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ASYMPTOTIC ANALYSIS OF ACOUSTIC WAVE DYNAMICS IN UNIFORM SHEAR FLOW

GRIGOL GOGOBERIDZE AND NATIA KEVLASHVILI

Abstract. We study linear equations describing the dynamics of acoustic waves and vortices in a uniform shear flow. Using the methods of asymptotic analysis, we derive Liouville–Green's asymptotic solutions. We show that in the flow with moderate and high shear rate there exist, besides the standard adiabatic dynamics, two additional phenomena: acoustic wave over-reflection and wave generation by vortices. Original asymptotic method is developed for derivation of generated acoustic wave intensity. Detailed analytical study of the problem is performed and the main quantitative and qualitative characteristics of the processes are obtained and analyzed.

1. Introduction

Linear mechanisms of wave generation by vortices in inhomogeneous flows have been studied intensively by different authors [2–4, 6, 9, 10, 13]. Many physical aspects of the phenomenon were analyzed in detail and interesting applications of the phenomena were proposedin the framework of these studies. However, the lack of quantitative analysis impedes future progress in applications to the concrete problems. The aim of the present paper is to develop analytical methods for studying the linear mechanism of wave generation by vortices. With this purpose in mind, we study the simplest system: the linear dynamics of perturbations in two dimensional constant shear flow. Dynamics of the Spatial Fourier harmonics (SFH) of linear perturbations in such a flow is governed by the following equations [4, 6]:

$$\frac{d\rho}{dt} = u + \gamma(t)v, \tag{1.1}$$

$$\frac{du}{dt} = -Av - \rho, \tag{1.2}$$

$$\frac{dv}{dt} = -\gamma(t)\rho. \tag{1.3}$$

Here ρ is the dimensionless density perturbation normalized by background density, u and v are dimensionless perturbations of the parallel and perpendicular velocity components, respectively, A is the dimensionless shear parameter, t is the dimensionless time and $\gamma(\tau) = k_y/k_x - At$, where k_x and k_y are parallel and perpendicular wave numbers, respectively.

Combining equations (1.1)–(1.3), one can get the following algebraic relation [4, 6]:

$$v - \gamma(\tau)u - A\rho = I,$$

where I is some constant of integration. With the use of this relation, the set of equations (1.1)–(1.3) can be reduced to the one second-order differential equation:

$$\frac{d^2u}{dt^2} + \omega^2(t)u = -\gamma(t)I, \tag{1.4}$$

where $\omega^2(t) = 1 + \gamma^2(t)$.

A general solution of this equation is the sum of the special solution of this equation and the general solution of the corresponding homogeneous equation

$$\frac{d^2u}{dt^2} + \omega^2(t)u = 0. \tag{1.5}$$

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Thus, if u_1 and u_2 are any independent solutions of the last equation, the general solution of equation (1.4) can be written as

$$u(t) = C_1 u_1(t) + C_2 u_2(t) + \frac{I}{W} \int_{-\infty}^{t} \gamma(t_1)[u_1(t)u_2(t_1) - u_1(t_1)u_2(t)]dt_1, \tag{1.6}$$

where W is the Wronskian of the linear solutions. $C_{1,2}$, as well as I, are defined by the initial conditions of the problem. It has to be noted that equation (1.5) is formally analogous to the Schrödinger equation that describes the problem of the so-called over-barrier reflection [11].

Further analysis becomes straightforward if the condition of Liouville–Green approximation is fulfilled, i.e., when $\omega^2(t)$ is a slowly varying function of time: $d\omega(t)/dt \ll \omega^2(t)$. Using the definition of $\omega^2(t)$ this condition takes the form

$$A|\gamma(t)| \ll [1 + \gamma^2(t)]^{3/2}. \tag{1.7}$$

Condition (1.7) holds for arbitrary γ if $A \ll 1$. If condition (1.7) is satisfied, then the asymptotic solutions of equation (1.5) given by the Liouville–Green method have the form [5, 7, 8, 12]

$$\tilde{u}_{1,2} = \frac{1}{\sqrt{\omega(t)}} e^{\pm i \int \omega(t)dt}. \tag{1.8}$$

Physically, these solutions correspond to shear modified acoustic waves that have positive and negative phase velocities along the x-axis [4, 6].

In the Liouville–Green approximation, the solution of the inhomogeneous equation (1.4) is also well known [12]:

$$u_{inh} \equiv I\tilde{u}_3 = I \sum_{m=0}^{\infty} A^m y_m(t), \tag{1.9}$$

with

$$y_0(t) = -\frac{\gamma(t)}{\omega^2(t)}, \quad y_{2n-1}(t) = 0, \quad y_{2n}(t) = -\frac{1}{\omega^2(t)} \frac{\partial^2 y_{2n-2}}{\partial \gamma^2} \tag{1.10}$$

and describes adiabatic evolution of vortical perturbations [6]. Thus, under condition (1.7), the general solution can be approximated as

$$u \approx C_1\tilde{u}_1 + C_2\tilde{u}_2 + I\tilde{u}_3;$$

it describes an independent evolution of different modes in the flow. $C_{1,2}$ and I represent the intensities of corresponding perturbations.

If $A \ll 1$, condition (1.7) holds for arbitrary $\gamma(t)$. In this case, the evolution of perturbations is adiabatic, i.e., the intensities $C_{1,2}$ and I remain constant. Comprehensive study of the adiabatic evolution of perturbations in the uniform shear flow has been performed by several authors [4, 6]. In the present paper, we focus on the non-adiabatic evolution of perturbations, i.e., we study the dynamics of perturbations in the flow with a relatively high normalized shear parameter A. In this case, the Liouville–Green condition (1.7) fails in the neighborhood of the point $\gamma(t) = 0$, but it remains valid for $|\gamma(t)| \gg \sqrt{A}$. The problem of asymptotic analysis can be formulated in the usual manner [5, 12]: assume initially that at $t = 0$, $\gamma(0) \equiv k_y/k_x \gg \sqrt{A}$ and the intensities of acoustic and vortical perturbations are $C_{1,2}$ and I, respectively. The problem is to determine intensities of the same perturbations $D_{1,2}$ and J after passing through the area of non-adiabatic evolution for $\gamma(t) \ll -\sqrt{A}$, i.e., for $t > 2k_y/k_x\sqrt{A}$.

First of all, we note that contrary to $C_{1,2}$, which are adiabatic invariants (i.e., they remain constant only during the adiabatic evolution), I is an absolute invariant of the governing set of equations (1.1)–(1.3) and therefore $J \equiv I$. Physically, this means that the intensity of vortical perturbations remain constant regardless of the shear parameter. When $I = 0$ initially, we have only acoustic waves. In the flow with a relatively strong shear, the intensities of the waves do not remain constant. It can be shown that during the non-adiabatic evolution the total energy of acoustic perturbations always increases (on the expense of the flow energy). This phenomenon is known as over-reflection (see, e.g., [7, 9] and references therein). Comprehensive study of the over-reflection of acoustic waves in the uniform shear flow has been performed in Ref. [6] and will not be presented here. In the next section, we study another non-adiabatic phenomenon, namely, acoustic wave generation by vortices.

2. Generation of Acoustic Waves by Vortices

The physical features of the generation of acoustic waves by vortices were already intensively studied by different authors [2–4,6]. Our study is concentrated on finding an analytical expression of intensity of generated acoustic waves.

Introducing a new variable $z = -\gamma(t)$, equation (1.4) can be presented as

$$\frac{d^2u}{dz^2} + \frac{1}{A^2} f(z)u = \frac{I}{A^2} z,$$

where $f(z) = 1 + z^2$. In this notation, equation (1.6) takes the form

$$u(z) = C_1 u_1(z) + C_2 u_2(z) + I u_3(z),$$

where

$$u_3(z) = \frac{1}{2iA} \int_{-\infty}^{z} \zeta Q(z, \zeta) d\zeta, \tag{2.1}$$

$Q(z, \zeta) \equiv u_1(z)u_2(\zeta) - u_1(\zeta)u_2(z)$ and $u_{1,2}$ are the solutions of the corresponding homogeneous equation

$$\frac{d^2u}{dz^2} + \frac{1}{A^2} f(z)u = 0, \tag{2.2}$$

chosen such that for $|z| \gg \sqrt{A}$ they converge to the Liouville–Green solutions (1.8).

Since we are interested in studying generation of acoustic waves by vortices, we assume that initially, at $z \ll -\sqrt{A}$, there exist only vortical perturbations $C_{1,2} = 0$ and $I \neq 0$. We need to determine the intensities of acoustic waves for $z \gg \sqrt{A}$. The generation process takes place in some vicinity of the point $z = -\gamma(t) = 0$. For $z \gg A^{1/2}$, we can rewrite equation (2.1) as

$$u(z) = iG_1 I u_1 + iG_2 I u_2 - \frac{I}{2iA} \int_{z}^{\infty} \zeta Q(z, \zeta) d\zeta, \tag{2.3}$$

where

$$G_{1,2} = \frac{\mp 1}{2A} \int_{-\infty}^{\infty} z u_{2,1} dz. \tag{2.4}$$

Separating solutions of the homogenous equation $u_{1,2}$ into odd and even parts, one can readily show that

$$\int_{-\infty}^{-z} \zeta Q(z, \zeta) d\zeta = \int_{z}^{\infty} \zeta Q(z, \zeta) d\zeta.$$

Thus the first two terms in equation (2.3) correspond to the generated acoustic waves with intensities $|G_{1,2} I|$ and therefore we interpret $|G_{1,2}|$ as acoustic wave generation coefficients. The aim of our further mathematical analysis is to find approximate expressions for the generation coefficients $G_{1,2}$.

We divide this derivation into two steps. First, we derive the solutions of homogeneous equation (2.2) in the form, suitable for our purposes, and afterwards derive expressions for the intensities.

Neither the Liouville–Green solutions (1.8), nor the formal exact solutions of the homogenous equation with application of the parabolic cylinder functions [1,6] are useful for the evaluation of $G_{1,2}$. As it is well known, the accuracy of the Liouville–Green solutions is not enough for the derivation of correct results in this kind of problems [5, 12]. The usage of the formal exact solutions does not allow us to get an exact evaluation of integrals in equation (2.4). Instead, we use another approximate solutions of equation (2.2), that is solutions in a form of formal series [12]. It is well known [12] that the Liouville–Green solutions represent the first term of the solutions in a form of formal series. The accuracy of those solutions in the limit $A \to 0$ is $O(A^{\infty})$. In the next subsection, we obtain the solutions of equation (2.2) in the form of formal series, and afterwards these solutions will be used for the evaluation of integrals in the expressions of generation coefficients (2.4).

2.1. **Solution in the form of the formal series.** Using the standard Liouville transform

$$\xi \equiv \int_0^z f(\zeta)d\zeta, \; V \equiv f^{1/4}u, \tag{2.5}$$

equation (1.5) can be rewritten as

$$\frac{d^2V}{d\xi^2} + \Big(\frac{1}{A^2} - \psi\Big)V = 0, \tag{2.6}$$

with $\psi = -f^{3/4}d^2(f^{-1/4})/dz^2$. Substituting the expansion

$$V_1 = e^{i\xi/A}\sum_{m=0}^{\infty} B_m(\xi)A^m \tag{2.7}$$

into equation (2.6), one can readily obtain the following recurrence relation for the expansion coefficients B_m:

$$B_m(\xi) = \frac{-1}{2i}\frac{dB_{m-1}}{d\xi} + \frac{1}{2i}\int \psi B_{m-1}d\xi,$$

or equivalently,

$$B_m(z) = \frac{-1}{2i}f^{-1/2}\frac{dB_{m-1}}{dz} + \frac{1}{2i}\int \Lambda B_{m-1}dz, \tag{2.8}$$

where

$$\Lambda \equiv \frac{4ff'' - 5(f')^2}{32f^{5/2}} = \frac{2-3z^2}{8(1+z^2)^{5/2}}, \tag{2.9}$$

the prime denotes z derivative, and $B_0 \equiv 1$.

The second independent solution is

$$V_2 = e^{-i\xi/A}\sum_{m=0}^{\infty}(-1)^m B_m(\xi)A^m. \tag{2.10}$$

Taking into consideration relations (2.5) it can be easily verified that the first term of the expansions (2.7) and (2.10) represent the standard Liouville–Green solutions (1.8). Note that the series given by equations (2.7) and (2.10) are convergent at least for such values of z and A that satisfy condition (1.7).

Combining equations (2.8) and (2.9), for the coefficients B_m, we derive

$$B_m(z) = \frac{b_m}{(1+z^2)^{3m/2}}z^{\frac{1-(-1)^m}{2}} + \frac{p_m(z)}{(1+z^2)^{3m/2-1}}. \tag{2.11}$$

Here, $p_m(z)$ is some polynomial, $b_0 = 1$ and $b_{m>0}$ satisfies the following recurrence relations:

$$b_m = (-1)^m i b_{m-1}\Big[\frac{3}{2}(m-1) + \frac{5}{24m}\Big]. \tag{2.12}$$

As we can see in the next subsection, only the first term in equation (2.11) contributes to the leading term of the generation coefficient in the limit $A \to 0$.

2.2. **The generation coefficient.** Now, we derive the leading term of asymptotics of the generation coefficients (2.4) in the limit $A \to 0$.

Using equations (2.7) and (2.10), it can be readily derived that $-G_1 = G_2 \equiv G$. Changing the variable of integration to ξ and substituting equation (2.7), the expressions of the generation coefficients (2.4) can be rewritten as:

$$G = \frac{1}{2A} \int_{-\infty}^{\infty} \frac{z(\xi)}{f^{3/4}(\xi)} \left(\sum_{m=0}^{\infty} B_m(\xi) A^m \right) e^{i\xi/A} d\xi. \tag{2.13}$$

Consider the integrand in the upper half of the complex ξ-plane. The only irregularity of the integrand is a brunch point at $\xi_0 = i\delta/2$, or equivalently, at $z(\xi_0) = i$, where δ is the standard phase integral. It is well known [12] that to construct single valued analytical continuation of the integrand in the upper half of the complex ξ-plane, one needs to make brunch cut that starts at the branch point ξ_0 and ends either at another brunch point, or at infinity. We choose to make brunch cut that tends to infinity along positive direction of the imaginary ξ-axis. Consider the integral (2.13) on the closed path γ that consists of the real ξ-axis, two quarters of a circle and a path ℓ that starts at $+i\infty$ along the left-hand side of the brunch cut, turns over ξ_0 and returns to $+i\infty$ along the right-hand side of the brunch cut. If one tends the radius of the circle to infinity, then it is easy to see that the integral on the both quarters of the circle tends to zero exponentially. Therefore, according the Cauchy theorem, the integral along the real ξ-axis is equal to the integral along ℓ. Taking into account that in the neighborhood of ξ_0, $z(\xi_0) = i$ and

$$f^{3/4} = \left[3i(\xi - i\delta/2)\right]^{1/2},$$

the leading term of asymptotics in equation (2.13) along ℓ, as well as the order of the remainder term, can be obtained by using Watson's lemma for the loop integrals [12]. After straightforward calculations one can obtain

$$G = A^{-1/2} e^{-\pi/4A} \left[\frac{2\pi}{3^{1/2}} \sum_{m=0}^{\infty} \frac{|b_m|}{3^m \Gamma\left(m + \frac{1}{2}\right)} + O(A^{2/3}) \right]. \tag{2.14}$$

As it was mentioned above, the accuracy of the Liouville–Green solutions is not enough to derive a correct expression of generation coefficient. Equation (2.14) shows that all terms of the solution in the form of formal series (2.7) contribute to the leading term of the asymptotics of generation coefficient (2.13).

Taking into account equation (2.12), it can be easily shown that the number series in equation (2.14) is rapidly convergent. Numerical evaluation of the series yields

$$\frac{2\pi}{3^{1/2}} \sum_{m=0}^{\infty} \frac{|b_m|}{3^m \Gamma\left(m + \frac{1}{2}\right)} \approx 1.2533,$$

and therefore

$$G \approx 1.2533 A^{-1/2} e^{-\pi/4A}. \tag{2.15}$$

The derived expression for the leading asymptotic term of the generation coefficient represents the main result of this paper.

3. Discussion and Conclusions

It has to be emphasized that the presented method of derivation of acoustic wave generation coefficient (2.15) is quite general. It can be successfully applied to the problems that are governed by the equation similar to equation (1.4):

$$\frac{d^2u}{dt^2} + \omega^2(t) u = \gamma(t) I,$$

with $\omega^2(t)$ and $\gamma(t)$ such that:

(a) $\omega^2(t)$ has only a pair of complex conjugated turning points in the complex t-plane;
(b) $\omega^2(t)$ satisfies the condition of adiabatic evolution on the real t-axis;
(c) special solutions of inhomogeneous equations (1.9), (1.10) are convergent on the real τ-axis.

Indeed, conditions (b) and (c) show that the solutions of the corresponding homogeneous equation (1.5) in a form of formal series, as well as the special solution (1.9), (1.10), are convergent on the real t-axis. If so, the generation coefficient defined similar to (2.4), after substitution of the solution in a form of formal series, represents a sum of converging integrals that can be readily calculated by the Watson lemma or the Cauchy theorem.

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Institute of Theoretical Physics, Ilia State University, 3/5 ave. Cholokashvili, Tbilisi 0162, Georgia
Email address: grigol_gogoberidze@iliauni.edu.ge