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CRITERION FOR THE EXISTENCE OF BOUNDED SOLUTIONS ON THE REAL AXIS $\mathbb{R}$ OF LINEAR SYSTEMS OF ORDINARY DIFFERENTIAL EQUATIONS

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Abstract. Effective necessary and sufficient conditions are established for the existence of bounded solutions of systems of linear ordinary differential equations on the real axis. Sufficient conditions are given for existence of bounded solutions satisfying the Nicoletti condition. Moreover, the method of the construction of such solutions is considered. Sufficient conditions for the existence of a unique solution and its positivity are established.

1. Statement of the Problem. Basic Notation and Definitions

For the linear system of the ordinary differential equations

$$\frac{dx}{dt} = P(t)\,x + q(t) \quad \text{for} \quad t \in \mathbb{R} \tag{1.1}$$

we consider the problem on the bounded solution

$$\sup\{\|x(t)\| : t \in \mathbb{R}\} < +\infty, \tag{1.2}$$

where $P = (p_{ik})_{i,k=1}^{n} \in L_{\rm loc}(\mathbb{R}, \mathbb{R}^{n\times n})$, and $q = (q_i)_{i=1}^{n} \in L_{\rm loc}(\mathbb{R}, \mathbb{R}^{n})$.

The same singular and another singular Nicoletti's [4] problems were considered in earlier works [2, 3], where effective sufficient conditions were obtained to guarantee the existence of a bounded solution of system (1.1). As we know, the criterion of the existence of a bounded solution was not investigated in earlier papers.

Analogous question has been investigated in [1] for systems of the so-called generalized ordinary differential equations.

In the present paper, our aim is to obtain a criterion for the existence of bounded solutions and, therefore, eliminate the existing gap.

The use in the paper will be made of the following notation and definitions.

$\mathbb{R} =]-\infty, +\infty[$, $\mathbb{R}_+ = [0, +\infty[$.

$\mathbb{R}^n$ and $\mathbb{R}^{n\times n}$ are the spaces of all real n-vectors $x = (x_i)_{i=1}^{n}$ and $n \times n$-matrices $X = (x_{ij})_{i,j=1}^{n}$ with the standard norms. I_n is the identity $n \times n$-matrix; δ_{ij} is the Kroneker symbol, i.e., $\delta_{ii} = 1$ and $\delta_{ij} = 0$ for $i \neq j$ $(i, j = 1, \dots)$.

$\operatorname{diag}(h_1, , \dots, h_n)$ is a diagonal matrix-functions with diagonal elements $h_1, \dots, h_n$.

$O_{n\times n}$ is the zero $n \times n$ matrix. We designate the zero n vector by O_n, as well.

The inequalities between the matrices are understood componentwise.

$AC_{\rm loc}(\mathbb{R}; \mathbb{R}^n)$ and $AC_{\rm loc}(\mathbb{R}; \mathbb{R}^{n\times n})$ are the sets of all vector-and matrix-functions, respectively, whose restrictions to an arbitrary closed interval from $\mathbb{R}$ are absolutely continuous.

$L_{\rm loc}(\mathbb{R}; \mathbb{R}^n)$ and $L_{\rm loc}(\mathbb{R}; \mathbb{R}^{n\times n})$ are the sets of all locally integrable vector-and matrix-functions, respectively.

A matrix-function has some property when each of its components has the same property.

By a solution of system (1.1) we mean a vector-function $x \in AC_{\rm loc}(\mathbb{R}; \mathbb{R}^n)$ such that

$$x'(t) = P(t)\,x(t) + q(t) \ \text{ for a.a. } \ t \in \mathbb{R}.$$

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1.1. **Formulation of the results.** Let for each $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$,

$$\mathcal{N}_0(t_1, \dots, t_n) = \{i : t_i \in \mathbb{R}\}.$$

Obviously, $\mathcal{N}_0(t_1, \dots, t_n) = \{1, \dots, n\}$ if $t_i \in \mathbb{R}$ $(i = 1, \dots, n)$, and $\mathcal{N}_0(t_1, \dots, t_n) = \varnothing$ if $t_i \in \{-\infty, +\infty\}$ $(i = 1, \dots, n)$.

In the case, where $t_i = -\infty$ $(t_i = +\infty)$, we assume $\operatorname{sgn}(t - t_i) = 1$ for $t \in \mathbb{R}$ $(\operatorname{sgn}(t - t_i) = -1$ for $t \in \mathbb{R})$.

Theorem 1.1. *Problem* (1.1), (1.2) *is solvable if and only if there exist $t_0 \in \mathbb{R}$ and a non-singular matrix-function $H = (h_{ik})_{i,k=1}^n \in AC_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n \times n})$ such that*

$$\sup\{\|H^{-1}(t)\| : t \in \mathbb{R}\} < +\infty, \tag{1.3}$$

$$s_{ik} = \sup\left\{\left|\int_{t_0}^{t} \exp\left(\int_{\tau}^{t} p_{ii}^*(s)ds\right)|p_{ik}^*(\tau)|d\tau\right| : t \in \mathbb{R}\right\} < +\infty$$
$$(i \neq k;\ i, k = 1, \dots, n), \tag{1.4}$$

$$\sup\left\{\left|\int_{t_0}^{t} \exp\left(\int_{\tau}^{t} p_{ii}^*(s)ds\right)|q_i^*(\tau)|d\tau\right| : t \in \mathbb{R}\right\} < +\infty \ \ (i = 1, \dots, n), \tag{1.5}$$

$$\sup\left\{\int_{t_0}^{t} p_{ii}^*(s)ds : t \in \mathbb{R}\right\} < +\infty \ \ (i = 1, \dots, n), \tag{1.6}$$

$$r(S) < 1, \tag{1.7}$$

where $r(S)$ is the spectral radius of S, $P^*(t) = (p_{ik}^*(t))_{i,k=1}^n \equiv \big(H'(t) + H(t)P(t)\big)H^{-1}(t)$, $q^*(t) = (q_i^*(t))_{i=1}^n \equiv H(t)q(t)$, and $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1, \dots, n)$.

From the proof of Theorem 1.1, it follows that in the theorem, we can assume without loss of generality that $H(t) \equiv X^{-1}(t)$, where X is the fundamental matrix of system (1.1). So, in this case, Theorem 1.1 has the following evident form:

Theorem 1.1′. *Problem* (1.1), (1.2) *is solvable if and only if*

$$\sup\left\{\|X(t)\| + \left|\int_{0}^{t} \|X^{-1}(\tau)q(\tau)\|d\tau\right| : t \in \mathbb{R}\right\} < +\infty.$$

We give here a simple generalization of the known results from [3].

Theorem 1.2. *Let $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ and let a non-singular matrix-function $H = (h_{ik})_{i,k=1}^n \in AC_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n \times n})$ be such that conditions* (1.3), (1.7) *and*

$$s_{ik} = \sup\left\{\left|\int_{t_i}^{t} \exp\left(\int_{\tau}^{t} p_{ii}^*(s)ds\right)|p_{ik}^*(\tau)|d\tau\right| : t \in \mathbb{R}\right\} < +\infty$$
$$(i \neq k;\ i, k = 1, \dots, n), \tag{1.8}$$

$$\sup\left\{\left|\int_{t_i}^{t} \exp\left(\int_{\tau}^{t} p_{ii}^*(s)ds\right)|q_i^*(\tau)|d\tau\right| : t \in \mathbb{R}\right\} < +\infty \ \ (i = 1, \dots, n), \tag{1.9}$$

$$\sup\left\{\int_{t_i}^{t} p_{ii}^*(s)ds : t \in \mathbb{R}\right\} < +\infty \ \ \textit{for} \ \ i \in \mathcal{N}_0(t_1, \dots, t_n) \tag{1.10}$$

hold, where the matrix- and vector-functions P^* and q^* are defined as in Theorem 1.1*, $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$. Let, moreover,*

$$h_{ik}(t_i) = 0 \ \textit{ for } \ i \in \mathcal{N}_0(t_1,\dots,t_n) \ \ (i \neq k;\ i,k = 1,\dots,n). \tag{1.11}$$

Then, for every $c_i \in \mathbb{R}$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$, system (1.1) *has at last one bounded on $\mathbb{R}$ solution $(x_i)_{i=1}^n$ satisfying the condition*

$$x_i(t_i) = c_i \ \textit{ for } \ i \in \mathcal{N}_0(t_1,\dots,t_n). \tag{1.12}$$

Theorem 1.2 has the following form if

$$H(t) \equiv \operatorname{diag}\bigg(\exp\bigg(-\int\limits_0^t p_{11}(s)ds\bigg),\dots,\exp\bigg(-\int\limits_0^t p_{nn}(s)ds\bigg)\bigg).$$

Corollary 1.1. *Let $t_i \in \mathbb{R} \cup \{-\infty,+\infty\}$ $(i = 1,\dots,n)$ be such that conditions* (1.7) *and*

$$s_{ik} = \sup\bigg\{\bigg|\int\limits_{t_i}^t \exp\bigg(\int\limits_0^\tau (p_{kk}(s) - p_{ii}(s))ds\bigg)|p_{ik}(\tau)|d\tau\bigg| : t \in \mathbb{R}\bigg\} < +\infty$$
$$(i \neq k;\ i,k = 1,\dots,n), \tag{1.13}$$

$$\sup\bigg\{\bigg|\int\limits_{t_i}^t \exp\bigg(-\int\limits_0^\tau p_{ii}(s)ds\bigg)|q_i(\tau)|d\tau\bigg| : t \in \mathbb{R}\bigg\} < +\infty \ \ (i = 1,\dots,n) \tag{1.14}$$

hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i,k = 1,\dots,n)$. Then, for every $c_i \in \mathbb{R}$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$, problem (1.1), (1.12) *has at last one bounded on $\mathbb{R}$ solution.*

Theorem 1.2 has the following form if $H(t) \equiv I_n$.

Corollary 1.2. *Let $t_i \in \mathbb{R} \cup \{-\infty,+\infty\}$ $(i = 1,\dots,n)$ be such that conditions* (1.7) *and*

$$s_{ik} = \sup\bigg\{\bigg|\int\limits_{t_i}^t \exp\bigg(\int\limits_\tau^t p_{ii}(s)ds\bigg)|p_{ik}(\tau)|d\tau\bigg| : t \in \mathbb{R}\bigg\} < +\infty$$
$$(i \neq k;\ i,k = 1,\dots,n), \tag{1.15}$$

$$\sup\bigg\{\bigg|\int\limits_{t_i}^t \exp\bigg(\int\limits_\tau^t p_{ii}(s)ds\bigg)|q_i(\tau)|d\tau\bigg| : t \in \mathbb{R}\bigg\} < +\infty \ \ (i = 1,\dots,n), \tag{1.16}$$

$$\sup\bigg\{\int\limits_{t_i}^t p_{ii}(s)ds : t \in \mathbb{R}\bigg\} < +\infty \ \textit{ for } \ i \in \mathcal{N}_0(t_1,\dots,t_n) \tag{1.17}$$

hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i,k = 1,\dots,n)$. Then, for every $c_i \in \mathbb{R}$ $(i \in \mathcal{N}_0(t_1,\dots,t_n))$, problem (1.1), (1.12) *has at last one bounded on $\mathbb{R}$ solution.*

Corollary 1.2 is proved in [3].

If $\mathcal{N}_0(t_1,\dots,t_n) = \varnothing$, condition (1.17) is eliminated and the theorem has the following form:

Corollary 1.3. *Let $t_i \in \{-\infty,+\infty\}$ $(i = 1,\dots,n)$ be such that conditions* (1.7), (1.15) *and* (1.16) *hold, where $S = (s_{ik})_{i,k=1}^n$, $s_{ii} = 0$ $(i = 1,\dots,n)$. Then system* (1.1) *has at last one bounded solution on $\mathbb{R}$.*

Remark 1.1. If $\mathcal{N}_0(t_1,\dots,t_n) = \{1,\dots,n\}$, i.e., $t_i \in \mathbb{R}$ for each $i \in \{1,\dots,n\}$, then condition (1.12) is the classical Cauchy–Nicoletti one. If, in addition, $t_1 = \dots = t_n$ we have the Cauchy problem. Moreover, if $\mathcal{N}_0(t_1,\dots,t_n) = \varnothing$, conditions (1.10) and (1.12) are eliminated and the results have the some forms as those given below.

Theorem 1.2′. *Let $t_i \in \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ and a non-singular matrix-function $H = (h_{ik})_{i,k=1}^n \in AC_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n\times n})$ be such that conditions* (1.3), (1.7), (1.8) *and* (1.9) *hold, where the matrix- and the vector-functions P^* and q^* and the matrix S are defined as in Theorem* 1.2. *Then system* (1.1) *has at last one solution bounded on $\mathbb{R}$.*

Corollary 1.4. *Let the conditions of Theorem* 1.2 *and*

$$p_{ik}^* \operatorname{sgn}(t - t_i) \geq 0 \ \text{ and } \ q_i^* \operatorname{sgn}(t - t_i) \geq 0 \ \text{ for a.a. } \ t \in \mathbb{R} \ \ (i, k = 1, \dots, n) \tag{1.18}$$

hold. Then, for every $c_i \in \mathbb{R}_+$ $(i \in \mathcal{N}_0(t_1, \dots, t_n))$, problem (1.1), (1.12) *has at last one solution x such that*

$$H(t)\, x(t) \geq O_n \ \text{ for } \ t \in \mathbb{R}. \tag{1.19}$$

If $\mathcal{N}_0(t_1, \dots, t_n) = \varnothing$, then Corollary 1.4 has the following form:

Corollary 1.4′. *Let the conditions of Theorem* 1.2′ *and* (1.18) *hold. Then problem* (1.1), (1.12) *has at least one bounded on $\mathbb{R}$ solution satisfying condition* (1.19).

Remark 1.2. Only the fulfillment of conditions of Theorem 1.2′ does not guarantee the uniqueness of a nonnegative solution, i.e., under the condition of the theorem, system (1.1) may be a solution whose components have differing signs. The corresponding example is constructed in [3] (see, p. 189, Remark 6.6).

Theorem 1.3. *Let $t_i \in \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ and a non-singular matrix-function $H = (h_{ik})_{i,k=1}^n \in AC_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n\times n})$ be such that conditions* (1.3), (1.7)–(1.10) *and*

$$\liminf_{t\to t_i} \int_t^0 p_{ii}^*(s) = -\infty \ \text{ for } \ i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n)$$

hold, where the matrix- and the vector-functions P^* and q^* and the matrix S are defined as in Theorem 1.1. *Then, for every $c_i \in \mathbb{R}$ $(i \in \mathcal{N}_0(t_1, \dots, t_n))$, problem* (1.1), (1.12) *has the unique and bounded on $\mathbb{R}$ solution $x = (x_i)_{i=1}^n$ and*

$$\|H(t)\, x(t) - x_m^*(t)\| \leq \rho_0 \alpha^m \ \text{ for } \ t \in \mathbb{R} \ \ (m = 1, 2, \dots),$$

where $x_m^* = (x_{im}^*)_{i=1}^n$ $(m = 0, 1, \dots)$ is the sequence of the vector-functions whosse components are defined by

$$x_{i0}^*(t) \equiv 0, \ \ x_{im}^*(t) \equiv u_i^*(t)$$
$$+ \sum_{k=1,\, k\neq i}^n \int_{t_i}^t \exp\bigg(\int_\tau^t p_{ii}^*(s)ds\bigg) p_{ik}^*(\tau) x_{k\,m-1}^*(\tau) d\tau \ \ (i = 1, \dots, n; \ m = 1, 2, \dots),$$

the functions u_i^* $(i = 1, \dots, n)$ are defined by

$$u_i^*(t) \equiv c_i \exp\bigg(\int_{t_i}^t p_{ii}^*(s)ds\bigg) + \int_{t_i}^t \exp\bigg(\int_\tau^t p_{ii}^*(s)ds\bigg) q_i^*(\tau)d\tau \ \text{ for } \ i \in \mathcal{N}_0(t_1, \dots, t_n),$$

$$u_i^*(t) \equiv \int_{t_i}^t \exp\bigg(\int_\tau^t p_{ii}^*(s)ds\bigg) q_i^*(\tau)d\tau \ \text{ for } \ i \in \{1, \dots, n\} \setminus \mathcal{N}_0(t_1, \dots, t_n),$$

and $\rho_0 > 0$ and $\alpha \in]0, 1[$ are the constant numbers, independent of m.

Corollary 1.5. *Let $t_i \in \mathbb{R} \cup \{-\infty, +\infty\}$ $(i = 1, \dots, n)$ and a non-singular matrix-function $H = (h_{ik})_{i,k=1}^n \in AC_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n\times n})$ be such that conditions* (1.3), (1.11) *and*

$$p_{ii}^*(t) \operatorname{sgn}(t - t_i) \leq \eta_{ii}, \ \ |p_{ik}^*(t)| \leq \eta_{ik} \ \ (i \neq k; \, i, k = 1, \dots, n) \tag{1.20}$$

hold on $\mathbb{R}$, where η_{ik} $(i,k=1,\dots,n)$ are the constants such that the real parts of characteristic values of the matrix $\Theta=(\eta_{ik})_{i,k=1}^n$ are negative. Let, moreover,

$$\sup\left\{\int\limits_t^{t+1}|q_i^*(s)|ds : t\in\mathbb{R}\right\}<\infty \ \ (i=1,\dots,n). \tag{1.21}$$

Then the conclusion of Theorem 1.3 *is true.*

Theorem 1.3 and Corollary 1.6 have for $\mathcal{N}_0(t_1,\dots,t_n)=\varnothing$ the following forms:

Theorem 1.3′. *Let $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$ and a non-singular matrix-function $H=(h_{ik})_{i,k=1}^n\in AC_{\rm loc}(\mathbb{R};\mathbb{R}^{n\times n})$ be such that conditions* (1.3), (1.7)–(1.9) *and*

$$\liminf_{t\to t_i}\int\limits_t^0 p_{ii}^*(s)=-\infty \ \ for \ \ i\in\{1,\dots,n\}$$

hold, where the matrix– and vector-functions P^* and q^* and the matrix S are defined as in Theorem 1.1. *Then system* (1.1) *has the unique and bounded on $\mathbb{R}$ solution $x=(x_i)_{i=1}^n$, and*

$$\|H(t)x(t)-x_m^*(t)\|\le\rho_0\alpha^m \ \ for \ \ t\in\mathbb{R} \ \ (m=1,2,\dots),$$

where $x_m^*=(x_{im}^*)_{i=1}^n$ $(m=0,1,\dots)$ is the sequence of vector-functions whose components are defined by

$$x_{i0}^*(t)\equiv 0,\ \ x_{im}^*(t)\equiv\int\limits_{t_i}^t\exp\bigg(\int\limits_\tau^t p_{ii}^*(s)ds\bigg)q_i^*(\tau)d\tau$$

$$+\sum_{k=1,\,k\neq i}^n\int\limits_{t_i}^t\exp\bigg(\int\limits_\tau^t p_{ii}^*(s)ds\bigg)p_{ik}^*(\tau)x_{k\,m-1}^*(\tau)d\tau \ \ (i=1,\dots,n;\ m=1,2,\dots),$$

and $\rho_0>0$ and $\alpha\in]0,1[$ are the constant numbers, independent of m.

Corollary 1.5′. *Let $t_i\in\{-\infty,+\infty\}$ $(i=1,\dots,n)$ and a non-singular matrix-function $H=(h_{ik})_{i,k=1}^n\in AC_{\rm loc}(\mathbb{R};\mathbb{R}^{n\times n})$ be such that conditions* (1.3) *and* (1.20) *hold on $\mathbb{R}$, where η_{ik} $(i,k=1,\dots,n)$ are the constants such that the real parts of characteristic values of the matrix $\Theta=(\eta_{ik})_{i,k=1}^n$ are negative. Let, moreover, condition* (1.21) *hold. Then the conclusion of Theorem* 1.3′ *is true.*

From Theorems 1.3 and 1.3′ and Corollaries 1.5 and 1.5′ immediately follow the following propositions.

Corollary 1.6. *Let the conditions of Theorem* 1.3 *or Corollary* 1.5 *and condition* (1.18) *be fulfilled. Then, for every $c_i\in\mathbb{R}_+$ $(i\in\mathcal{N}_0(t_1,\dots,t_n))$, system* (1.1) *has the unique boundary solution on $\mathbb{R}$, satisfying condition* (1.12), *and this solution is nonnegative.*

Corollary 1.6′. *Let the conditions of Theorem* 1.5′ *or Corollary* 1.5′ *and condition* (1.18) *hold. Then system* (1.1) *has the unique boundary solution on $\mathbb{R}$ and this solution is nonnegative.*

Remark 1.3. If H is the diagonal matrix described above, then theorems and corollaries have the forms, where condition (1.6) is eliminated, and conditions (1.3), (1.4) and (1.5) have the forms of conditions (1.13) and (1.14), respectively.

2. Short Description of the Proofs

The results given above, with the exception of Theorem 1.1, were obtained in [3] for $H(t)\equiv I_n$. We will need some from the last paper for the matrix-and vector-functions P^* and q^* defined as in Theorem 1.1. We use also the following simple lemma.

Lemma 2.1. *Let the nonsingular matrix-function* $H = (h_{ik})_{i,k=1}^{n} \in AC_{\rm loc}(\mathbb{R};\mathbb{R}^{n\times n})$ *be such that condition* (1.3) *hold. Then problem* (1.1), (1.2) *is solvable if and only if the system*

$$\frac{dy}{dt} = P^*(t)\,y + q^*(t), \quad \text{for} \quad t\in\mathbb{R}, \tag{2.1}$$

is solvable under the condition

$$\sup\{\|y(t)\| : t\in\mathbb{R}\} < +\infty,$$

where $P^*(t)$ *and* $q^*(t)$ *are defined as in Theorem* 1.1. *Moreover, if* $t_i \in \mathbb{R}\cup\{-\infty,+\infty\}$ $(i=1,\dots,n)$ *are such that condition* (1.11) *holds, then the solvability of problem* (1.1), (1.12) *is equivalent to that of system* (2.1) *under the condition*

$$y_i(t_i) = c_i\, h_{ii}(t_i) \ \text{ for } \ i\in\mathcal{N}_0(t_1,\dots,t_n),$$

Consider Theorem 1.1. Owing to Lemma 2.1 and Corollary 1.2, problem (1.1), (1.2) is solvable.

Now, consider the necessity question. By Corollary 1.2, the homogeneous system

$$\frac{dx}{dt} = P(t)\,x \quad \text{for} \quad t\in\mathbb{R}, \tag{2.2}$$

under condition (1.2), has at least a solution $\overline{x}_i$ satisfying the condition

$$x(t_0) = c_i \tag{2.3}$$

for every $i=1,\dots,n$, where $c_i = (\delta_{il})_{l=1}^{n}$ (the Kroneker symbol).

Let $X(t) \equiv (\overline{x}_1(t),\dots,\overline{x}_n(t))$. Then due to (2.3) we have $\det X(t_0) = 1$ and, therefore, $X(t)$ is a fundamental matrix of system (2.2). Moreover, by (1.2) we find

$$\sup\{\|X(t)\| : t\in\mathbb{R}\} < +\infty.$$

Let now $H(t) \equiv X^{-1}(t)$. Then by the last estimate matrix-function H satisfies estimate (1.3). In addition, by property of fundamental matrix X, we get $H'(t) + H(t)P(t) \equiv O_{n\times n}$. So, $P^*(t) \equiv O_{n\times n}$ and all conditions (1.4)–(1.7) are fulfilled.

Due to Lemma 2.1, Theorem 1.2 evidently follows from Corollary 1.2.

Corollary 1.1 follows immediately from Theorem 1.2 because for the diagonal matrix given in the proposition, we conclude that $p^*_{ii}(t) \equiv 0$ $(i=1,\dots,n)$, $p^*_{ik}(t) \equiv \exp\Big(\int\limits_0^t (p_{kk}(s) - p_{ii}(s)ds\Big)p_{ik}(t)$ $(i\neq k,\ i,k=1,\dots,n)$ and the matrix $S = (s_{ik})_{i,k=1}^{n}$ has the form given in the corollary. □

Theorems $1.2, 1.2', 1.3, 1.3'$ and all corollaries follow immediately from Lemma 2.1 and from the corresponding results circumscribed above [3], i.e., for the case, when $H(t) \equiv I_n$.

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