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VARIABLE EXPONENT $p(x)$-LAPLACIAN-LIKE DIRICHLET PROBLEM WITH CONVECTION

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Abstract. In this paper, we investigate a Dirichlet problems with $p(x)$-Laplacian-like operators of the form

$$\begin{cases} -\operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big) = \omega |u|^{\varsigma(x)-2}u + \varpi f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

in the setting of the variable-exponent Sobolev spaces $W_0^{1,p(x)}(\Omega)$, where Ω is a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, ω and ϖ are two real parameters and $p(\cdot), \varsigma(\cdot) \in C_+(\overline{\Omega})$. Under the suitable nonstandard growth conditions on f and using the topological degree for a class of demicontinuous operators of generalized (S_+) type, we establish the existence of "a weak solution" for the above problem.

1. Introduction

In recent years, partial differential equations with nonlinearities and nonconstant exponents have received a lot of attention (see [11]). The impulse of this topic would come from the new search field that reflects a new type of physical phenomena, a class of nonlinear problems with variable exponents. Modeling with classic Lebesgue and Sobolev spaces has been demonstrated to be limited for a number of materials with inhomogeneities. In the subject of fluid mechanics, for example, great emphasis has been paid to the study of electrorheological fluids, which have the ability to modify their mechanical properties when exposed to an electric field (see [1,2,12,13]). Rajagopal and Růžička recently developed a very interesting model for these fluids in [14] (see also [16]), taking into account the delicate interaction between the electric field $E(x)$ and the moving liquid. This type of problem's energy is provided by $\int\limits_\Omega |\nabla u|^{p(x)}dx$.

Let Ω be a smooth bounded domain in $\mathbb{R}^N (N \geq 2)$, with a Lipschitz boundary denoted by $\partial\Omega$. In this paper we deal with the question of the existence of a weak solution for a class of $p(x)$-Laplacian-like Dirichlet problems of the following form:

$$\begin{cases} -\operatorname{div}\Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big) = \omega |u|^{\varsigma(x)-2}u + \varpi f(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $p(\cdot), \varsigma(\cdot) \in C_+(\overline{\Omega})$ with $p(\cdot)$ is a log-Hölder continuous function (in a sense to be precised in Section 2 below), ω and ϖ are two real parameters and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a Carathéodory function. The expression $f(x,u,\nabla u)$ is often referred to as a convection term. Note that, since the nonlinearity f depends on the gradient ∇u, problem (1.1) does not have a variational structure, so the variational methods cannot be applied directly.

The motivation for this research comes from the application of similar problems in physics to model the behavior of elasticity [5–7,19] and electrorheological fluids (see [14,16]), which have the ability to modify their mechanical properties when exposed to an electric field (see [1–3,8,12,13]).

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Our aim in this paper is to prove the existence of a weak solution for problem (1.1) by using another approach based on the topological degree for a class of demicontinuous operators of generalized (S_+) type of [4] and the theory of the variable-exponent Sobolev spaces.

The remainder of the article is organized as follows. In Section 2, we review some fundamental preliminaries about the functional framework where we will treat our problem. In Section 3, we introduce some classes of operators of generalized (S_+) type, as well as the Berkovits topological degrees. Finaly, in Section 4, we give our basic assumptions, some technical lemmas, and also state and prove the main result of the paper.

2. Preliminaries

For convenience, we only recall some basic facts with will be used later (we refer to [3,8,9,18] for more details).

Let Ω be a smooth bounded domain in $\mathbb{R}^N$ $(N \geq 2)$, with a Lipschitz boundary denoted by $\partial\Omega$. Set

$$C_+(\overline{\Omega}) = \big\{p : p \in C(\overline{\Omega}) \text{ such that } p(x) > 1 \text{ for any } x \in \overline{\Omega}\big\}.$$

For each $p \in C_+(\overline{\Omega})$, we define

$$p^+ := \max\big\{p(x),\ x \in \overline{\Omega}\big\} \quad \text{and} \quad p^- := \min\big\{p(x),\ x \in \overline{\Omega}\big\}.$$

For every $p \in C_+(\overline{\Omega})$, we define

$$L^{p(x)}(\Omega) = \left\{u : \Omega \to \mathbb{R} \text{ is measurable such that } \int_\Omega |u(x)|^{p(x)}dx < +\infty\right\},$$

equipped with the Luxemburg norm

$$|u|_{p(x)} = \inf\left\{\lambda > 0 : \varrho_{p(x)}\Big(\frac{u}{\lambda}\Big) \leq 1\right\},$$

where

$$\varrho_{p(x)}(u) = \int_\Omega |u(x)|^{p(x)}dx, \quad \text{for all } u \in L^{p(x)}(\Omega).$$

Proposition 2.1 ([9]). *If $(u_n) \subset L^{p(x)}(\Omega)$ and $u \in L^{p(x)}(\Omega)$, then*

$$|u|_{p(x)} < 1\,(resp. = 1; > 1) \iff \varrho_{p(x)}(u) < 1\,(resp. = 1; > 1),$$

$$|u|_{p(x)} > 1 \implies |u|^{p^-}_{p(x)} \leq \varrho_{p(x)}(u) \leq |u|^{p^+}_{p(x)}, \tag{2.1}$$

$$|u|_{p(x)} < 1 \implies |u|^{p^+}_{p(x)} \leq \varrho_{p(x)}(u) \leq |u|^{p^-}_{p(x)}, \tag{2.2}$$

$$\lim_{n\to\infty} |u_n - u|_{p(x)} = 0 \iff \lim_{n\to\infty} \varrho_{p(x)}(u_n - u) = 0. \tag{2.3}$$

Remark 2.1. According to (2.1) and (2.2), we have

$$|u|_{p(x)} \leq \varrho_{p(x)}(u) + 1, \tag{2.4}$$

$$\varrho_{p(x)}(u) \leq |u|^{p^-}_{p(x)} + |u|^{p^+}_{p(x)}. \tag{2.5}$$

Proposition 2.2 ([9]). *The space $\big(L^{p(x)}(\Omega), |\cdot|_{p(x)}\big)$ is a separable and reflexive Banach space.*

Proposition 2.3 ([9]). *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$ for all $x \in \Omega$. For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have the following Hölder-type inequality:*

$$\left|\int_\Omega uv\ dx\right| \leq \Big(\frac{1}{p^-} + \frac{1}{p'^-}\Big)|u|_{p(x)}|v|_{p'(x)} \leq 2|u|_{p(x)}|v|_{p'(x)}. \tag{2.6}$$

Remark 2.2. If $p_1,\ p_2 \in C_+(\overline{\Omega})$ with $p_1(x) \leq p_2(x)$ for any $x \in \overline{\Omega}$, then there exists the continuous embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$.

Now, let $p \in C_+(\overline{\Omega})$ and we define $W^{1,p(x)}(\Omega)$ as

$$W^{1,p(x)}(\Omega) = \big\{u \in L^{p(x)}(\Omega) \text{ such that } |\nabla u| \in L^{p(x)}(\Omega)\big\},$$

equipped with the norm

$$||u|| = |u|_{p(x)} + |\nabla u|_{p(x)}.$$

We also define $W_0^{1,p(x)}(\Omega)$ as the subspace of $W^{1,p(x)}(\Omega)$, which is the closure of $C_0^\infty(\Omega)$ with respect to the norm $||\cdot||$.

Proposition 2.4 ([18])**.** *If the exponent $p(x)$ satisfies the log-Hölder continuity condition, i.e., there is a constant $a > 0$ such that for every $x,\ y \in \Omega$, $x \neq y$ with $|x-y| \leq \frac{1}{2}$ one has*

$$|p(x) - p(y)| \leq \frac{a}{-\log|x-y|}\,, \tag{2.7}$$

then we have the Poincaré inequality, i.e., there exists a constant $C > 0$ depending only on Ω and the function p such that

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)}, \quad \textit{for all} \ \ u \in W_0^{1,p(x)}(\Omega). \tag{2.8}$$

In this paper, we use on $W_0^{1,p(x)}(\Omega)$ the following equivalent norm:

$$|u|_{1,p(x)} = |\nabla u|_{p(x)},$$

which is equivalent to $||\cdot||$.

Proposition 2.5 ([9])**.** *The space $\big(W_0^{1,p(x)}(\Omega), |\cdot|_{1,p(x)}\big)$ is a separable and reflexive Banach space.*

Remark 2.3. The dual space of $W_0^{1,p(x)}(\Omega)$ is the space $W^{-1,p'(x)}(\Omega)$ defined by

$$W^{-1,p'(x)}(\Omega) := \left\{u = u_0 - \sum_{i=1}^{N} D_i u_i \ \text{ with } \ (u_0, u_1, \ldots, u_N) \in (L^{p'(x)}(\Omega))^N\right\},$$

equipped with the norm

$$|u|_{-1,p'(x)} = \inf\left\{|u_0|_{p'(x)} + \sum_{i=1}^{N} |u_i|_{p'(x)}\right\}.$$

3. Topological Degree Theory

The readers can find more information about the history of this theory in [4, 10]. In what follows, let X be a real separable reflexive Banach space and X^* be its dual space with a dual pairing $\langle\cdot\,,\cdot\rangle$ and the given nonempty subset $\mathscr{D}$ of X. Strong (weak) convergence is represented by the symbol $\to$ ($\rightharpoonup$).

Definition 3.1. Let Y be the real Banach space. An operator $F : \mathscr{D} \subset X \to Y$ is said to be

(1) bounded, if it takes any bounded set into a bounded set;
(2) demicontinuous, if for any sequence $(u_n) \subset \mathscr{D}$, $u_n \to u$ implies that $F(u_n) \rightharpoonup F(u)$;
(3) compact, if it is continuous and the image of any bounded set is relatively compact.

Definition 3.2. A mapping $F : \mathscr{D} \subset X \to X^*$ is said to be

(1) of class (S_+), if for any sequence $(u_n) \subset \mathscr{D}$ with $u_n \rightharpoonup u$ and $\limsup\limits_{n\to\infty}\langle Fu_n, u_n - u\rangle \leq 0$, we have $u_n \to u$;
(2) quasimonotone, if for any sequence $(u_n) \subset \mathscr{D}$ with $u_n \rightharpoonup u$, we have $\limsup\limits_{n\to\infty}\langle Fu_n,\ u_n - u\rangle \geq 0$.

Definition 3.3. Let $T : \mathscr{D}_1 \subset X \to X^*$ be a bounded operator such that $\mathscr{D} \subset \mathscr{D}_1$. For any operator $F : \mathscr{D} \subset X \to X$, we say that

(1) F is of class $(S_+)_T$, if for any sequence $(u_n) \subset \mathscr{D}$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$ and $\limsup\limits_{n\to\infty}\langle Fu_n,\ y_n - y\rangle \leq 0$, we have $u_n \to u$;

(2) F has the property $(QM)_T$, if for any sequence $(u_n) \subset \mathscr{D}$ with $u_n \rightharpoonup u$, $y_n := Tu_n \rightharpoonup y$, we have $\limsup_{n\to\infty}\langle Fu_n, y - y_n\rangle \geq 0$.

In the sequel, for any $T \in \mathscr{F}_1(\mathscr{D})$, we consider the following classes of operators:

$$\mathscr{F}_1(\mathscr{D}) := \big\{F : \mathscr{D} \subset D(F) \to X^* : F \text{ is bounded, demicontinuous and of class } (S_+)\big\},$$

$$\mathscr{F}_{T,B}(\mathscr{D}) := \big\{F : \mathscr{D} \subset D(F) \to X : F \text{ is bounded, demicontinuous and of class } (S_+)_T\big\},$$

$$\mathscr{F}_T(\mathscr{D}) := \big\{F : \mathscr{D} \subset D(F) \to X : F \text{ is demicontinuous and of class } (S_+)_T\big\}.$$

Now, let $\mathscr{O}$ be the collection of all bounded open sets in X and we define

$$\mathscr{F}(X) := \big\{F \in \mathscr{F}_T(\overline{E}) : E \in \mathscr{O},\ \mathrm{T} \in \mathscr{F}_1(\overline{\mathrm{E}})\big\}.$$

Lemma 3.1 ([10, Lemma 2.3])**.** *Let $T \in \mathscr{F}_1(\overline{E})$ be continuous and $S : D(S) \subset X^* \to X$ be demicontinuous such that $T(\overline{E}) \subset D(S)$, where E is a bounded open set in a real reflexive Banach space X. Then the following statements are true:*

(1) *If S is quasimonotone, then $I + S \circ T \in \mathscr{F}_T(\overline{E})$, where I denotes the identity operator.*
(2) *If S is of class (S_+), then $S \circ T \in \mathscr{F}_T(\overline{E})$.*

Definition 3.4. Suppose that E is a bounded open subset of the real reflexive Banach space X, $T \in \mathscr{F}_1(\overline{E})$ is continuous and $F, S \in \mathscr{F}_T(\overline{E})$. The affine homotopy $\mathscr{H} : [0,1] \times \overline{E} \to X$ defined by

$$\mathscr{H}(t,u) := (1-t)Fu + tSu, \quad \text{for all} \quad (t,u) \in [0,1] \times \overline{E}$$

is called an admissible affine homotopy with the common continuous essential inner map T.

Theorem 3.1. *Let $M = \big\{(F,E,h) :\ E \in \mathscr{O},\ F \in \mathscr{F}_{T,B}(\overline{E}),\ h \notin F(\partial E)\big\}$. Then there exists a unique degree function $d : M \longrightarrow \mathbb{Z}$ that satisfies the following properties:*

(1) (Normalization) *For any $h \in F(E)$, we have $d(I,E,h) = 1$.*
(2) (Homotopy invariance) *If $\mathscr{H} : [0,1] \times \overline{E} \to X$ is a bounded admissible affine homotopy with a common continuous essential inner map and h: $[0,1] \to X$ is a continuous path in X such that $h(t) \notin \mathscr{H}(t, \partial E)$ for all $t \in [0,1]$, then $d(\mathscr{H}(t,\cdot), E, h(t)) = C$ for all $t \in [0,1]$.*
(3) (Existence) *If $d(F,E,h) \neq 0$, then the equation $Fu = h$ has a solution in E.*

Definition 3.5 ([10, Definition 3.3])**.** The above degree is defined as follows:

$$d(F,E,h) := d_B(F|_{\overline{E}_0}, E_0, h),$$

where d_B is the Berkovits degree [4] *and E_0 is any open subset of E with $F^{-1}(h) \subset E_0$ and F is bounded on $\overline{E}_0$.*

4. Main Results

In this section, we discuss the existence of a weak solution of (1.1).

We assume that $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with a Lipschitz boundary $\partial\Omega$, $p \in C_+(\overline{\Omega})$ satisfies (2.7), $\varsigma \in C_+(\overline{\Omega})$ with $2 \leq \varsigma^- \leq \varsigma(x) \leq \varsigma^+ < p^- \leq p(x) \leq p^+ < \infty$ and $f : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is a function such that:

(A_1) f satisfies the Carathéodory condition.
(A_2) There exists $C_1 > 0$ and $\vartheta \in L^{p'(x)}(\Omega)$ such that

$$|f(x,y,z)| \leq C_1(\vartheta(x) + |y|^{q(x)-1} + |z|^{q(x)-1})$$

for a.e. $x \in \Omega$ and all $(y,z) \in \mathbb{R} \times \mathbb{R}^N$, where $q \in C_+(\overline{\Omega})$ with $2 \leq q^- \leq q(x) \leq q^+ < p^-$.

Remark 4.1. • Note that for all $u, \psi \in W_0^{1,p(x)}(\Omega)$,

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\psi dx$$

is well defined (see [15]).

- $\omega|u|^{\varsigma(x)-2}u \in L^{p'(x)}(\Omega)$ and $\varpi f(x,u,\nabla u) \in L^{p'(x)}(\Omega)$ under $u \in W_0^{1,p(x)}(\Omega)$ and the given hypotheses about the exponents p, ς and q and assumption (A_2) because: $\vartheta \in L^{p'(x)}(\Omega)$, $r(x) = (q(x)-1)p'(x) \in C_+(\overline{\Omega})$ with $r(x) < p(x)$ and $\beta(x) = (\varsigma(x)-1)p'(x) \in C_+(\overline{\Omega})$ with $\beta(x) < p(x)$. Then by Remark 2.2, we can conclude that $L^{p(x)} \hookrightarrow L^{r(x)}$ and $L^{p(x)} \hookrightarrow L^{\beta(x)}$.
  Hence, since $\psi \in L^{p(x)}(\Omega)$, we have $\big(\omega|u|^{\varsigma(x)-2}u + \varpi f(x,u,\nabla u)\big)\psi \in L^1(\Omega)$.
  This implies that the integral $\int_\Omega \big(\omega|u|^{\varsigma(x)-2}u + \varpi f(x,\ u,\ \nabla u)\big)\psi dx$ exists.

Now, let us introduce the definition of a weak solution for (1.1).

Definition 4.1. We say that $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution of (1.1) if

$$\int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\psi dx = \int_\Omega \big(\omega|u|^{\varsigma(x)-2}u + \varpi f(x,u,\nabla u)\big)\psi dx,$$

for all $\psi \in W_0^{1,p(x)}(\Omega)$.

Let us now give some lemmas that will be used later. First, let us consider the following functional:

$$\mathscr{J}(u) := \int_\Omega \frac{1}{p(x)}\Big(|\nabla u|^{p(x)} + \sqrt{1+|\nabla u|^{2p(x)}}\Big)dx.$$

From [15], it is clear that the derivative operator of the functional $\mathscr{J}$ in the weak sense at the point $u \in W_0^{1,p(x)}(\Omega)$ is the functional $\mathscr{L} := \mathscr{J}'(u) \in W^{-1,p'(x)}(\Omega)$ such that

$$\langle \mathscr{L}u, \psi\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\psi dx,$$

for all $u, \psi \in W_0^{1,p(x)}(\Omega)$, where $\langle\cdot,\cdot\rangle$ means the duality pairing between $W^{-1,p'(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$. In addition, the following lemma summarizes the properties of the operator $\mathscr{L}$.

Lemma 4.1 ([15])**.** *The mapping $\mathscr{L} : W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle \mathscr{L}u, \psi\rangle = \int_\Omega \Big(|\nabla u|^{p(x)-2}\nabla u + \frac{|\nabla u|^{2p(x)-2}\nabla u}{\sqrt{1+|\nabla u|^{2p(x)}}}\Big)\nabla\psi dx,$$

is a continuous, bounded, strictly monotone operator and is of class (S_+).

Lemma 4.2. *If $(A_1)-(A_2)$ hold, then the operator $\mathscr{V} : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ defined by*

$$\langle \mathscr{V}u, \psi\rangle = -\int_\Omega \big(\omega|u|^{\varsigma(x)-2}u + \varpi f(x,u,\nabla u)\big)\psi dx,$$

is compact.

Proof. We follow four steps to prove this lemma.

Step 1: Let us define the operator $\Upsilon : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Upsilon u(x) := -\omega|u(x)|^{\varsigma(x)-2}u(x).$$

At this step, we prove that Υ is bounded and continuous. It is clear that Υ is continuous. Next, we show that Υ is bounded. Let $u \in W_0^{1,p(x)}(\Omega)$, and using (2.4) and (2.5), we obtain

$$\begin{aligned}
|\Upsilon u|_{p'(x)} &\le \varrho_{p'(x)}(\Upsilon u) + 1 \\
&= \int_\Omega |\omega|u|^{\varsigma(x)-2}u|^{p'(x)}dx + 1 \\
&= \int_\Omega |\omega|^{p'(x)}|u|^{(\varsigma(x)-1)p'(x)}dx + 1 \\
&\le \big(|\omega|^{p'^-} + |\omega|^{p'^+}\big)\varrho_{\beta(x)}(u) + 1 \\
&\le \big(|\omega|^{p'^-} + |\omega|^{p'^+}\big)\big(|u|^{\beta^-}_{\beta(x)} + |u|^{\beta^+}_{\beta(x)}\big) + 1.
\end{aligned}$$

Hence we deduce from $L^{p(x)} \hookrightarrow L^{\beta(x)}$ and (2.8) that

$$|\Upsilon u|_{p'(x)} \le C\Big(|u|^{\beta^-}_{1,p(x)} + |u|^{\beta^+}_{1,p(x)}\Big) + 1.$$

Consequently, Υ is bounded on $W_0^{1,p(x)}(\Omega)$.

Step 2: We define the operator $\Phi : W_0^{1,p(x)}(\Omega) \to L^{p'(x)}(\Omega)$ by

$$\Phi u(x) := -\varpi f(x, u, \nabla u).$$

We will show that Φ is bounded and continuous. Let $u \in W_0^{1,p(x)}(\Omega)$. According to (A_2) and inequalities (2.4) and (2.5), we obtain

$$\begin{aligned}
|\Phi u|_{p'(x)} &\le \varrho_{p'(x)}(\Phi u) + 1 \\
&= \int_\Omega |\varpi f(x, u(x), \nabla u(x))|^{p'(x)}dx + 1 \\
&= \int_\Omega |\varpi|^{p'(x)}|f(x, u(x), \nabla u(x))|^{p'(x)}dx + 1 \\
&\le \big(|\varpi|^{p'^-} + |\varpi|^{p'^+}\big)\int_\Omega |C_1\big(\vartheta(x) + |u|^{q(x)-1} + |\nabla u|^{q(x)-1}\big)|^{p'(x)}dx + 1 \\
&\le C\big(|\varpi|^{p'^-} + |\varpi|^{p'^+}\big)\int_\Omega \big(\vartheta(x)^{p'(x)} + |u|^{(q(x)-1)p'(x)} + |\nabla u|^{(q(x)-1)p'(x)}\big)dx + 1 \\
&\le C\big(|\varpi|^{p'^-} + |\varpi|^{p'^+}\big)\big(\varrho_{p'(x)}(\vartheta) + \varrho_{r(x)}(u) + \varrho_{r(x)}(\nabla u)\big) + 1 \\
&\le C\big(|\vartheta|^{p'^+}_{p(x)} + |u|^{r^+}_{r(x)} + |u|^{r^-}_{r(x)} + |\nabla u|^{r^+}_{r(x)} + |\nabla u|^{r^-}_{r(x)}\big) + 1.
\end{aligned}$$

Taking into account that $L^{p(x)} \hookrightarrow L^{r(x)}$ and (2.8), we have then

$$|\Phi u|_{p'(x)} \le C\big(|\vartheta|^{p'^+}_{p(x)} + |u|^{r^+}_{1,p(x)} + |u|^{r^-}_{1,p(x)}\big) + 1,$$

and, consequently, Φ is bounded on $W_0^{1,p(x)}(\Omega)$.

Let us show that Φ is continuous. Let $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, we need to show that $\Phi u_n \to \Phi u$ in $L^{p'(x)}(\Omega)$. Towards this end, we apply the Lebesgue theorem.

Note that if $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$, then $u_n \to u$ in $L^{p(x)}(\Omega)$ and $\nabla u_n \to \nabla u$ in $(L^{p(x)}(\Omega))^N$. Consequently, there exist a subsequence (u_k) of (u_n) and ϕ in $L^{p(x)}(\Omega)$ and ψ in $(L^{p(x)}(\Omega))^N$ such that

$$\begin{aligned}
u_k(x) \to u(x) \quad &\text{and} \quad \nabla u_k(x) \to \nabla u(x), \\
|u_k(x)| \le \phi(x) \quad &\text{and} \quad |\nabla u_k(x)| \le |\psi(x)|.
\end{aligned} \tag{4.1}$$

Hence from (A_2) and (4.1), we have

$$|f(x, u_k(x), \nabla u_k(x))| \le C_1(\vartheta(x) + |\phi(x)|^{q(x)-1} + |\psi(x)|^{q(x)-1}).$$

On the other hand, thanks to (A_1), as $k \longrightarrow \infty$, we get

$$f(x, u_k(x), \nabla u_k(x)) \to f(x, u(x), \nabla u(x)) \quad \text{a.e.} \quad x \in \Omega.$$

Seeing that

$$\vartheta + |\phi|^{q(x)-1} + |\psi(x)|^{q(x)-1} \in L^{p'(x)}(\Omega),$$

and

$$\varrho_{p'(x)}(\Phi u_k - \Phi u) = \int_\Omega |f(x, u_k(x), \nabla u_k(x)) - f(x, u(x), \nabla u(x))|^{p'(x)} dx,$$

then from the Lebesgue theorem and equivalence (2.3), we have $\Phi u_k \to \Phi u$ in $L^{p'(x)}(\Omega)$, and, consequently, $\Phi u_n \to \Phi u$ in $L^{p'(x)}(\Omega)$.

Step 3: Let $I^* : L^{p'(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$ be the adjoint operator of the natural embedding mapping $I : W_0^{1,p(x)}(\Omega) \to L^{p(x)}(\Omega)$. We then define

$$I^* \circ \Upsilon : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega)$$

and

$$I^* \circ \Phi : W_0^{1,p(x)}(\Omega) \to W^{-1,p'(x)}(\Omega).$$

On the other hand, taking into account that I is compact, then I^* is compact. Thus the compositions $I^* \circ \Upsilon$ and $I^* \circ \Phi$ are compact, that means $\mathscr{V} = I^* \circ \Upsilon + I^* \circ \Phi$ is compact. With this last step the proof of Lemma 4.2 is completed. □

We are now in the position to get the existence result of a weak solution for (1.1).

Theorem 4.1. *If the assumptions* (A_1)–(A_2) *hold, then problem* (1.1) *possesses at least one weak solution* u *in* $W_0^{1,p(x)}(\Omega)$.

Proof. The basic idea of our proof is to reduce problem (1.1) to a new one governed by a Hammerstein equation, and apply the theory of topological degree introduced in Section 3 to show the existence of a weak solution to the problem under consideration. For all $u, \psi \in W_0^{1,p(x)}(\Omega)$, we define the operators $\mathscr{Z}$ and $\mathscr{V}$ by

$$\begin{aligned}
\mathscr{Z} &: W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\
\langle \mathscr{Z} u, \psi \rangle &= \int_\Omega \Big(|\nabla u|^{p(x)-2} \nabla u + \frac{|\nabla u|^{2p(x)-2} \nabla u}{\sqrt{1 + |\nabla u|^{2p(x)}}} \Big) \nabla \psi dx, \\
\mathscr{V} &: W_0^{1,p(x)}(\Omega) \longrightarrow W^{-1,p'(x)}(\Omega) \\
\langle \mathscr{V} u, \psi \rangle &= - \int_\Omega \big(\omega |u|^{\varsigma(x)-2} u + \varpi f(x,\ u,\ \nabla u) \big) \psi dx.
\end{aligned}$$

Consequently, problem (1.1) is equivalent to the equation

$$\mathscr{Z} u = -\mathscr{V} u, \quad u \in W_0^{1,p(x)}(\Omega). \tag{4.2}$$

Taking into account that by Lemma 4.1, the operator $\mathscr{Z}$ is continuous, bounded, strictly monotone and of class (S_+), then by [17, Theorem 26 A], the inverse operator

$$\mathscr{K} := \mathscr{Z}^{-1} : W^{-1,p'(x)}(\Omega) \to W_0^{1,p(x)}(\Omega),$$

is also bounded, continuous, strictly monotone and of class (S_+).

On the other side, according to Lemma 4.2, we find that the operator $\mathscr{V}$ is bounded, continuous and quasimonotone.

Consequently, following Zeidler's terminology [17], equation (4.2) is equivalent to the following abstract Hammerstein equation:

$$u = \mathscr{K} \psi \ \text{ and } \ \psi + \mathscr{V} \circ \mathscr{K} \psi = 0, \quad u \in W_0^{1,p(x)}(\Omega) \text{ and } \psi \in W^{-1,p'(x)}(\Omega). \tag{4.3}$$

Due to the equivalence of (4.2) and (4.3), it suffics to solve (4.3). Towards this end, we apply the Berkovits topological degree introduced in Section 3.

First, let us set

$$\mathscr{B} := \big\{\psi \in W^{-1,p'(x)}(\Omega) : \exists\, t \in [0,1] \ \text{ such that } \ \psi + t\mathscr{V} \circ \mathscr{K}\psi = 0\big\}.$$

Next, we show that $\mathscr{B}$ is bounded in $\in W^{-1,p'(x)}(\Omega)$.

Let us put $u := \mathscr{K}\psi$ for all $\psi \in \mathscr{B}$. If $|\nabla u|_{p(x)} \leq 1$, then $|\mathscr{K}\psi|_{1,p(x)} \leq 1$, that means $\big\{\mathscr{K}\psi : \psi \in \mathscr{B}\big\}$ is bounded. If $|\nabla u|_{p(x)} > 1$, then from (2.1), (A_2), inequalities (2.6) and (2.5) and Young's inequality, we deduce that

$$\begin{aligned}
|\mathscr{K}\psi|^{p^-}_{1,p(x)} &= |\nabla u|^{p-}_{p(x)} \\
&\leq \varrho_{p(x)}(\nabla u) \\
&\leq \langle \mathscr{Z} u,\ u\rangle \\
&= \langle \psi,\ \mathscr{K}\psi\rangle \\
&= -t\langle \mathscr{V} \circ \mathscr{K}\psi,\ \mathscr{K}\psi\rangle \\
&= \ t\int_\Omega \big(\omega|u|^{\varsigma(x)-2}u + \varpi f(x,\ u,\ \nabla u)\big)u dx \\
&\leq t\max(|\omega|, C_1|\varpi|)\bigg(\int_\Omega |u|^{\varsigma(x)}dx + \int_\Omega |\vartheta(x)u(x)|dx + \int_\Omega |u(x)|^{q(x)}dx + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\bigg) \\
&= t\max(|\omega|, C_1|\varpi|)\bigg(\varrho_{\varsigma(x)}(u) + \int_\Omega |\vartheta(x)u(x)|dx + \varrho_{q(x)}(u) + \int_\Omega |\nabla u|^{q(x)-1}|u|dx\bigg) \\
&\leq C\big(|u|^{\varsigma^-}_{\varsigma(x)} + |u|^{\varsigma^+}_{\varsigma(x)} + |\vartheta|_{p'(x)}|u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + \frac{1}{q'^-}\varrho_{q(x)}(\nabla u) + \frac{1}{q-}\varrho_{q(x)}(u)\big) \\
&\leq C\big(|u|^{\varsigma^-}_{\varsigma(x)} + |u|^{\varsigma^+}_{\varsigma(x)} + |u|_{p(x)} + |u|^{q^+}_{q(x)} + |u|^{q^-}_{q(x)} + |\nabla u|^{q^+}_{q(x)}\big).
\end{aligned}$$

Then, according to $L^{p(x)} \hookrightarrow L^{\varsigma(x)}$ and $L^{p(x)} \hookrightarrow L^{q(x)}$, we get

$$|\mathscr{K}\psi|^{p^-}_{1,p(x)} \leq C\big(|\mathscr{K}\psi|^{\varsigma^+}_{1,p(x)} + |\mathscr{K}\psi|_{1,p(x)} + |\mathscr{K}\psi|^{q^+}_{1,p(x)}\big),$$

which implies that $\big\{\mathscr{K}\psi : \psi \in \mathscr{B}\big\}$ is bounded.

On the other hand, we have that the operator is $\mathscr{V}$ is bounded, then $\mathscr{V} \circ \mathscr{K}\psi$ is bounded. Thus owing to (4.3), we have that $\mathscr{B}$ is bounded in $W^{-1,p'(x)}(\Omega)$.

However, there exists a constant $b > 0$ such that

$$|\psi|_{-1,p'(x)} < b \ \text{ for all } \ \psi \in \mathscr{B},$$

which leads to

$$\psi + t\mathscr{V} \circ \mathscr{K}\psi \neq 0, \quad \psi \in \partial\mathscr{B}_b(0) \ \text{ and } \ t \in [0,1],$$

where $\mathscr{B}_b(0)$ is the ball of center 0 and radius b in $W^{-1,p'(x)}(\Omega)$.

Moreover, by Lemma 3.1, we conclude that

$$I + \mathscr{V} \circ \mathscr{K} \in \mathscr{F}_{\mathscr{K}}(\overline{\mathscr{B}_b(0)}) \ \text{ and } \ I = \mathscr{Z} \circ \mathscr{K} \in \mathscr{F}_{\mathscr{K}}(\overline{\mathscr{B}_b(0)}).$$

On the other side, taking into account that I, $\mathscr{V}$ and $\mathscr{K}$ are bounded, then $I + \mathscr{V} \circ \mathscr{K}$ is bounded. Hence we infer that

$$I + \mathscr{V} \circ \mathscr{K} \in \mathscr{F}_{\mathscr{K},B}(\overline{\mathscr{B}_b(0)}) \ \text{ and } \ I = \mathscr{Z} \circ \mathscr{K} \in \mathscr{F}_{\mathscr{K},B}(\overline{\mathscr{B}_b(0)}).$$

Next, we define the homotopy

$$\begin{aligned}
\mathscr{H} : [0,1]\times\overline{\mathscr{B}_b(0)} &\to W^{-1,p'(x)}(\Omega) \\
(t,\psi) &\mapsto \mathscr{H}(t,\psi) := \psi + t\mathscr{V} \circ \mathscr{K}\psi.
\end{aligned}$$

Thus thanks to the properties of the degree d mentioned in Theorem 3.1, we obtain

$$d(I+\mathscr{V}\circ\mathscr{K},\mathscr{B}_b(0),0)=d(I,\mathscr{B}_b(0),\ 0)=1\neq 0,$$

which implies that there exists $\psi\in\mathscr{B}_b(0)$ which verifies

$$\psi+\mathscr{V}\circ\mathscr{K}\psi=0.$$

Finally, we infer that $u=\mathscr{K}\psi$ is a weak solution of (1.1). Thus the proof of this theorem is completed. □

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