Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 2, 317–319

ON THE SOLVABILITY OF THE BOUNDARY VALUE PROBLEM FOR ONE CLASS OF HIGHER-ORDER NONLINEAR HYPERBOLIC SYSTEMS

TEONA BIBILASHVILI[1] AND SERGO KHARIBEGASHVILI[1,2]

Abstract. The boundary value problem for one class of higher-order nonlinear hyperbolic systems is considered. The theorems on the existence and uniqueness of solutions of the boundary value problem are proved. The question of the nonexistence of a solution to this problem is also considered.

On a plane of variables x and t, we consider the following fourth−order hyperbolic system

$$\square^2 u_i + f_i(u_1,\dots,u_N) = F_i(x,t), \quad i=1,\dots,N, \tag{1}$$

where $\square := \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}$, $f=(f_1,\dots,f_N)$ and $F=(F_1,\dots,F_N)$ are the given vector functions, while $u=(u_1,\dots,u_N)$ is an unknown N−dimensional vector function, $N \geq 2$.

Denote by $D_T: 0<x<l,\ 0<t<T$ the rectangular domain bounded by the sides $\gamma_1: t=0$, $0\leq x\leq l$, $\gamma_2: t=T$, $0\leq x\leq l$, and $\gamma_3: x=0$, $0\leq t\leq T$, $\gamma_4: x=l$, $0\leq t\leq T$.

For system (1) in the domain D_T, consider the boundary value problem: find in the domain D_T a solution $u=(u_1(x,t),\dots,u_N(x,t))$ of system (1) according to the boundary conditions

$$u|_{\gamma_i} = u_t|_{\gamma_i} = 0, \quad i=1,2; \qquad u|_{\gamma_j} = u_x|_{\gamma_j} = 0, \quad j=3,4. \tag{2}$$

Note that some multidimensional analogues of problem (1), (2) in the scalar case for one equation were considered in [1] and for a system in [2].

Introduce the Hilbert space $\overset{o}{W}{}^1_{2,\square}(D_T)$ as a completion with respect to the norm

$$\|u\|^2_{\overset{o}{W}{}^1_{2,\square}(D_T)} = \int\limits_{D_T} \Big[u^2 + \Big(\frac{\partial u}{\partial x}\Big)^2 + \Big(\frac{\partial u}{\partial t}\Big)^2 + (\square u)^2\Big] dxdt \tag{3}$$

of the classical space

$$\overset{o}{C}{}^k(\overline{D}_T, \partial D_T) := \Big\{u \in C^k(\overline{D_T}) : u|_{\gamma_i} = u_t|_{\gamma_i} = 0,\ i=1,2; \quad u|_{\gamma_j} = u_x|_{\gamma_j} = 0,\ j=3,4\Big\}$$

for $k=2$.

It follows from (3) that if $u \in \overset{o}{W}{}^1_{2,\square}(D_T)$, then $u \in \overset{o}{W}{}^1_2(D_T)$ and $\square u \in L_2(D_T)$. Here, $W^1_2(D_T)$ is the well−known Sobolev space consisting of the elements of $L_2(D_T)$, having the first order generalized derivatives from $L_2(D_T)$, and $\overset{o}{W}{}^1_2(D_T) := \{u \in W^1_2(D_T) : u|_{\partial D_T} = 0\}$, where the equality $u|_{\partial D_T}=0$ is understood in the sense of the trace theory.

Remark 1. Below, the vector function f in system (1) requires that

$$f \in C(\mathbb{R}^{\mathbb{N}}), \quad |f(u)| \leq M_1 + M_2|u|^\alpha, \quad \alpha = \mathrm{const} > 1,\ \ u \in \mathbb{R}^{\mathbb{N}}, \tag{4}$$

where $|\cdot|$ is the norm of the space $\mathbb{R}^n$, $M_i = \mathrm{const} \geq 0$, $i=1,2$. As is known, since the dimension of the domain $D_T \subset \mathbb{R}^2$ is equal to two, the embedding operator $I: W^1_2(D_T) \to L_q(D_T)$ is linear and compact for any fixed $q = \mathrm{const} > 1$. At the same time, the Nemitskii operator $K: L_q(D_T) \to L_2(D_T)$ acting by formula $Ku = f(u)$, where $u \in L_q(D_T)$ and the vector function f satisfies condition (4), is bounded and continuous for $q \geq 2\alpha$. Therefore, if we take $q = 2\alpha$, then the operator

$$K_0 = KI : W^1_2(D_T) \to L_2(D_T)$$

2020 *Mathematics Subject Classification.* 35G30, 35L55.
Key words and phrases. Nonlinear higher−order hyperbolic systems; Boundary value problem, existence; Uniqueness and nonexistence of solutions.

is continuous and compact. Whence, in particular, we find that if $u \in W_2^1(D_T)$, then $f(u) \in L_2(D_T)$.

Definition 1. Let the vector function f satisfy condition (4) and $F \in L_2(D_T)$. The vector function $u \in \overset{o}{W}{}^1_{2,\square}(D_T)$ is said to be a weak generalized solution of problem (1), (2), if for any vector function $\varphi = (\varphi_1, \dots, \varphi_N) \in \overset{o}{W}{}^1_{2,\square}(D_T)$ the integral equality

$$\int\limits_{D_T} \square u \square \varphi dxdt + \int\limits_{D_T} f(u)\varphi dxdt = \int\limits_{D_T} F\varphi dxdt \ \ \forall \varphi \in \overset{o}{W}{}^1_{2,\square}(D_T) \tag{5}$$

is valid.

Note that due to Remark 1, the integral $\int_{D_T} f(u)\varphi dxdt$ in the left−hand side of equality (5) is defined correctly, since from $u \in \overset{o}{W}{}^1_{2,\square}(D_T)$ it follows that $f(u) \in L_2(D_T)$, and since $\varphi \in L_2(D_T)$, therefore $f(u)\varphi \in L_1(D_T)$.

It is easy to verify that the classical solution $u \in \overset{o}{C}{}^4(\overline{D}_T, \partial D_T)$ of problem (1), (2) represents a weak generalized solution according to Definition 1, i.e., it satisfies the integral identity (5), on the other hand, if the weak generalized solution of problem (1), (2) belongs to the class $\overset{o}{C}{}^4(\overline{D}_T, \partial D_T)$, then it will be the classical solution of this problem.

As will be noted below, if the nonlinear vector function f is not required to satisfy other conditions in addition to (4), then problem (1), (2) may not have a solution. At the same time, if the additional condition

$$\lim_{|u|\to\infty} \inf \frac{uf(u)}{|u|^2} \geq 0 \tag{6}$$

is satisfied, where $uf(u) = \sum_{i=1}^N u_i f_i$, $|u|^2 = \sum_{i=1}^N u_i^2$, then an a priori estimate

$$\|u\|_{\overset{o}{W}{}^1_{2,\square}(D_T)} \leq c_1 \|F\|_{L_2(D_T)} + c_2$$

is proved for a weak generalized solution u of problem (1), (2), where the constants $c_1 > 0$ and $c_2 \geq 0$, independent of u and F. Thence, taking into account Remark 1, it follows that there exists a weak generalized solution of problem (1), (2). Thus, the following theorem holds.

Theorem 1. *Let conditions* (4) *and* (6) *be fulfilled. Then for any* $F \in L_2(D_T)$, *problem* (1), (2) *has at least one weak generalized solution* u *in the space* $\overset{o}{W}{}^1_{2,\square}(D_T)$ *in the sense of Definition* 1.

Regarding the uniqueness of a weak generalized solution of the boundary value problem (1), (2), the following theorem is true.

Theorem 2. *Let the vector function* f *satisfy conditions* (4) *and*

$$(f(u) - f(v))(u - v) \geq 0 \quad \forall u, v \in \mathbb{R}^N. \tag{7}$$

Then for any vector function $F \in L_2(D_T)$, *the boundary value problem* (1), (2) *cannot have more than one weak generalized solution* $u = (u_1, \dots, u_N)$ *in the space* $\overset{o}{W}{}^1_{2,\square}(D_T)$ *in the sense of Definition* 1.

Theorems 1 and 2 result in the following

Theorem 3. *Let the vector function* f *satisfy conditions* (4), (6) *and* (7). *Then for any vector function* $F = (F_1, \dots, F_N) \in L_2(D_T)$, *the boundary value problem* (1), (2) *has a unique weak generalized solution* $u = (u_1, \dots, u_N)$ *in the space* $\overset{o}{W}{}^1_{2,\square}(D_T)$ *in the sense of Definition* 1.

Now, let us give one class of vector functions f, when condition (4) is satisfied, but condition (6) is violated, and in this case, for a sufficiently wide class of vector functions $F = (F_1, \dots, F_N) \in L_2(D_T)$, the problem (1), (2) has no weak generalized solution. This class is given by the following formula:

$$f_i(u_1, \dots, u_N) = \sum_{j=1}^N \alpha_{ij} |u_j|^{\beta_{ij}} + b_i, \quad i = 1, \dots, N, \tag{8}$$

where the constants α_{ij}, β_{ij} and b_i satisfy the following inequalities:

$$\alpha_{ij} > 0, \quad \beta_{ij} = \text{const} > 1, \quad \sum_{i=1}^{N} b_i > 0, \quad i, j = 1, \dots, N. \tag{9}$$

The following theorem holds.

Theorem 4. *Let the vector function* $f = (f_1, \dots, f_N)$ *satisfy conditions* (8) *and* (9), $F^0 = (F_1^0, \dots, F_N^0) \in l_2(D_T)$, $G = \sum_{i=1}^{N} F_i^0 < 0$, *and* $F = \gamma F^0$, $\gamma = \text{const} > 0$. *Then there exists a number* $\gamma_0 = \gamma_0(G, \beta_{ij}) > 0$ *such that problem* (1), (2) *has no weak generalized solution* $u \in \overset{o}{W}{}^{1}_{2,\Box}(D_T)$ *in the sense of Definition* 1, *when* $\gamma > \gamma_0$.

References

1. S. Kharibegashvili, B. Midodashvili, On the existence, uniqueness and nonexistence of solutions to a boundary value problem for a quasilinear hyperbolic equation. (Russian) *translated from Ukraïn. Mat. Zh.* **71** (2019), no. 8, 1123–1132; *Ukrainian Math. J.* **71** (2020), no. 8, 1282–1293.
2. S. Kharibegashvili, B. Midodashvili, On the solvability of one boundary value problem for one class of higher-order semilinear hyperbolic systems. *Lith. Math. J.* **62** (2022), no. 3, 360–371.

(Received 22.02.2024)

[1]Georgian Technical University, 77 Kostava Str., Tbilisi, Georgia

[2]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

Email address: teonabibilashvili12@gmail.com

Email address: kharibegashvili@yahoo.com