

Remark 5. It can be shown that for a system of vectors $\{e_1, e_2, \dots, e_k\}$ of Theorem 2, there is an algorithm of complexity $O(k\ln(k))$ which enables one to find a pair (e_i, e_j) with the minimal value of the angle between e_i and e_j.

Theorem 3. *Let $k \geq 4$ be a natural number, $\{e_1, e_2, \dots, e_k\}$ be a system of nonzero non-coplanar vectors in the space $\mathbf{R}^3$, no two of which are of the same direction, and let*

$$e_1 + e_2 + \cdots + e_k = 0.$$

Then there exists a permutation ϕ of the set $\{1, 2, \dots, k\}$ such that the vectors $e_{\phi(1)}, e_{\phi(2)}, \dots, e_{\phi(k)}$ form an oriented Hamiltonian cycle in the 1-skeleton graph of some convex polyhedron $P \subset \mathbf{R}^3$ with k vertices.

The proof of Theorem 3 is based on the result of Theorem 2. Using induction on m, a similar result can be obtained for finite systems of vectors in the Euclidean space $\mathbf{R}^m$, where $m \geq 4$. The algorithm mentioned in Theorem 2 produces (by induction) an appropriate algorithm for constructing an m-dimensional convex polyhedron P in $\mathbf{R}^m$ having the property that there is an oriented Hamiltonian cycle in the 1-skeleton graph of P, which is formed by all members of a given finite system of vectors in $\mathbf{R}^m$ (here we mean that the sum of vectors of the system is zero and no two of them are of the same direction).

Remark 6. It is not difficult to present an example of a system

$$\{e_1, e_2, e_3, e_4, e_5\}$$

of nonzero distinct vectors in the space $\mathbf{R}^3$ such that:

(a) $e_1 = 2e_2$,

(b) the plane generated by $\{e_1, e_2, e_3\}$ differs from that of generated by $\{e_4, e_5\}$ (so, the vectors e_1, e_2, e_3, e_4, e_5 are not coplanar);

(c) $e_1 + e_2 + e_3 + e_4 + e_5 = 0$;

(d) there exists no convex polyhedron P in $\mathbf{R}^3$ with a Hamiltonian cycle in the 1-skeleton graph of P formed by all vectors from $\{e_1, e_2, e_3, e_4, e_5\}$.

Remark 7. In the space $\mathbf{R}^3$, there is a system of nonzero distinct vectors

$$\{e_1, e_2, e_3, e_4, e_5, e_6\}$$

which satisfies the assumptions of Theorem 3 and for which these three conditions are also fulfilled:

(a) there exists a permutation ϕ of the set $\{1, 2, \dots, 6\}$ such that the vectors $e_{\phi(1)}, e_{\phi(2)}, \dots, e_{\phi(6)}$ form an oriented Hamiltonian cycle in the 1-skeleton graph of some convex polyhedron $P_1 \subset \mathbf{R}^3$ with 6 vertices;

(b) there exists a permutation ψ of the set $\{1, 2, \dots, 6\}$ such that the vectors $e_{\psi(1)}, e_{\psi(2)}, \dots, e_{\psi(6)}$ form an oriented Hamiltonian cycle in the 1-skeleton graph of some convex polyhedron $P_2 \subset \mathbf{R}^3$ with 6 vertices;

(c) P_1 is combinatorially isomorphic to a trigonal prism in $\mathbf{R}^3$, and P_2 is combinatorially isomorphic to an octahedron in $\mathbf{R}^3$.

In particular, condition (c) shows that it makes no sense to speak on the uniqueness of a polyhedron P in the formulation of Theorem 3.

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