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ON SOME VERSION OF MEASURABLE UNIFORMIZATIONS OF PLANE SETS

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Abstract. It is shown that if G is a non-absolute null subset of the plane $\mathbf{R}^2$, then either G or G^{-1} admits a measurable uniformization in some generalized sense.

Let E be a ground (base) set and let $\mathbf{R}$ denote the real line.

A function $f : E \to \mathbf{R}$ is called relatively measurable if there exists a nonzero σ-finite continuous (i.e., vanishing at all singletons in E) measure μ on E such that f becomes μ-measurable.

Otherwise, f is called absolutely nonmeasurable (see, e.g., [3, 4, 6]).

Throughout this short communication, we identify any function with its graph.

Recall that a topological space T is absolute null if there exists no nonzero σ-finite Borel continuous measure on T (this definition assumes that all singletons in T are Borel subsets of T).

It is provable within the **ZFC** theory that there are uncountable absolute null subspaces of $\mathbf{R}$. Moreover, any Luzin subset of $\mathbf{R}$ or $\mathbf{R}^2$ is absolute null (cf. [1, 2, 7, 9, 10]). However, it is well known that the existence of Luzin sets needs additional set-theoretical assumptions (see [1]).

Remark 1. As was shown by Sierpiński and Zygmund in [12], there exist many functions $h : \mathbf{R} \to \mathbf{R}$ such that for every set $X \subset \mathbf{R}$ of cardinality continuum, the restricted function $h|X$ is not continuous on X. In this context, absolutely nonmeasurable functions acting from $\mathbf{R}$ into $\mathbf{R}$ are rather similar to Sierpiński–Zygmund functions. Namely, if a function $f : \mathbf{R} \to \mathbf{R}$ is absolutely nonmeasurable and Y is any non-absolute null subset of $\mathbf{R}$, then the restricted function $f|Y$ is not continuous and even is not Borel on Y. Note also that:

(a) there exist Sierpiński–Zygmund functions which are not absolutely nonmeasurable;

(b) the assertion that there exist absolutely nonmeasurable functions from $\mathbf{R}$ into $\mathbf{R}$, which are not Sierpiński–Zygmund functions, is consistent with the **ZFC** theory (e.g., holds true under **MA**);

(c) the assertion that there exist absolutely nonmeasurable functions from $\mathbf{R}$ into $\mathbf{R}$, which are simultaneously Sierpiński–Zygmund functions, is consistent with the **ZFC** theory (e.g., holds true under **MA**).

It should be underlined that (a) is provable within the **ZFC** theory, while assertions (b) and (c) cannot be established without using additional set-theoretical hypotheses.

Here, we formulate a complete characterization of absolutely nonmeasurable functions.

Lemma 1. *For a given function $f : E \to \mathbf{R}$, the following two assertions are equivalent:*

(1) *f is absolutely nonmeasurable;*

(2) *the range of f is an absolute null subset of $\mathbf{R}$ and the set $f^{-1}(r)$ is at most countable for each $r \in \mathbf{R}$.*

The proof of this lemma can be found in [3] and [6].

Remark 2. If $E = \mathbf{R}$, then it follows from Lemma 1 that the existence of an absolutely nonmeasurable function $f : \mathbf{R} \to \mathbf{R}$ cannot be proved within the **ZFC** theory. On the other hand, the existence of such f can be established by assuming **MA** (in fact, it suffices to assume the existence of a generalized Luzin set in $\mathbf{R}$ of cardinality continuum).

If G is a subset of the plane $\mathbf{R}^2$ having some nice measurability properties, then there naturally arises the problem of the existence of a function $g : \mathrm{pr}_1(G) \to \mathbf{R}$ entirely contained in G and having

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analogous nice measurability properties. This g is called an appropriate uniformization (or a selector) of G.

The uniformization problem was first formulated in the framework of classical descriptive set theory (cf. [9]). This important problem was extensively investigated from various points of view (see, for example, $[1, 2, 7, 8]$).

In the present note, we intend to describe those sets $G \subset \mathbf{R}^2$ which admit a relatively measurable uniformization. In other words, we wish to find the conditions for G which guarantee the existence of a function

$$g : \mathrm{pr}_1(G) \to \mathbf{R},$$

entirely contained in G and measurable with respect to some nonzero σ-finite continuous measure defined on a σ-algebra of subsets of $\mathrm{pr}_1(G)$.

First, let us formulate one result which shows that in a certain situation all uniformizations of a set $G \subset \mathbf{R}^2$ may be absolutely nonmeasurable (although G belongs to the projective hierarchy of Luzin and Sierpiński).

Theorem 1. *It is consistent with the* **ZFC** *theory that there exists a subset G of the plane $\mathbf{R}^2$ satisfying the following relations:*

(1) *G is a projective set in $\mathbf{R}^2$ and $\mathrm{pr}_1(G) = \mathbf{R}$;*

(2) *$\mathrm{pr}_2(G)$ is an absolute null subspace of $\mathbf{R}$;*

(3) *all horizontal sections of G are at most countable;*

(4) *every function $g : \mathbf{R} \to \mathbf{R}$ contained in G is absolutely nonmeasurable.*

The proof of this statement modifies the argument given in [5] and [6]. In the proof, one delicate circumstance is taken into account, namely, the existence of projective Luzin sets in $\mathbf{R}$ does not contradict the axioms of the **ZFC** theory (actually, such sets exist in the Constructible Universe $\mathbf{L}$ of Gödel).

Lemma 2. *Assume* **MA**. *Let G be a subset of $\mathbf{R}^2$ equipped with a nonzero Borel σ-finite continuous measure μ. Suppose also that all vertical sections of G are of μ-measure zero.*

Then there exists a function $g : \mathrm{pr}_1(G) \to \mathbf{R}$ satisfying the following two conditions:

(1) *g is entirely contained in G;*

(2) *g is μ-thick in G (i.e., g has common points with every μ-measurable subset of G having strictly positive μ-measure).*

Lemma 3. *Preserve the assumptions of Lemma 2 and suppose in addition that all horizontal sections of G are of μ-measure zero.*

Then any function g described in Lemma 2 is relatively measurable.

Lemma 4. *Let E_1 and E_2 be two topological spaces, X be an absolute null subset of E_1, and let for every point $x \in X$, a set Y_x be given which is absolute null in E_2.*

Then the set $Z = \cup\{\{x\} \times Y_x : x \in X\}$ is absolute null in the topological product space $E_1 \times E_2$.

Remark 3. Actually, this lemma slightly generalizes the well-known fact that the topological product of two absolute null spaces is again an absolute null space. The same holds true for the topological product of any finite family of absolute null spaces. In general, the above-mentioned fact fails to be valid for the topological products of countably many absolute null spaces.

Lemma 5. *Let G be a subset of $\mathbf{R}^2$ such that some horizontal section of G is not absolute null.*

Then there exists a relatively measurable function $f : \mathrm{pr}_1(G) \to \mathbf{R}$, entirely contained in G.

Remark 4. One can say more about the function f of Lemma 5. Namely, there is a nonzero σ-finite continuous measure μ on some σ-algebra of subsets of $\mathrm{pr}_1(G)$ such that f is μ-almost constant everywhere on $\mathrm{pr}_1(G)$.

Theorem 2. *Assume* **MA** *and let G be a subset of $\mathbf{R}^2$ which is not absolute null.*

Then the disjunction of these two assertions takes place:

(1) *there exists a relatively measurable function $f : \mathrm{pr}_1(G) \to \mathbf{R}$ entirely contained in G;*

(2) *there exists a relatively measurable function $g : \mathrm{pr}_2(G) \to \mathbf{R}$ entirely contained in G^{-1}.*

Remark 5. In $\mathbf{R}^2$, consider the product set $G = \mathbf{R} \times Y$, where Y is any countably infinite set in $\mathbf{R}$. Clearly, this G is a Borel non-absolute null subset of $\mathbf{R}^2$. Also, it is easy to see that all functions $g : Y \to \mathbf{R}$ are absolutely nonmeasurable. Consequently, in the formulation of Theorem 2 the disjunction of assertions (1) and (2) is necessary.

Remark 6. According to the well-known result of Sierpiński [11], there exists a bijective function

$$f : \mathbf{R} \to \mathbf{R}$$

such that its graph is thick in the plane $\mathbf{R}^2$ with respect to the Lebesgue standard measure on $\mathbf{R}^2$. So, the graph of f is not absolute null in the plane, although all vertical and horizontal sections of f are singletons. Both functions f and f^{-1} are relatively measurable. Moreover, there exist the measures μ and ν on $\mathbf{R}$ which satisfy the following three relations:

(a) both μ and ν extend the Lebesgue standard measure on $\mathbf{R}$;
(b) f is measurable with respect to μ;
(c) f^{-1} is measurable with respect to ν.

Remark 7. Consider any uncountable subset G of $\mathbf{R}^2$ having the following two properties:

(a) all vertical and horizontal sections of G are at most countable;
(b) both projections $\mathrm{pr}_1(G)$ and $\mathrm{pr}_2(G)$ are absolute nulls.

Note that there are many examples of such sets G within the **ZFC** theory. In view of Lemma 1, every uniformization of G (respectively, every uniformization of G^{-1}) is absolutely nonmeasurable. This circumstance implies that for such a G, the disjunction of assertions (1) and (2) of Theorem 2 fails to be true.

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