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(β,γ)-SECOND HANKEL–CLIFFORD LIPSCHITZ FUNCTIONS IN THE SPACE L^2_μ

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Abstract. In this paper, using a generalized translation operator, we obtain an analogue of the Younis theorem 5.2 (see in [17] for the second Hankel–Clifford transform on the half-line for the functions satisfying the (β,γ)-second Hankel–Clifford Lipschitz condition in the space $L^2_\mu(0,+\infty)$.

1. Introduction and Preliminaries

Younis [17, Theorem 5.2], characterized the set of functions in $L^2(\mathbb{R})$ satisfying the Dini–Lipschitz condition by means of an asymptotic estimate growth of the norm of their Fourier transforms, namely, we have following

Theorem 1.1 ([17, Theorem 5.2]). *Let $f \in L^2(\mathbb{R})$. Then the following are equivalent:*

1. $\|f(.+h)-f(.)\|_{L^2(\mathbb{R})} = O\left(\dfrac{h^\alpha}{(\log\frac{1}{h})^\beta}\right)$ *as* $h \longrightarrow 0$, $0<\alpha<1$, $\beta>0$,
2. $\displaystyle\int\limits_{|\lambda|\geq r} |\mathcal{F}(f)(\lambda)|^2 d\lambda = O\left(\frac{r^{-2\alpha}}{(\log r)^{2\beta}}\right)$ *as* $r \longrightarrow +\infty$,

where $\mathcal{F}$ stands for the Fourier transform of f.

There are many analogues of this result: for the Dunkl transform, for Fourier–Bessel transform on $\mathbb{R}^n_+$, for generalized Fourier–Bessel transform, for generalized Fourier–Dunkl transform, for the first Hankel–Clifford transform (see, for example, [3–7, 11, 12]).

The main aim of this paper is to establish an analogue of Theorem 1.1 in the second Hankel–Clifford transform.

We briefly overview the theory of second Hankel–Clifford transformation and related harmonic analysis [13–15].

We define the space $L^p_\mu = L^p_\mu(0,+\infty)$, $1 \leq p < \infty$ and $\mu \geq 0$, as the space of all those real-valued measurable functions f on $(0,+\infty)$ such that

$$\|f\|_{L^p_\mu} = \left(\int\limits_0^{+\infty} |f(x)|^p x^\mu dx\right)^{\frac{1}{p}} < \infty.$$

The Bessel–Clifford function of the first kind of order $\mu \geq 0$ (see [8])

$$C_\mu(x) = \sum_{k=0}^{+\infty} \frac{(-1)^k x^k}{k!\Gamma(\mu+k+1)}$$

is a solution of the differential equation

$$xy'' + (\mu+1)y' + y = 0$$

and we have

$$C_\mu(x) = x^{-\frac{\mu}{2}} J_\mu(2\sqrt{x}), \tag{1}$$

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where J_μ is the Bessel function of the first kind.

For $f \in L^1_\mu$, Hayek [10] introduced the second Hankel–Clifford transformation by

$$h_{2,\mu}(f)(\lambda) = \int_0^{+\infty} C_\mu(\lambda x) f(x) x^\mu dx$$

and its inversion formula defined by

$$f(x) = \int_0^{+\infty} C_\mu(\lambda x) h_{2,\mu}(f)(\lambda) \lambda^\mu d\lambda.$$

The corresponding Parseval's equality now takes the form [14]

$$\int_0^{+\infty} f(x) g(x) x^\mu dx = \int_0^{+\infty} F_2(\lambda) G_2(\lambda) \lambda^\mu d\lambda,$$

where $F_2(\lambda) = h_{2,\mu}(f)(\lambda)$ and $G_2(\lambda) = h_{2,\mu}(g)(\lambda)$, i.e., for $f \in L^2_\mu$, we have

$$\|f\|_{L^2_\mu} = \|h_{2,\mu}(f)\|_{L^2_\mu}.$$

Let $\Delta = \Delta(x, y, z)$ be the area of a triangle with sides x, y, z if such a triangle exists (see [9, 16]). Set

$$D_\mu(x, y, z) = \frac{\Delta^{2\mu+1}}{2^{2\mu}(xyz)^\mu \Gamma(\mu + \frac{1}{2})\sqrt{\pi}}$$

if Δ exists and zero otherwise. We note that $D_\mu(x, y, z) \geq 0$ and that $D_\mu(x, y, z)$ is symmetric in x, y, z.

The generalized translation operator on L^2_μ is defined by

$$T_h(f)(x) = \int_0^{+\infty} f(z) D_\mu(h, x, z) z^\mu dz, \quad 0 < x, \ \ h < \infty.$$

From Lemma 1.3 in [15], we have

$$h_{2,\mu}(T_h(f))(\lambda) = C_\mu(\lambda h) h_{2,\mu}(f)(\lambda), \tag{2}$$

where $f \in L^2_\mu$.

For $\mu \geq -\frac{1}{2}$, we introduce the normalized spherical Bessel function j_μ defined by

$$j_\mu(x) = \frac{2^\mu \Gamma(\mu + 1) J_\mu(x)}{x^\mu}. \tag{3}$$

From [1], we have the following

Lemma 1.1. *Let $\mu \geq -\frac{1}{2}$. The following inequalities hold:*

1. $|j_\mu(x)| \leq 1$.
2. $1 - j_\mu(x) = O(x^2)$; $0 \leq x \leq 1$.
3. $\sqrt{x} J_\mu(x) = O(1)$.

Lemma 1.2. *For $|x| \geq 1$,*

$$|1 - j_\mu(x)| \geq c,$$

where $c > 0$ is a constant.

Proof. (Analogue of Lemma 2.9 in [2]). □

It follows from (1) and (3) that

$$C_\mu(x) = \frac{1}{\Gamma(\mu + 1)} j_\mu(2\sqrt{x}).$$

2. Main Result

In this section, we give the main result of this paper. We first need to define the (β,γ)-second Hankel–Clifford Lipschitz class.

Definition 2.1. Let $\beta \in (0,1)$ and $\gamma \geq 0$. A function $f \in L^2_\mu$ is said to be in the (β,γ)-second Hankel–Clifford Lipschitz class, denoted by $SHLip(\beta,2,\gamma)$ if

$$\left\| T_h f(x) - \frac{1}{\Gamma(\mu+1)} f(x) \right\|_{L^2_\mu} = O\left(\frac{h^\beta}{\left(\log \frac{1}{h}\right)^\gamma} \right) \text{ as } h \longrightarrow 0.$$

Our main result is as follows.

Theorem 2.1. *Let $f \in L^2_\mu$, then the following are equivalent:*

1. $f \in SHLip(\beta,2,\gamma)$,
2. $\int_r^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(\frac{r^{-2\beta}}{(\log 4r)^{2\gamma}} \right)$ *as* $r \longrightarrow +\infty$.

Proof. 1) ⇒ 2). Assume that $f \in SHLip(\beta,2,\gamma)$. Then

$$\left\| T_h f(x) - \frac{1}{\Gamma(\mu+1)} f(x) \right\|_{L^2_\mu} = O\left(\frac{h^\beta}{\left(\log \frac{1}{h}\right)^\gamma} \right) \text{ as } h \longrightarrow 0.$$

From (2), we have

$$\begin{aligned} h_{2,\mu}\Big(T_h f - \frac{1}{\Gamma(\mu+1)} f\Big)(\lambda) &= \Big(C_\mu(\lambda h) - \frac{1}{\Gamma(\mu+1)}\Big) h_{2,\mu}(f)(\lambda) \\ &= \frac{1}{\Gamma(\mu+1)} \big(j_\mu(2\sqrt{\lambda h}) - 1\big) h_{2,\mu}(f)(\lambda). \end{aligned}$$

Parseval's identity yields

$$\begin{aligned} &\left\| T_h f(x) - \frac{1}{\Gamma(\mu+1)} f(x) \right\|^2_{L^2_\mu} \\ &= \frac{1}{(\Gamma(\mu+1))^2} \int_0^{+\infty} \left| 1 - j_\mu(2\sqrt{\lambda h}) \right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda. \end{aligned} \tag{4}$$

If $\lambda \in \left[\frac{1}{4h}, \frac{2}{4h}\right]$, then $2\sqrt{\lambda h} \geq 1$ and Lemma 1.2 implies that

$$1 \leq \frac{1}{c^2} \left| 1 - j_\mu(2\sqrt{\lambda h}) \right|^2.$$

Then

$$\begin{aligned} &\frac{1}{(\Gamma(\mu+1))^2} \int_{\frac{1}{4h}}^{\frac{2}{4h}} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\ &\leq \frac{1}{c^2 (\Gamma(\mu+1))^2} \int_{\frac{1}{4h}}^{\frac{2}{4h}} \left| 1 - j_\mu(2\sqrt{\lambda h}) \right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\ &\leq \frac{1}{c^2 (\Gamma(\mu+1))^2} \int_0^{+\infty} \left| 1 - j_\mu(2\sqrt{\lambda h}) \right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\ &\leq \frac{1}{c^2} \left\| T_h f(x) - \frac{1}{\Gamma(\mu+1)} f(x) \right\|^2_{L^2_\mu} \\ &= O\left(\frac{h^{2\beta}}{\left(\log \frac{1}{h}\right)^{2\gamma}} \right) \end{aligned}$$

holds and we obtain

$$\int_r^{2r} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(\frac{r^{-2\beta}}{(\log 4r)^{2\gamma}}\right) \text{ as } r \longrightarrow +\infty.$$

Thus there exists $c_1 > 0$ such that

$$\int_r^{2r} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \le c_1 \frac{r^{-2\beta}}{(\log 4r)^{2\gamma}}.$$

So,

$$\begin{aligned}\int_r^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda &= \left[\int_r^{2r} + \int_{2r}^{4r} + \int_{4r}^{8r} + \cdots\right] |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\ &\le c_1 \frac{r^{-2\beta}}{(\log 4r)^{2\gamma}} + c_1 \frac{(2r)^{-2\beta}}{(\log 8r)^{2\gamma}} + c_1 \frac{(4r)^{-2\beta}}{(\log 16r)^{2\gamma}} + \cdots \\ &\le c_1 \frac{r^{-2\beta}}{(\log 4r)^{2\gamma}} \left(1 + 2^{-2\beta} + \left(2^{-2\beta}\right)^2 + \left(2^{-2\beta}\right)^3 + \cdots\right) \\ &\le c_1 K_\beta \frac{r^{-2\beta}}{(\log 4r)^{2\gamma}},\end{aligned}$$

where $K_\beta = \left(1 - 2^{-2\beta}\right)^{-1}$, since $2^{-2\beta} < 1$.

This proves that

$$\int_r^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(\frac{r^{-2\beta}}{(\log 4r)^{2\gamma}}\right).$$

2) $\Rightarrow$ 1). Now, suppose that

$$\int_r^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(\frac{r^{-2\beta}}{(\log 4r)^{2\gamma}}\right) \text{ as } r \longrightarrow +\infty.$$

Then we write

$$\int_0^{+\infty} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = I_1 + I_2,$$

where

$$I_1 = \int_0^{\frac{1}{4h}} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda$$

and

$$I_2 = \int_{\frac{1}{4h}}^{+\infty} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda.$$

Estimate the summands I_1 and I_2.

From (1) of Lemma 1.1, we have

$$\begin{aligned} I_2 &= \int_{\frac{1}{4h}}^{+\infty} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\ &\leq 4 \int_{\frac{1}{4h}}^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda \\ &= O\Big(\frac{h^{2\beta}}{\left(\log \frac{1}{h}\right)^{2\gamma}}\Big). \end{aligned}$$

Then

$$\frac{1}{\Gamma(\mu+1)} \int_{\frac{1}{4h}}^{+\infty} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\Big(\frac{h^{2\beta}}{\left(\log \frac{1}{h}\right)^{2\gamma}}\Big).$$

Set

$$\varphi(x) = \int_x^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda.$$

We know from (2) of Lemma 1.1 that

$$1 - j_\mu(2\sqrt{\lambda h}) = O(\lambda h) \ \text{ for } \ 0 \leq 2\sqrt{\lambda h} \leq 1.$$

Thus there exists $c_2 > 0$ such that

$$\left|1 - j_\mu(2\sqrt{\lambda h})\right| \leq c_2 \lambda h \ \text{ for } \ 0 \leq 2\sqrt{\lambda h} \leq 1.$$

Then

$$I_1 \leq -c_2 h^2 \int_0^{\frac{1}{4h}} x^2 \varphi'(x) dx.$$

Using integration by parts, we obtain

$$\begin{aligned} I_1 &\leq -c_2 h^2 \int_0^{\frac{1}{4h}} x^2 \varphi'(x) dx \\ &\leq -c_2 \varphi\Big(\frac{1}{4h}\Big) + 2c_2 h^2 \int_0^{\frac{1}{4h}} x\varphi(x) dx \\ &\leq 2c_2 h^2 \int_0^{\frac{1}{4h}} x\varphi(x) dx \\ &\leq c_3 h^2 \int_0^{\frac{1}{4h}} \frac{x^{1-2\beta}}{(\log 4x)^{2\gamma}} dx \\ &\leq c_3 h^2 \int_0^{\frac{1}{h}} \frac{x^{1-2\beta}}{(\log x)^{2\gamma}} dx \\ &\leq c_3 \frac{h^{2\beta}}{\left(\log \frac{1}{h}\right)^{2\gamma}}\,. \end{aligned}$$

Then

$$\frac{1}{\Gamma(\mu+1)}\int_0^{\frac{1}{4h}}\left|1-j_\mu(2\sqrt{\lambda h})\right|^2|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda=O\bigg(\frac{h^{2\beta}}{\left(\log\frac{1}{h}\right)^{2\gamma}}\bigg).$$

Hence

$$\left\|T_hf(x)-\frac{1}{\Gamma(\mu+1)}f(x)\right\|_{L^2_\mu}=O\bigg(\frac{h^{\beta}}{\left(\log\frac{1}{h}\right)^{\gamma}}\bigg)\ \text{ as }\ h\longrightarrow 0$$

and this completes the proof. □

Definition 2.2. A function $f\in L^2_\mu$ is said to be in the (ψ,γ)-second Hankel–Clifford Lipschitz class, denoted by $SHLip(\psi,2,\gamma)$, if

$$\left\|T_hf(x)-\frac{1}{\Gamma(\mu+1)}f(x)\right\|_{L^2_\mu}=O\bigg(\frac{\psi(h)}{\left(\log\frac{1}{h}\right)^{\gamma}}\bigg),\quad \gamma>0\ \text{ as }\ h\longrightarrow 0,$$

where

1. $\psi(t)$ is a continuous increasing function on $[0,+\infty)$,
2. $\psi(0)=0$,
3. $\psi(ts)=\psi(t)\psi(s)\quad$ for all $s,t\in[0,+\infty)$,
4. $\int_0^{\frac{1}{h}}x\frac{\psi(x^{-2})}{(\log x)^{2\gamma}}dx=O\bigg(\frac{1}{h^2}\frac{\psi(h^2)}{\left(\log\frac{1}{h}\right)^{2\gamma}}\bigg)$.

Theorem 2.2. *Let $f\in L^2_\mu$ and let ψ be a fixed function satisfying the conditions of Definition 2.2. Then the following are equivalent:*

1. $f\in SHLip(\psi,2,\gamma)$,
2. $\int_r^{+\infty}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda=O\bigg(\frac{\psi(r^{-2})}{(\log 4r)^{2\gamma}}\bigg)$ *as;* $r\longrightarrow+\infty$.

Proof. 1) ⇒ 2). Assume that $f\in SHLip(\psi,2,\gamma)$. Then we have

$$\left\|T_hf(x)-\frac{1}{\Gamma(\mu+1)}f(x)\right\|_{L^2_\mu}=O\bigg(\frac{\psi(h)}{\left(\log\frac{1}{h}\right)^{\gamma}}\bigg)\ \text{ as }\ h\longrightarrow 0.$$

If $\lambda\in\left[\frac{1}{4h},\frac{2}{4h}\right]$, then $2\sqrt{\lambda h}\geq 1$, and in a similar manner as in the proof of Theorem 2.1, we obtain

$$\begin{aligned}\int_{\frac{1}{4h}}^{\frac{2}{4h}}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda&\leq\frac{1}{c^2}\left\|T_hf(x)-\frac{1}{\Gamma(\mu+1)}f(x)\right\|^2_{L^2_\mu}\\&=O\bigg(\frac{\psi(h^2)}{\left(\log\frac{1}{h}\right)^{2\gamma}}\bigg).\end{aligned}$$

Thus there exists a positive constant $c_4>0$ such that

$$\int_r^{2r}|h_{2,\mu}(f)(\lambda)|^2\lambda^\mu d\lambda\leq c_4\frac{\psi(r^{-2})}{(\log 4r)^{2\gamma}}\,.$$

Hence

$$\int_r^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = \left[\int_r^{2r} + \int_{2r}^{4r} + \int_{4r}^{8r} + \cdots\right] |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda$$

$$\le c_4 \frac{\psi\left(r^{-2}\right)}{(\log 4r)^{2\gamma}} + c_4 \frac{\psi\left((2r)^{-2}\right)}{(\log 8r)^{2\gamma}} + c_4 \frac{\psi\left((4r)^{-2}\right)}{(\log 16r)^{2\gamma}} + \cdots$$

$$\le c_4 \frac{\psi\left(r^{-2}\right)}{(\log 4r)^{2\gamma}} + c_4 \frac{\psi\left((2r)^{-2}\right)}{(\log 4r)^{2\gamma}} + c_4 \frac{\psi\left((4r)^{-2}\right)}{(\log 4r)^{2\gamma}} + \cdots$$

$$\le c_4 \frac{\psi\left(r^{-2}\right)}{(\log 4r)^{2\gamma}} \left(1 + \psi\left(2^{-2}\right) + \left(\psi\left(2^{-2}\right)\right)^2 + \left(\psi\left(2^{-2}\right)\right)^3 + \cdots\right)$$

$$\le c_4 K \frac{\psi\left(r^{-2}\right)}{(\log 4r)^{2\gamma}},$$

where $K = \left(1 - \psi\left(2^{-2}\right)\right)^{-1}$. Since by (1) and (3) of Definition 2.2, it follows that $\psi\left(2^{-2}\right) < 1$.

This proves that

$$\int_r^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(\frac{\psi\left(r^{-2}\right)}{(\log 4r)^{2\gamma}}\right) \text{ as } r \longrightarrow +\infty.$$

2) $\Rightarrow$ 1). Now, suppose that

$$\int_r^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(\frac{\psi\left(r^{-2}\right)}{(\log 4r)^{2\gamma}}\right) \text{ as } r \longrightarrow +\infty.$$

By (4) of Definition 2.2, it follows that we have to show that

$$\int_0^{+\infty} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = O\left(\frac{\psi(h^2)}{\left(\log \frac{1}{h}\right)^{2\gamma}}\right) \text{ as } h \longrightarrow 0$$

and we write

$$\int_0^{+\infty} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda = I_1 + I_2$$

where

$$I_1 = \int_0^{\frac{1}{4h}} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda$$

and

$$I_2 = \int_{\frac{1}{4h}}^{+\infty} \left|1 - j_\mu(2\sqrt{\lambda h})\right|^2 |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda.$$

First, from (1) of Lemma 1.1, we see that

$$I_2 \le 4 \int_{\frac{1}{4h}}^{+\infty} |h_{2,\mu}(f)(\lambda)|^2 \lambda^\mu d\lambda$$

$$= O\left(\frac{\psi(h^2)}{\left(\log \frac{1}{h}\right)^{2\gamma}}\right) \text{ as } h \longrightarrow 0.$$

In proving Theorem 2.1, we can see that there exists a positive constant c_2 such that

$$I_1 \leq 2c_2h^2 \int_0^{\frac{1}{4h}} x\varphi(x)dx.$$

Then

$$I_1 \leq c_4h^2 \int_0^{\frac{1}{4h}} x\frac{\psi(x^{-2})}{(\log 4x)^{2\gamma}}dx.$$

By (4) of Definition 2.2, it follows that

$$I_1 = O\bigg(\frac{\psi(h^2)}{\left(\log \frac{1}{h}\right)^{2\gamma}}\bigg),$$

where c_4 is a positive constant and this completes the proof. □

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