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# COMPATIBLE STRUCTURE IN IDEAL $m$-SPACES

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**Abstract.** In this paper, an extensive study of ideal on $m$-spaces $(X, m)$ is given and some new types of sets are introduced with the help of local functions. Several characterizations of these sets are also discussed through this paper. Moreover, characterizations of $f_\psi$-operator and $\psi$-codense on the $m$ are obtained and the notion of $\psi$-compatibility with an ideal $\mathscr{I}$ is investigated.

## 1. Introduction and Preliminaries

An ideal $\mathscr{I}$ on a space $X$ is a nonempty collection of subsets of $X$ which satisfies the following properties:

(1) $A \in \mathscr{I}$ and $B \subseteq A$ implies that $B \in \mathscr{I}$.
(2) $A \in \mathscr{I}$ and $B \in \mathscr{I}$ imply that $A \cup B \in \mathscr{I}$.

An ideal topological space is a topological space $(X, \tau)$ with an ideal $\mathscr{I}$ on $X$ and we denote it by $(X, \tau, \mathscr{I})$ (see [7,8]).

**Definition 1.1** ([12])**.** A subfamily $m$ of the power set $\mathscr{P}(X)$ of a nonempty set $X$ is called a *minimal structure* (briefly, *$m$-structure*) on $X$ if $m$ satisfies the following conditions:
(1) $\emptyset \in m$ and $X \in m$.
(2) The union of any family of subsets belonging to $m$ belongs to $m$.

By $(X, m)$ we denote a nonempty set $X$ with a minimal structure $m$ on $X$ and call it an *$m$-space*. Each member of $m$ is said to be *$m$-open* and the complement of an $m$-open set is said to be *$m$-closed*. For a point $x \in X$, the family $\{U : x \in U \text{ and } U \in m\}$ is denoted by $m(x)$.

Let $(X, m)$ be an $m$-space and $A$ be a subset of $X$. The *$m$-closure* $m\operatorname{Cl}(A)$ and the *$m$-interior* $m\operatorname{Int}(A)$ of $A$ [9] are defined as follows:

(1) $m\operatorname{Cl}(A) = \cap\{F \subset X : A \subset F, X \setminus F \in m\}$.
(2) $m\operatorname{Int}(A) = \cup\{U \subset X : U \subset A, U \in m\}$.

**Definition 1.2** ([11])**.** Let $(X, m, \mathscr{I})$ be an ideal $m$-space. For a subset $A$ of $X$, the *minimal local function* $A^*(\mathscr{I}, m)$ of $A$ is defined as follows:

$$A^*(\mathscr{I}, m) = \{x \in X : U \cap A \notin \mathscr{I} \text{ for every } U \in m(x)\}.$$

Hereafter, $A^*(\mathscr{I}, m)$ is denoted simply by $A^*$. An ideal $m$-space $(X, m, \mathscr{I})$ is said to be $\mathscr{I}$-resolvable if X has two disjoint $\mathscr{I}$-dense subsets, where a subset $A$ of $X$ is $\mathscr{I}$-dense if $A^* = X$. Also, papers [1–5] introduce some property related to the ideal $m$-spaces.

**Definition 1.3** ([6])**.** Let $(X, m)$ be an $m$-space. A function $\psi : m \to \mathscr{P}(X)$ is called a $\psi$-operation on $m$ if $\psi(U) \subseteq U$ for every proper subset $U \in m$ and $\psi(X) = X$. A subset $A$ of $X$ is said to be $\psi$-open if there exists a proper subset $U \in m$ such that $A \subseteq \psi(U)$ or $A = \psi(X) = X$. We put $\Psi_m = \{A \subseteq X : A \subseteq \psi(U) \text{ for some proper subset } U \in m \text{ or } A = X\}$. Then $\Psi_m$ is the family of all $\psi$-open sets. The complement of a $\psi$-open set is said to be $\psi$-closed.

In this paper, the characterizations of $f_\psi$-operator and $\psi$-codense on the $m$ are given and the notion of $\psi$-compatibility with an ideal $\mathscr{I}$ is investigated.

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**Lemma 1.1** ([6]). *Let $(X, m)$ be an $m$-space. For $\Psi_m$, the following properties hold:*

(1) $\emptyset, X \in \Psi_m$.
(2) *If $A_\alpha \in \Psi_m$ for each $\alpha \in \Lambda$, then $\cap_{\alpha\in\Lambda} A_\alpha \in \Psi_m$.*

**Definition 1.4** ([6]). Let $(X, m, \mathscr{I})$ be an ideal $m$-space. For a subset $A$ of $X$, we define the following set: $A_\psi(\mathscr{I}, m) = \{x \in X : A \cap U \notin \mathscr{I}$ for every $U \in \Psi_m(x)\}$, where $\Psi_m(x) = \{U \in \Psi_m : x \in U\}$. In case there is no confusion $A_\psi(\mathscr{I}, m)$ is briefly denoted by $A_\psi$ and is called the $\psi$-local function of $A$ with respect to $\mathscr{I}$ and $m$.

We set $\mathrm{Int}_\psi(A) = \cup\{U : U \subseteq A, U \in \Psi_m\}$ and $\mathrm{Cl}_\psi(A) = \cap\{F : A \subseteq F, X - F \in \Psi_m\}$.

**Lemma 1.2** ([6]). *Let $(X, m)$ be an $m$-space, $\mathscr{I}$ and $\mathscr{J}$ be ideals on $X$ and let $A$ and $B$ be subsets of $X$. Then the following properties hold:*

(1) *If $A \subseteq B$, then $A_\psi \subseteq B_\psi$.*
(2) *If $\mathscr{I} \subseteq \mathscr{J}$, then $A_\psi(\mathscr{I}) \supseteq A_\psi(\mathscr{J})$.*
(3) $A_\psi = \mathrm{Cl}_\psi(A_\psi) \subseteq \mathrm{Cl}_\psi(A)$.
(4) *If $A \subseteq A_\psi$, then $A_\psi = \mathrm{Cl}_\psi(A_\psi) = \mathrm{Cl}_\psi(A)$.*
(5) *If $A \in \mathscr{I}$, then $A_\psi = \emptyset$.*
(6) $(A \cap B)_\psi \subseteq A_\psi \cap B_\psi$.

**Corollary 1.1** ([6]). *Let $(X, m, \mathscr{I})$ be an ideal $m$-space and $A, B$ be subsets of $X$ with $B \in \mathscr{I}$. Then $(A \cup B)_\psi = A_\psi = (A - B)_\psi$.*

**Theorem 1.1** ([6]). *Let $(X, m, \mathscr{I})$ be an ideal $m$-space and $A$, $B$ be any subsets of $X$. Then the following properties hold:*

(1) $(\emptyset)_\psi = \emptyset$.
(2) $(A_\psi)_\psi \subseteq A_\psi$.
(3) $A_\psi \cup B_\psi = (A \cup B)_\psi$.

**Theorem 1.2** ([6]). *Let $(X, m, \mathscr{I})$ be an ideal $m$-space, $\mathrm{Cl}^*_\psi(A) = A_\psi \cup A$ and $A$, $B$ be subsets of $X$. Then*

(1) $\mathrm{Cl}^*_\psi(\emptyset) = \emptyset$.
(2) $A \subseteq \mathrm{Cl}^*_\psi(A)$.
(3) $\mathrm{Cl}^*_\psi(A \cup B) = \mathrm{Cl}^*_\psi(A) \cup \mathrm{Cl}^*_\psi(B)$.
(4) $\mathrm{Cl}^*_\psi(A) = \mathrm{Cl}^*_\psi(\mathrm{Cl}^*_\psi(A))$.
(5) *If $A \subseteq B$, then $\mathrm{Cl}^*_\psi(A) \subseteq \mathrm{Cl}^*_\psi(B)$.*

By Theorem 1.2, we find that $\mathrm{Cl}^*_\psi(A) = A \cup A_\psi$ is a Kuratowski closure operator. We denote by $\Psi^*_\psi(\mathscr{I}) = \Psi^*_\psi$ the topology generated by $\mathrm{Cl}^*_\psi$, that is, $\Psi^*_\psi = \{U \subseteq X : \mathrm{Cl}^*_\psi(X - U) = X - U\}$. A subset $A$ of $X$ is said to be $\Psi^*_\psi$-closed if and only if $A_\psi \subseteq A$.

**Theorem 1.3** ([6]). *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. Then $\beta(\Psi_m, \mathscr{I}) = \{V - I : V \in \Psi_m, I \in \mathscr{I}\}$ is a basis for $\Psi^*_\psi$.*

The following example shows that $\beta(\Psi_m, \mathscr{I})$ is not a topology, in general.

**Example 1.1.** Let $X = \{a, b, c, d\}$ and $m = \{\emptyset, X, \{a, b\}, \{a, c, d\}\}$ with $\mathscr{I} = \{\emptyset, \{a\}\}$. A function $\psi : m \to \mathscr{P}(X)$ is defined as $\psi(X) = X$, $\psi(\{a, b\}) = \{a\}$, $\psi(\{a, c, d\}) = \{c, d\}$ and $\psi(\emptyset) = \emptyset$. Then $\Psi_m = \{\emptyset, X, \{a\}, \{c\}, \{d\}, \{c, d\}\}$ and $\beta(\Psi_m, \mathscr{I}) = \{\emptyset, X, \{a\}, \{c\}, \{d\}, \{c, d\}, \{b, c, d\}\}$ and $\Psi^*_\psi = \{\emptyset, X, \{a\}, \{c\}, \{d\}, \{c, d\}, \{a, c\}, \{a, d\}, \{a, c, d\}, \{b, c, d\}\}$.

It is clear that $m$ and $\Psi_m$ are independent, and we have $\Psi_m \subseteq \beta(\Psi_m, \mathscr{I}) \subseteq \Psi^*_\psi$.

We recall that $\mathscr{I}$ is $\psi$-codense in an ideal $m$-space if $\Psi_m \cap \mathscr{I} = \emptyset$.

**Example 1.2.** Let $X = \{a, b, c, d\}$ and $m = \{\emptyset, X, \{a, b\}, \{a, c, d\}\}$ with $\mathscr{I} = \{\emptyset, \{b\}\}$. A function $\psi : m \to \mathscr{P}(X)$ is defined as $\psi(X) = X$, $\psi(\{a, b\}) = \{a\}$, $\psi(\{a, c, d\}) = \{c, d\}$ and $\psi(\emptyset) = \emptyset$. Then $\Psi_m = \{\emptyset, X, \{a\}, \{c\}, \{d\}, \{c, d\}\}$. It is clear that $\mathscr{I}$ is $\psi$-codense.

**Example 1.3.** Let $X = \{a, b, c, d\}$ and $m = \{\emptyset, X, \{a, b\}, \{a, c, d\}\}$ with $\mathscr{I} = \{\emptyset, \{a\}, \{b\}\}$. A function $\psi : m \to \mathscr{P}(X)$ is defined as $\psi(X) = X$, $\psi(\{a, b\}) = \{a\}$, $\psi(\{a, c, d\}) = \{c, d\}$ and $\psi(\emptyset) = \emptyset$. Then $\Psi_m = \{\emptyset, X, \{a\}, \{c\}, \{d\}, \{c, d\}\}$. It is clear that $\mathscr{I}$ is not $\psi$-codense.

**Theorem 1.4** ([6]). *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. Then the following properties are equivalent:*

(1) *$\mathscr{I}$ is $\psi$-codense;*
(2) *If $I \in \mathscr{I}$, then $\text{Int}_\psi(I) = \emptyset$;*
(3) *For every $G \in \Psi_m$, $G \subseteq G_\psi$;*
(4) *$X = X_\psi$.*

**Lemma 1.3.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space and $A, B$ be subsets of $X$. Then $A_\psi - B_\psi = (A - B)_\psi - B_\psi$.*

## 2. $f_\psi$-operator in Ideal $m$-spaces

**Definition 2.1.** Let $(X, m, \mathscr{I})$ be an ideal $m$-space. An operator $f_\psi : \mathscr{P}(X) \to \mathscr{P}(X)$ is defined as follows for every $A \in X$, $f_\psi(A) = \{x \in X : \text{there exists } U \in \Psi_m(x) \text{ such that } U - A \in \mathscr{I}\}$ and we observe that $f_\psi(A) = X - (X - A)_\psi$.

Several basic facts concerning the behavior of the operator $f_\psi$ are included in the following theorem.

**Theorem 2.1.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. Then the following properties hold:*

(1) *If $A \subseteq X$, then $f_\psi(A)$ is $\psi$-open.*
(2) *If $A \subseteq B$, then $f_\psi(A) \subseteq f_\psi(B)$.*
(3) *If $A, B \in \mathscr{P}(X)$, then $f_\psi(A \cap B) = f_\psi(A) \cap f_\psi(B)$.*
(4) *If $U \in \Psi^*_\psi$, then $U \subseteq f_\psi(U)$.*
(5) *If $A \subseteq X$, then $f_\psi(A) \subseteq f_\psi(f_\psi(A))$.*
(6) *If $A \subseteq X$, then $f_\psi(A) = f_\psi(f_\psi(A))$ if and only if $(X - A)_\psi = ((X - A)_\psi)_\psi$.*
(7) *If $A \in \mathscr{I}$, then $f_\psi(A) = X - X_\psi$.*
(8) *If $A \subseteq X$, then $A \cap f_\psi(A) = \text{Int}_\psi(A)$.*
(9) *If $A \subseteq X$, $I \in \mathscr{I}$, then $f_\psi(A - I) = f_\psi(A)$.*
(10) *If $A \subseteq X$, $I \in \mathscr{I}$, then $f_\psi(A \cup I) = f_\psi(A)$.*
(11) *If $(A - B) \cup (B - A) \in \mathscr{I}$, then $f_\psi(A) = f_\psi(B)$.*

*Proof.* (1) This follows from Lemma 1.2 (3).
(2) This follows from Lemma 1.2 (1).
(3) It follows from (2) that $f_\psi(A \cap B) \subseteq f_\psi(A)$ and $f_\psi(A \cap B) \subseteq f_\psi(B)$. Hence $f_\psi(A \cap B) \subseteq f_\psi(A) \cap f_\psi(B)$. Now, let $x \in f_\psi(A) \cap f_\psi(B)$. There exist $U, V \in \Psi_m(x)$ such that $U - A \in \mathscr{I}$ and $V - B \in \mathscr{I}$. Let $G = U \cap V \in \Psi_m(x)$ and we have $G - A \in \mathscr{I}$ and $G - B \in \mathscr{I}$ by the assumption. Thus $G - (A \cap B) = (G - A) \cup (G - B) \in \mathscr{I}$ by additivity, and hence $x \in f_\psi(A \cap B)$. We have shown $f_\psi(A) \cap f_\psi(B) \subseteq f_\psi(A \cap B)$ and thus the proof is complete.
(4) If $U \in \Psi^*_\psi$, then $X - U$ is $\Psi^*_\psi$-closed which implies $(X - U)_\psi \subseteq X - U$ and hence $U \subseteq X - (X - U)_\psi = f_\psi(U)$.
(5) This follows from (4).
(6) This follows from the facts:

(1) $f_\psi(A) = X - (X - A)_\psi$.
(2) $f_\psi(f_\psi(A)) = X - [X - (X - (X - A)_\psi)]_\psi = X - ((X - A)_\psi)_\psi$.

(7) By Corollary 1.1, we obtain $(X - A)_\psi = X_\psi$ if $A \in \mathscr{I}$.
(8) If $x \in A \cap f_\psi(A)$, then $x \in A$ and there exists a $U_x \in \Psi_m(x)$ such that $U_x - A \in \mathscr{I}$. Then by Theorem 1.3, $U_x - (U_x - A)$ is an $\Psi^*_\psi$-open neighborhood of $x$ and $x \in \text{Int}_\psi(A)$. On the other hand, if $x \in \text{Int}_\psi(A)$, there exists a basic $\Psi^*_\psi$-open neighborhood $V_x - I$ of $x$, where $V_x \in \Psi_m$ and $I \in \mathscr{I}$, such that $x \in V_x - I \subseteq A$ which implies $V_x - A \subseteq I$ and hence $V_x - A \in \mathscr{I}$. So, $x \in A \cap f_\psi(A)$.
(9) This follows from Corollary 1.1 and $f_\psi(A - I) = X - [X - (A - I)]_\psi = X - [(X - A) \cup I]_\psi = X - (X - A)_\psi = f_\psi(A)$.
(10) This follows from Corollary 1.1 and $f_\psi(A \cup I) = X - [X - (A \cup I)]_\psi = X - [(X - A) - I]_\psi = X - (X - A)_\psi = f_\psi(A)$.

(11) Assume $(A - B) \cup (B - A) \in \mathscr{I}$. Let $A - B = I$ and $B - A = J$. Observe that $I, J \in \mathscr{I}$ by the assumption. Observe also that $B = (A - I) \cup J$. Thus $f_\psi(A) = f_\psi(A - I) = \Psi[(A - I) \cup J] = f_\psi(B)$ by (9) and (10). □

**Corollary 2.1.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. Then $U \subseteq f_\psi(U)$ for every $\psi$-open set $U \in \Psi_m$.*

*Proof.* We know that $f_\psi(U) = X - (X - U)_\psi$. Now, $(X - U)_\psi \subseteq \mathrm{Cl}_\psi(X - U) = X - U$, since $X - U$ is $\psi$-closed. Therefore $U = X - (X - U) \subseteq X - (X - U)_\psi = f_\psi(U)$. □

**Theorem 2.2.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space and $A \subseteq X$. Then the following properties hold:*

(1) $f_\psi(A) = \cup\{U \in \Psi_m : U - A \in \mathscr{I}\}$.
(2) $f_\psi(A) \supseteq \cup\{U \in \Psi_m : (U - A) \cup (A - U) \in \mathscr{I}\}$.

*Proof.* (1) This follows immediately from the definition of $f_\psi$-operator.
(2) Since $\mathscr{I}$ is heredity, it is obvious that $\cup\{U \in \Psi_m : (U - A) \cup (A - U) \in \mathscr{I}\} \subseteq \cup\{U \in \Psi_m : U - A \in \mathscr{I}\} = f_\psi(A)$ for every $A \subseteq X$. □

**Theorem 2.3.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. If $\sigma = \{A \subseteq X : A \subseteq f_\psi(A)\}$, then $\sigma$ is a topology for $X$ and $\sigma = \Psi^*_\psi$.*

*Proof.* Let $\sigma = \{A \subseteq X : A \subseteq f_\psi(A)\}$. First, we show that $\sigma$ is a topology. Observe that $\emptyset \subseteq f_\psi(\emptyset)$ and $X \subseteq f_\psi(X) = X$, and thus $\emptyset$ and $X \in \sigma$. Now, if $A, B \in \sigma$, then $A \cap B \subseteq f_\psi(A) \cap f_\psi(B) = f_\psi(A \cap B)$ which implies that $A \cap B \in \sigma$. If $\{A_\alpha : \alpha \in \Delta\} \subseteq \sigma$, then $A_\alpha \subseteq f_\psi(A_\alpha) \subseteq f_\psi(\cup A_\alpha)$ for every $\alpha$ and hence $\cup A_\alpha \subseteq f_\psi(\cup A_\alpha)$. This shows that $\sigma$ is a topology. Now, if $U \in \Psi^*_\psi$ and $x \in U$, then by Theorem 1.3, there exist $V \in \Psi_m(x)$ and $I \in \mathscr{I}$ such that $x \in V - I \subseteq U$. Clearly, $V - U \subseteq I$ so, $V - U \in \mathscr{I}$ by the assumption and hence $x \in f_\psi(U)$. Thus $U \subseteq f_\psi(U)$ and we have shown that $\Psi^*_\psi \subseteq \sigma$. Now, let $A \in \sigma$, then we have $A \subseteq f_\psi(A)$, that is, $A \subseteq X - (X - A)_\psi$ and $(X - A)_\psi \subseteq X - A$. This shows that $X - A$ is $\Psi^*_\psi$-closed and hence $A \in \Psi^*_\psi$. Thus $\sigma \subseteq \Psi^*_\psi$ and hence $\sigma = \Psi^*_\psi$. □

## 3. Some Properties of $\psi$-compatible in Ideal $m$-spaces

**Definition 3.1** ([6])**.** Let $(X, m, \mathscr{I})$ be an ideal $m$-space. The $m$-structure $m$ is said to be $\psi$-compatible with the ideal $\mathscr{I}$, denoted by $m \sim_\psi \mathscr{I}$, if for every $A \subseteq X$, the following holds: if for every $x \in A$, there exists $U \in \Psi_m(x)$ such that $U \cap A \in \mathscr{I}$, then $A \in \mathscr{I}$.

**Lemma 3.1** ([6])**.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space, then $m \sim_\psi \mathscr{I}$ if and only if $A - A_\psi \in \mathscr{I}$ for every $A \subseteq X$.*

**Example 3.1.** Let $X = \{a, b, c, d\}$ and $m = \{\emptyset, X, \{a, b\}, \{a, c, d\}\}$ with $\mathscr{I} = \{\emptyset, \{a\}\}$. A function $\psi : m \to \mathscr{P}(X)$ is defined as $\psi(X) = X$, $\psi(\{a, b\} = \{a\}, \psi(\{a, c, d\} = \{a\}$ and $\psi(\emptyset) = \emptyset$. Then $\Psi = \{\emptyset, X, \{a\}\}$. Since $A - A_\psi \in \mathscr{I}$ for every $A \subseteq X$, therefore the $m$-structure $m$ is $\psi$-compatible with the ideal $\mathscr{I}$. Also, $\beta(\Psi, \mathscr{I}) = \{\emptyset, X, \{a\}, \{b, c, d\}\}$ and $m^*_\psi = \{\emptyset, X, \{a\}, \{b, c, d\}\}$.

**Example 3.2.** Let $X = \mathbb{R}$ and let us consider the $m$-structure $m = \{A \subseteq \mathbb{R} : 1 \notin A\} \cup \{\mathbb{R}\}$ with the ideal of finite subsets of $X$ which are denoted by $\mathscr{I}_{Fin}$. A function $\psi : m \to \mathscr{P}(X)$ is defined as $\psi(A) = A$, for all $A \subseteq X$ . Then $\Psi = m = \{A \subseteq \mathbb{R} : 1 \notin A\} \cup \{\mathbb{R}\}$. Now, for any $A \in m$, $A_\psi = \emptyset$ or $A_\psi = \{1\}$. Since for some $A \subseteq X$ we have $A - A_\psi \notin \mathscr{I}_{Fin}$, therefore the $m$-structure $m$ is not $\psi$-compatible with the ideal $\mathscr{I}_{Fin}$.

**Theorem 3.1.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space, $m$ be $\psi$-compatible with $\mathscr{I}$ is $\psi$-codense. Let $G$ be a $\Psi^*_\psi$-open set such that $G = U - A$, where $U \in \Psi_m$ and $A \in \mathscr{I}$. Then $\mathrm{Cl}_\psi(G_\psi) = \mathrm{Cl}_\psi(G) = G_\psi = U_\psi = \mathrm{Cl}_\psi(U) = \mathrm{Cl}_\psi(U_\psi)$.*

*Proof.* (1) Let $G = U - A$, where $U \in \Psi_m$ and $A \in \mathscr{I}$. Since $\mathscr{I}$ is $\psi$-codense, by Theorem 1.4, we have $U \subseteq U_\psi$. Hence by Lemma 1.2, $U_\psi = \mathrm{Cl}_\psi(U_\psi) = \mathrm{Cl}_\psi(U)$.
(2) Since $G$ is $\Psi^*_\psi$-open, $X - G = \mathrm{Cl}^*_\psi(X - G)$ and hence $(X - G)_\psi \subseteq X - G$. By Lemma 1.3, $X_\psi - G_\psi \subseteq (X - G)_\psi$. But $\Psi_m \cap \mathscr{I} = \emptyset$ and by Theorem 1.4, $X_\psi = X$ and hence $X - G_\psi \subseteq (X - G)_\psi \subseteq X - G$. Therefore $G \subseteq G_\psi$. Hence $\mathrm{Cl}_\psi(G) \subseteq \mathrm{Cl}_\psi(G_\psi)$. Hence by Lemma 1.2, $G_\psi = \mathrm{Cl}_\psi(G) = \mathrm{Cl}_\psi(G_\psi)$.
(3) Again, $G \subseteq U$ implies that $G_\psi \subseteq U_\psi$. By Lemma 1.3, $G_\psi = (U - A)_\psi \supseteq U_\psi - A_\psi = U_\psi$ since $A \in \mathscr{I}$. Thus $U_\psi = G_\psi$.
By (1), (2) and (3), we obtain the result. □

**Theorem 3.2.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. Then $m \sim_\psi \mathscr{I}$ if and only if $f_\psi(A) - A \in \mathscr{I}$ for every $A \subseteq X$.*

*Proof. Necessity.* Assume $m \sim_\psi \mathscr{I}$ and let $A \subseteq X$. Observe that $x \in f_\psi(A) - A \in \mathscr{I}$ if and only if $x \notin A$ and $x \notin (X - A)_\psi$ if and only if $x \notin A$ and there exists $U_x \in \Psi_m(x)$ such that $U_x - A \in \mathscr{I}$ if and only if there exists $U_x \in \Psi_m(x)$ such that $x \in U_x - A \in \mathscr{I}$. Now, for each $x \in f_\psi(A) - A$ and $U_x \in \Psi_m(x)$, $U_x \cap (f_\psi(A) - A) \in \mathscr{I}$ by the assumption and hence $f_\psi(A) - A \in \mathscr{I}$ by the assumption that $m \sim_\psi \mathscr{I}$.

*Sufficiency.* Let $A \subseteq X$ and assume that for each $x \in A$, there exists $U_x \in \Psi_m(x)$ such that $U_x \cap A \in \mathscr{I}$. Observe that $f_\psi(X - A) - (X - A) = \{x :$ there exists $U_x \in \Psi_m(x)$ such that $x \in U_x \cap A \in \mathscr{I}\}$. Thus we have $A \subseteq f_\psi(X - A) - (X - A) \in \mathscr{I}$ and hence $A \in \mathscr{I}$ by the assumption of $\mathscr{I}$. □

**Lemma 3.2.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space such that $m \sim_\psi \mathscr{I}$ and $A \subseteq X$, then $A$ is a $\Psi^*_\psi$-closed if and only if $A = B \cup I$ such that $B$ is $\psi$-closed and $I \in \mathscr{I}$.*

*Proof.* If $A$ is a $\Psi^*_\psi$-closed set, then $A_\psi \subseteq A$ which implies that $A = A \cup A_\psi = (A - A_\psi) \cup A_\psi$. Then by Lemma 1.2, $A_\psi$ is a $\psi$-closed set and by Lemma 3.1, $A - A_\psi \in \mathscr{I}$. Conversely, if $A = B \cup I$ such that $B$ is an $\psi$-closed set and $I \in \mathscr{I}$, then by Corollary 1.1, we get $A_\psi = (B \cup I)\psi = B_\psi \cup I_\psi = B_\psi \subseteq \mathrm{Cl}_\psi(B) = B \subseteq A$ which implies that $A$ is a $\Psi^*_\psi$-closed. □

**Corollary 3.1.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space such that $m \sim_\psi \mathscr{I}$. Then $\beta(\mathscr{I}, m)$ is a topology on $X$ and hence $\beta(\mathscr{I}, m) = \Psi^*_\psi$.*

*Proof.* Let $A \in \Psi^*_\psi$. Then by Lemma 3.2, $X - A = F \cup I$, where $F$ is $\psi$-closed and $I \in \mathscr{I}$. Then $A = X - (F \cup I) = (X - F) \cap (X - I) = (X - F) - I = V - I$, where $V = X - F \in \Psi_m$. Thus every $\psi$-open set is of the form $V - I$, where $V \in \Psi_m$ and $I \in \mathscr{I}$. The result follows by Theorem 1.3. □

**Proposition 3.1.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space with $m \sim_\psi \mathscr{I}$, $A \subseteq X$. If $N$ is a nonempty $\psi$-open subset of $A_\psi \cap f_\psi(A)$, then $N - A \in \mathscr{I}$ and $N \cap A \notin \mathscr{I}$.*

*Proof.* If $N \subseteq A_\psi \cap f_\psi(A)$, then $N - A \subseteq f_\psi(A) - A \in \mathscr{I}$ by Theorem 3.2, and hence $N - A \in \mathscr{I}$, by the assumption. Since $N \in \Psi_m - \{\emptyset\}$ and $N \subseteq A_\psi$, we have $N \cap A \notin \mathscr{I}$ by the definition of $A_\psi$. □

As a consequence of the above proposition, we have the following

**Corollary 3.2.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space with $m \sim_\psi \mathscr{I}$. Then $f_\psi(f_\psi(A)) = f_\psi(A)$ for every $A \subseteq X$.*

*Proof.* $f_\psi(A) \subseteq f_\psi(f_\psi(A))$ follows from Theorem 2.1 (5). Since $m \sim_\psi \mathscr{I}$, it follows from Theorem 3.2 that $f_\psi(A) \subseteq A \cup I$ for some $I \in \mathscr{I}$ and hence $f_\psi(f_\psi(A)) = f_\psi(A)$ by Theorem 2.1 (10). □

**Theorem 3.3.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space with $m \sim_\psi \mathscr{I}$. Then $f_\psi(A) = \cup\{f_\psi(U) : U \in \Psi_m, f_\psi(U) - A \in \mathscr{I}\}$.*

*Proof.* Let $\Phi(A) = \cup\{f_\psi(U) : U \in \Psi_m, f_\psi(U) - A \in \mathscr{I}\}$. Clearly, $\Phi(A) \subseteq f_\psi(A)$. Now, let $x \in f_\psi(A)$. Then there exists $U \in \Psi_m(x)$ such that $U - A \in \mathscr{I}$. By Corollary 2.1, $U \subseteq f_\psi(U)$ and $f_\psi(U) - A \subseteq [f_\psi(U) - U] \cup [U - A]$. By Theorem 3.2, $f_\psi(U) - U \in \mathscr{I}$ and hence $f_\psi(U) - A \in \mathscr{I}$. Thus $x \in \Phi(A)$ and $\Phi(A) \supseteq f_\psi(A)$. Consequently, we obtain $\Phi(A) = f_\psi(A)$. □

In [10], Newcomb defines $A = B$ [mod $\mathscr{I}$] if $(A - B) \cup (B - A) \in \mathscr{I}$ and observes that [mod $\mathscr{I}$] is an equivalence relation. By Theorem 2.1 (11), we have that if $A = B$ [mod $\mathscr{I}$], then $f_\psi(A) = f_\psi(B)$.

**Definition 3.2.** Let $(X, m, \mathscr{I})$ be an ideal $m$-space. A subset $A$ of $X$ is called a Baire set with respect to $\Psi_m$ and $\mathscr{I}$, if there exists an $\psi$-open set $U \in \Psi_m$ such that $A = U$ [mod $\mathscr{I}$], where the collection of all Baire sets is denoted by $\mathscr{W}_r(X, m, \mathscr{I})$.

**Lemma 3.3.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space with $m \sim_\psi \mathscr{I}$. If $U, V \in \Psi_m$ and $f_\psi(U) = f_\psi(V)$, then $U = V$ [mod $\mathscr{I}$].*

*Proof.* Since $U \in \Psi_m$, we have $U \subseteq f_\psi(U)$ and hence $U - V \subseteq f_\psi(U) - V = f_\psi(V) - V \in \mathscr{I}$ by Theorem 3.2. Similarly, $V - U \in \mathscr{I}$. Now, $(U - V) \cup (V - U) \in \mathscr{I}$ by additivity. Hence $U = V$ [mod $\mathscr{I}$]. □

**Theorem 3.4.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space with $m \sim_\psi \mathscr{I}$. If $A$, $B \in \mathscr{W}_r(X, m, \mathscr{I})$ and $f_\psi(A) = f_\psi(B)$, then $A = B$ $[\bmod\ \mathscr{I}]$.*

*Proof.* Let $U, V \in \Psi_m$ such that $A = U$ $[\bmod\ \mathscr{I}]$ and $B = V$ $[\bmod\ \mathscr{I}]$. Now, $f_\psi(A) = f_\psi(U)$ and $f_\psi(B) = f_\psi(V)$ by Theorem 2.1(11). Since $f_\psi(A) = f_\psi(U)$ implies that $f_\psi(U) = f_\psi(V)$, hence $U = V$ $[\bmod\ \mathscr{I}]$ by Lemma 3.3. Thus $A = B$ $[\bmod\ \mathscr{I}]$ by transitivity. □

## 4. $\psi$-codense in Ideal $m$-spaces

**Lemma 4.1.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. If $A$ is a $\psi$-open set, then it is $\psi$-codense if and only if $A_\psi = \mathrm{Cl}_\psi(A)$.*

*Proof.* Let $A$ be nonempty $\psi$-open sets, then by Lemma 1.2, we have $A_\psi \subseteq \mathrm{Cl}_\psi(A)$. Let $x \in \mathrm{Cl}_\psi(A)$, then for all $\psi$-open set $U_x$ containing $x$, we have $U_x \cap A \neq \phi$. Again, $U_x \cap A$ is a nonempty $\psi$-open set, so $U_x \cap A \notin \mathscr{I}$, since $\mathscr{I}$ is $\psi$-codense. Hence $x \in A_\psi$. Therefore $A_\psi = \mathrm{Cl}_\psi(A)$. Conversely, for any $\psi$-open set $A$, we have $A_\psi = \mathrm{Cl}_\psi(A)$. Then $X = X_\psi$ and this implies that $\mathscr{I}$ is $\psi$-codense by Theorem 1.4. □

**Proposition 4.1.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space.*

(1) *If $B \in \mathscr{W}_r(X, m, \mathscr{I}) - \mathscr{I}$, then there exists $A \in \Psi_m - \{\emptyset\}$ such that $B = A$ $[\bmod\ \mathscr{I}]$.*
(2) *If $\mathscr{I}$ is $\psi$-codense, then $B \in \mathscr{W}_r(X, m, \mathscr{I}) - \mathscr{I}$ if and only if there exists $A \in \Psi_m - \{\emptyset\}$ such that $B = A$ $[\bmod\ \mathscr{I}]$.*

*Proof.* (1) Assume $B \in \mathscr{W}_r(X, m, \mathscr{I}) - \mathscr{I}$, then $B \in \mathscr{W}_r(X, m, \mathscr{I})$. Now, if there does not exist $A \in \Psi_m - \{\emptyset\}$ such that $B = A$ $[\bmod\ \mathscr{I}]$, we have $B = \emptyset$ $[\bmod\ \mathscr{I}]$. This implies that $B \in \mathscr{I}$ which is a contradiction.
(2) Assume there exists $A \in \Psi_m - \{\emptyset\}$ such that $B = A$ $[\bmod\ \mathscr{I}]$. Then $A = (B - J) \cup I$, where $J = B - A, I = A - B \in \mathscr{I}$. If $B \in \mathscr{I}$, then $A \in \mathscr{I}$ by the assumption and additivity, which contradicts that $\mathscr{I}$ is $\psi$-codense. □

**Proposition 4.2.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space with $\mathscr{I}$ is $\psi$-codense. If $B \in \mathscr{W}_r(X, m, \mathscr{I}) - \mathscr{I}$, then $f_\psi(B) \cap \mathrm{Int}_\psi(B_\psi) \neq \emptyset$.*

*Proof.* Assume $B \in \mathscr{W}_r(X, m, \mathscr{I}) - \mathscr{I}$, then by Proposition 4.1(1), there exists $A \in \Psi_m - \{\emptyset\}$ such that $B = A$ $[\bmod\ \mathscr{I}]$. This implies that $\emptyset \neq A \subseteq A_\psi = ((B - J) \cup I)_\psi = B_\psi$, where $J = B - A, I = A - B \in \mathscr{I}$ by Theorem 1.1 and Corollary 1.1. Also, $\emptyset \neq A \subseteq f_\psi(A) = f_\psi(B)$ by Theorem 2.1 (11), so, $A \subseteq f_\psi(B) \cap \mathrm{Int}_\psi(B_\psi)$. □

Given an ideal $m$-space $(X, m, \mathscr{I})$, let $\mathscr{U}(X, m, \mathscr{I})$ denote $\{A \subseteq X :$ there exists $B \in \mathscr{W}_r(X, m, \mathscr{I}) - \mathscr{I}$ such that $B \subseteq A\}$.

**Proposition 4.3.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space with $\mathscr{I}$ is $\psi$-codense. The following properties are equivalent:*

(1) $A \in \mathscr{U}(X, m, \mathscr{I})$.
(2) $f_\psi(A) \cap \mathrm{Int}_\psi(A_\psi) \neq \emptyset$.
(3) $f_\psi(A) \cap A_\psi \neq \emptyset$.
(4) $f_\psi(A) \neq \emptyset$.
(5) $\mathrm{Int}_\psi(A) \neq \emptyset$.
(6) *There exists $N \in \Psi_m - \{\emptyset\}$ such that $N - A \in \mathscr{I}$ and $N \cap A \notin \mathscr{I}$.*

*Proof.* (1) ⇒ (2): Let $B \in \mathscr{W}_r(X, m, \mathscr{I}) - \mathscr{I}$ such that $B \subseteq A$. Then $\mathrm{Int}_\psi(B_\psi) \subseteq \mathrm{Int}_\psi(A_\psi)$ and $f_\psi(B) \subseteq f_\psi(A)$ and hence $\mathrm{Int}_\psi(B_\psi) \cap f_\psi(B) \subseteq \mathrm{Int}_\psi(A_\psi) \cap f_\psi(A)$. By Proposition 4.2, we have $f_\psi(A) \cap \mathrm{Int}_\psi(A_\psi) \neq \emptyset$.
(2) ⇒ (3): The proof is obvious.
(3) ⇒ (4): The proof is obvious.
(4) ⇒ (5): If $f_\psi(A) \neq \emptyset$, then there exists $U \in \Psi_m - \{\emptyset\}$ such that $U - A \in \mathscr{I}$. Since $U \notin \mathscr{I}$ and $U = (U - A) \cup (U \cap A)$, we have $U \cap A \notin \mathscr{I}$. By Theorem 2.1, $\emptyset \neq (U \cap A) \subseteq f_\psi(U) \cap A = f_\psi((U - A) \cup (U \cap A)) \cap A = f_\psi(U \cap A) \cap A \subseteq f_\psi(A) \cap A = \mathrm{Int}_\psi(A)$. Hence $\mathrm{Int}_\psi(A) \neq \emptyset$.
(5) ⇒ (6): If $\mathrm{Int}_\psi(A) \neq \emptyset$, then by Theorem 1.3, there exists $N \in \Psi_m - \{\emptyset\}$ and $I \in \mathscr{I}$ such that

$\emptyset \neq N - I \subseteq A$. We have $N - A \in \mathscr{I}$, $N = (N - A) \cup (N \cap A)$ and $N \notin \mathscr{I}$. This implies that $N \cap A \notin \mathscr{I}$.
(6) $\Rightarrow$ (1): Let $B = N \cap A \notin \mathscr{I}$ with $N \in \Psi_m - \{\emptyset\}$ and $N - A \in \mathscr{I}$. Then $B \in \mathscr{W}_r(X, m, \mathscr{I}) - \mathscr{I}$ since $B \notin \mathscr{I}$ and $(B - N) \cup (N - B) = N - A \in \mathscr{I}$. □

**Theorem 4.1.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space, where $\mathscr{I}$ is $\psi$-codense. Then for $A \subseteq X$, $f_\psi(A) \subseteq A_\psi$.*

*Proof.* Suppose $x \in f_\psi(A)$ and $x \notin A_\psi$. Then there exists a nonempty neighborhood $U_x \in \Psi_m(x)$ such that $U_x \cap A \in \mathscr{I}$. Since $x \in f_\psi(A)$, by Theorem 2.2, $x \in \cup\{U \in \Psi_m : U - A \in \mathscr{I}\}$ and there exists $V \in \Psi_m$ such that $x \in V$ and $V - A \in \mathscr{I}$. Now, we have $U_x \cap V \in \Psi_m(x)$, $U_x \cap V \cap A \in \mathscr{I}$ and $(U_x \cap V) - A \in \mathscr{I}$ by the assumption. Hence by a finite additivity, we have $(U_x \cap V \cap A) \cup (U_x \cap V - A) = (U_x \cap V) \in \mathscr{I}$. Since $(U_x \cap V) \in \Psi_m(x)$, this contradicts to $\mathscr{I}$ is $\psi$-codense. Therefore $x \in A_\psi$. This implies that $f_\psi(A) \subseteq A_\psi$. □

**Corollary 4.1.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space, where $\mathscr{I}$ is $\psi$-codense. Then for $A \subseteq X$, $f_\psi(A) \subseteq \mathrm{Cl}_\psi(A_\psi)$.*

**Theorem 4.2.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. Then the following properties are equivalent:*

(1) *$\mathscr{I}$ is $\psi$-codense.*
(2) *$f_\psi(\emptyset) = \emptyset$.*
(3) *If $A \subseteq X$ is $\psi$-closed, then $f_\psi(A) - A = \emptyset$.*
(4) *If $I \in \mathscr{I}$, then $f_\psi(I) = \emptyset$.*

*Proof.* (1) $\Rightarrow$ (2): Since $\mathscr{I}$ is $\psi$-codense, by Theorem 2.2, we have $f_\psi(\emptyset) = \cup\{U \in \Psi_m : U \in \mathscr{I}\} = \emptyset$.
(2) $\Rightarrow$ (3): Suppose $x \in f_\psi(A) - A$, then there exists $U_x \in \Psi_m(x)$ such that $x \in U_x - A \in \mathscr{I}$ and $U_x - A \in \Psi_m$. But $U_x - A \in \{U \in \Psi_m : U \in \mathscr{I}\} = f_\psi(\emptyset)$ which implies that $f_\psi(\emptyset) = \emptyset$. Hence $f_\psi(A) - A = \emptyset$.
(3) $\Rightarrow$ (4): Let $I \in \mathscr{I}$ and since $\emptyset_m$ is $\psi$-closed, therefore $f_\psi(I) = f_\psi(I \cup \emptyset) = f_\psi(\emptyset) = \emptyset$.
(4) $\Rightarrow$ (1): Suppose $A \in \Psi_m \cap \mathscr{I}$, then $A \in \mathscr{I}$ and by (4), $f_\psi(A) = \emptyset$. Since $A \in \Psi_m$, by Corollary 2.1, we have $A \subseteq f_\psi(A) = \emptyset$. Hence $\mathscr{I}$ is $\psi$-codense. □

**Theorem 4.3.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. Then $\mathscr{I}$ is $\psi$-codense if and only if $[f_\psi(A)]_\psi = \mathrm{Cl}_\psi[f_\psi(A)]$ for every $A \subseteq X$.*

*Proof.* Let $\mathscr{I}$ be $\psi$-codense. It is obvious that $[f_\psi(A)]_\psi \subseteq \mathrm{Cl}_\psi[f_\psi(A)]$. For the reverse inclusion, let $x \in \mathrm{Cl}_\psi[f_\psi(A)]$. Then for every $\psi$-open sets $U_x$ containing $x$, $U_x \cap f_\psi(A) \neq \emptyset$ and $U_x \cap f_\psi(A) \in \Psi$ implies that $U_x \cap f_\psi(A) \notin \mathscr{I}$, since $\mathscr{I}$ is $\psi$-codense. Hence $x \in [f_\psi(A)]_\psi$. Thus $[f_\psi(A)]_\psi = \mathrm{Cl}_\psi[f_\psi(A)]$. Conversely, suppose that $[f_\psi(A)]_\psi = \mathrm{Cl}_\psi[f_\psi(A)]$, for every $A \subseteq X$. Then for $X \subseteq X$, $[f_\psi(X)]_\psi = \mathrm{Cl}_\psi[f_\psi(X)]$. Hence $[X - (X - X)_\psi]_\psi = \mathrm{Cl}_\psi[X - (X - X)_\psi]$ implies that $X_\psi = \mathrm{Cl}_\psi(X) = X$. Thus $\mathscr{I}$ is $\psi$-codense. □

**Theorem 4.4.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space such that $m \sim_\psi \mathscr{I}$ and $\mathscr{I}$ is $\psi$-codense. Then*

(1) *$(X, \Psi_m)$ is Hausdorff or Urysohn if and only if $(X, \Psi^*_\psi)$ is respectively so.*
(2) *If $(X, \Psi^*_\psi)$ is regular, then $\Psi_m = \Psi^*_\psi$.*
(3) *$(X, \Psi_m)$ is connected if and only if $(X, \Psi^*_\psi)$ is connected.*

*Proof.* (1) Let $(X, \Psi^*_\psi)$ be Hausdorff and $x$, $y$ be any two distinct points of $X$. Then there exist disjoint $\Psi^*_\psi$-open sets $G$ and $H$ containing $x$ and $y$, respectively. Then by Corollary 3.1, $G = U - I_1$ and $H = V - I_2$, where $U$, $V \in \Psi_m$ and $I_1, I_2 \in \mathscr{I}$. Since $U$ and $V$ are $\psi$-open sets containing $x$ and $y$, respectively, it remains to show that $U \cap V = \emptyset$. Now, $G \cap H = [U - I_1] \cap [V - I_2] = [U \cap V] - [I_1 \cup I_2] = \emptyset$, then $U \cap V \subseteq I_1 \cup I_2$ and hence $[U \cap V]_\psi \subseteq [I_1 \cup I_2]_\psi = [I_1]_\psi \cup [I_2]_\psi = \emptyset$ by Lemma 1.2 and Theorem 1.1. Since $\mathscr{I}$ is $\psi$-codense, we have by Lemma 4.1 that $U \cap V \subseteq [U \cap V]_\psi = \emptyset$, so, $U \cap V = \emptyset$. The converse is trivial.

Next, let $(X, \Psi^*_\psi)$ be Urysohn and $x, y$ be two distinct points of $X$. Then there exist $\Psi^*_\psi$-open sets $G$ and $H$ containing $x$ and $y$, respectively, and $\mathrm{Cl}^*_\psi(G) \cap \mathrm{Cl}^*_\psi(H) = \emptyset$, where, by Corollary 3.1, we can take $G = U - I_1$ and $H = V - I_2$, where $U$, $V \in \Psi$ and $I_1, I_2 \in \mathscr{I}$. Then $x \in U$, $y \in V$ and by Theorem 3.1, $\mathrm{Cl}_\psi(U) \cap \mathrm{Cl}_\psi(V) = \emptyset$. Hence $(X, \Psi_m)$ is Urysohn.

Conversely, if $(X, \Psi_m)$ is Urysohn, then for $x, y \in X$ with $x \neq y$, there exist $U, V \in \Psi_m$ such that $\mathrm{Cl}_\psi(U) \cap \mathrm{Cl}_\psi(V) = \emptyset$. Then $U, V$ are $\Psi^*_\psi$-open and by Theorem 3.1, $\mathrm{Cl}_\psi(U) = \mathrm{Cl}^*_\psi(U)$ and $\mathrm{Cl}_\psi(V) = \mathrm{Cl}^*_\psi(V)$ and hence $(X, \Psi^*_\psi)$ is Urysohn.

(2) For any $A \subseteq X$, we clearly have $\mathrm{Cl}^*_\psi(A) \subseteq \mathrm{Cl}_\psi(A)$. Let $x \notin \mathrm{Cl}^*_\psi(A)$. Then for some $\Psi^*_\psi$-open neighbourhood of $G$ of $x$ and $G \cap A = \emptyset$. By regularity of $(X, \Psi^*_\psi)$, there exists $H \in \Psi^*_\psi$ with $H = U - I$, where $U \in \Psi_m$ and $I \in \mathscr{I}$ such that $x \in H \subseteq \mathrm{Cl}^*_\psi(H) \subseteq G$. Now, $U \cap A \subseteq \mathrm{Cl}_\psi(U) \cap A = \mathrm{Cl}^*_\psi(H) \cap A \subseteq G \cap A = \emptyset$ by Theorem 3.1, and hence $U \cap A = \emptyset$, where $x \in U \in \Psi_m$, then $x \notin \mathrm{Cl}_\psi(A)$. Hence $\mathrm{Cl}^*_\psi(A) = \mathrm{Cl}_\psi(A)$ for each $A \subseteq X$ and $\Psi_m = \Psi^*_\psi$.

(3) If $(X, \Psi^*_\psi)$ is connected, then so is $(X, \Psi_m)$. Suppose $(X, \Psi^*_\psi)$ is not connected, then there exists a nonempty $\Psi^*_\psi$-clopen set $A \neq X$ and $X = A \cup (X - A)$, then $X = X_\psi = [A \cup (X - A)]_\psi = A_\psi \cup (X - A)_\psi$. Now, $A$ and $X - A$ are $\Psi^*_\psi$-closed, $A_\psi \cup (X - A)_\psi \subseteq A \cup (X - A)$ and hence $A_\psi \cup (X - A)_\psi = \emptyset$. Again, as $A$ is $\Psi^*_\psi$-open, by Theorem 3.1, $A_\psi = \mathrm{Cl}_\psi(A) \neq \emptyset$. Similarly, $(X - A)_\psi = \mathrm{Cl}_\psi(X - A) \neq \emptyset$. Thus $X = \mathrm{Cl}_\psi(A) \cup \mathrm{Cl}_\psi(X - A)$ and $\mathrm{Cl}_\psi(A) \cap \mathrm{Cl}_\psi(X - A) = \emptyset$ and $\mathrm{Cl}_\psi(A) \neq \emptyset \neq \mathrm{Cl}_\psi(X - A)$ and hence $(X, \Psi_m)$ is not connected. □

We recall that a topological space $X$ is called quasi $H$-closed ($QHC$, for short) [13] if every open cover of $X$ has a finite subcollection, the union of its closures cover of $X$.

**Theorem 4.5.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space such that $\mathscr{I}$ is $\psi$-codense. Then $(X, \Psi_m)$ is $QHC$ if and only if $(X, \Psi^*_\psi)$ is $QHC$.*

*Proof.* Let $(X, \Psi_m)$ be $QHC$ and let $\mathscr{U} = \{U_\alpha - I_\alpha : U_\alpha \in \Psi_m, I_\alpha \in \mathscr{I}, \alpha \in \Lambda\}$ be a $\Psi^*_\psi$-basic open cover of $X$. Then $\{U_\alpha : \alpha \in \Lambda\}$ is a $\psi$-open cover of $X$. By a quasi $H$-closedness of $(X, \Psi_m)$, there exist finitely many $\alpha$, say, $\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda$ such that $X = \cup_{i=1}^n \mathrm{Cl}_\psi(U_{\alpha_i})$. We have to show that $X = \cup_{i=1}^n \mathrm{Cl}^*_\psi(U_{\alpha_i} - I_{\alpha_i})$. Suppose $x \in X = \cup_{i=1}^n \mathrm{Cl}_\psi(U_{\alpha_i})$ such that $x \notin \cup_{i=1}^n \mathrm{Cl}^*_\psi(U_{\alpha_i} - I_{\alpha_i})$. Then $x \notin \mathrm{Cl}^*_\psi(U_{\alpha_i} - I_{\alpha_i})$ for each $i = 1, 2, \ldots, n$, while for some $\alpha_k \in \{\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda\}$, $x \in \mathrm{Cl}_\psi(U_{\alpha_k})$. Since $x \notin \mathrm{Cl}^*_\psi(U_{\alpha_i} - I_{\alpha_i})$, we get $G_i = V_i - I_i$ with $V_i \in \Psi_m$ and $I_i \in \mathscr{I}$ such that $x \in G_i$ and $G_i \cap [U_{\alpha_i} - I_{\alpha_i}] = \emptyset$ for $i = 1, 2, \ldots, n$. Now, $x \in G = G_1 \cap G_2 \cap \cdots \cap G_n = [V_1 \cap V_2 \cap \cdots \cap V_n] - [I_1 \cup I_2 \cup \cdots \cup I_n] \in \Psi^*_\psi$. Then this implies that $G \cap [U_{\alpha_k} - I_{\alpha_k}] = \emptyset$ and $U_{\alpha_k} \cap [V_1 \cap V_2 \cap \cdots \cap V_n] \neq \emptyset$ and so, $U_{\alpha_k} \cap [V_1 \cap V_2 \cap \cdots \cap V_n] \notin \mathscr{I}$. To arrive at a contradiction, we only show that $U_{\alpha_k} \cap [V_1 \cap V_2 \cap \cdots \cap V_n] \subseteq I_{\alpha_k} \cup [I_1 \cup I_2 \cup \cdots \cup I_n] \in \mathscr{I}$. Let $z \in U_{\alpha_k} \cap [V_1 \cap V_2 \cap \cdots \cap V_n]$. Then as $\emptyset = G \cap [U_{\alpha_k} - I_{\alpha_k}] = [(V_1 \cap V_2 \cap \cdots \cap V_n) - (I_1 \cup I_2 \cup \cdots \cup I_n)] \cap [U_{\alpha_k} - I_{\alpha_k}]$, we have $z \in (I_1 \cup I_2 \cup \cdots \cup I_n)$ or $z \in I_{\alpha_k}$ and hence $z \in (I_1 \cup I_2 \cup \cdots \cup I_n) \cup I_{\alpha_k}$. This completes the proof. □

**Definition 4.1.** A subset $A$ in an ideal $m$-space $(X, m, \mathscr{I})$ is said to be $\mathscr{I}_\psi$-dense if $A_\psi = X$.

An ideal $m$-space is $\psi$-hyperconnected if every nonempty $\psi$-open set is $\mathscr{I}_\psi$-dense in $X$.

**Proposition 4.4.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. Then the following properties are equivalent:*

(1) *Every nonempty $\psi$-open set is $\mathscr{I}_\psi$-dense;*
(2) *$(X, m, \mathscr{I})$ is $\psi$-hyperconnected and $\mathscr{I}$ is $\psi$-codense.*

*Proof.* (1) ⇒ (2): Since every nonempty $\psi$-open set is $\mathscr{I}_\psi$-dense, then $(X, m, \mathscr{I})$ is $\psi$-hyperconnected. Let $A$ be $\psi$-open, nonempty and a member of the ideal. By (1), $A_\psi = X$. On the other hand, since $A \in \mathscr{I}$, $A_\psi = \emptyset$. Hence $X = \emptyset$. By the contradiction, $\mathscr{I}$ is $\psi$-codense.
(2) ⇒ (1): Let $\emptyset \neq A \in \Psi_m$. Let $x \in X$. Due to the $\psi$-hyperconnectedness of $(X, m, \mathscr{I})$, every $\psi$-open neighborhood $V$ of $x$ meets $A$. Moreover, $A \cap V$ is a $\psi$-open non-ideal set, since $\mathscr{I}$ is $\psi$-codense. Thus $x \in A_\psi$. This shows that $A_\psi = X$ and $A$ is $\mathscr{I}_\psi$-dense. □

**Definition 4.2.** An ideal $m$-space $(X, m, \mathscr{I})$ is said to be $\mathscr{I}_\psi$-resolvable if X has two disjoint $\mathscr{I}_\psi$-dense subsets.

**Lemma 4.2.** *If $(X, m, \mathscr{I})$ is $\mathscr{I}_\psi$-resolvable, then $\mathscr{I}$ is $\psi$-codense.*

*Proof.* If $X = A \cup B$, where $A$ and $B$ are disjoint $\mathscr{I}_\psi$-dense, then $A_\psi = X$ and $B_\psi = X$. Therefore $\Psi_m \cap A \notin \mathscr{I}$ and $\Psi_m \cap B \notin \mathscr{I}$. Hence $\Psi_m \cap \mathscr{I} = \emptyset$, and $\mathscr{I}$ is $\psi$-codense. □

**Proposition 4.5.** *Every $\mathscr{I}_\psi$-resolvable ideal $m$-space $(X, m, \mathscr{I})$ is $\mathscr{I}$-resolvable.*

*Proof.* If $X = A \cup B$, where $A$ and $B$ are disjoint $\mathscr{I}_\psi$-dense, then $A_\psi = X$ and $B_\psi = X$. Therefore $X = A_\psi \subseteq A^*$ and $X = B_\psi \subseteq B^*$, we get $X = A^*$ and $X = B^*$. Hence $X = A \cup B$, where $A$ and $B$ are disjoint $\mathscr{I}$-dense and $(X, \tau, \mathscr{I})$ is $\mathscr{I}$-resolvable. □

The collection of all $\mathscr{I}_\psi$-dense in $(X, m, \mathscr{I})$ is denoted by $\mathscr{I}_\psi D(X, \Psi_m)$. The collection of all $m$-dense sets in $(X, m)$ is denoted by $D(X, m)$. Now, we show that the collection of $m$-dense sets in $m$-space $(X, \Psi^*_\psi)$ and the collection of $\mathscr{I}_\psi$-dense sets in ideal $m$-space $(X, m, \mathscr{I})$ are equal if $\mathscr{I}$ is $\psi$-codense.

**Theorem 4.6.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. If $\mathscr{I}$ is $\psi$-codense, then $\mathscr{I}_\psi D(X, \Psi_m) = D(X, \Psi^*_\psi)$.*

*Proof.* Let $D \in \mathscr{I}_\psi D(X, \Psi_m)$. Then $\mathrm{Cl}^*_\psi(D) = D \cup D_\psi = X$, i.e., $D \in D(X, \Psi^*_\psi)$. Therefore $\mathscr{I}_\psi D(X, \Psi_m) \subseteq D(X, \Psi^*_\psi)$.

Conversely, let $D \in D(X, \Psi^*_\psi)$. Then $\mathrm{Cl}^*_\psi(D) = D \cup D_\psi = X$. We prove that $D_\psi = X$. Let $x \in X$ such that $x \notin D_\psi$. Therefore there exists $\emptyset \neq U \in \Psi_m$ such that $U \cap D \in \mathscr{I}$. Since $U \notin \mathscr{I}$, $U \cap (X - D) \notin \mathscr{I}$, hence $U \cap (X - D) \neq \emptyset$. Let $x_0 \in U \cap (X - D)$. Then $x_0 \notin D$ and also $x_0 \notin D_\psi$. Since $x_0 \in D_\psi$ implies that $U \cap D \notin \mathscr{I}$, this contradicts to $U \cap D \in \mathscr{I}$. Thus $x_0 \notin D \cup D_\psi = \mathrm{Cl}^*_\psi(D) = X$. This is a contradiction. Therefore we obtain $D \in \mathscr{I}_\psi D(X, \Psi_m)$. Thus $D(X, \Psi^*_\psi) \subseteq \mathscr{I}_\psi D(X, \Psi)$. Hence $\mathscr{I}_\psi D(X, \Psi_m) = D(X, \Psi^*_\psi)$. □

**Theorem 4.7.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. Then for $x \in X$, $X - \{x\}$ is $\mathscr{I}_\psi$-dense if and only if $f_\psi(\{x\}) = \emptyset$.*

*Proof.* The proof follows from the definition of $\mathscr{I}_\psi$-dense sets, since $f_\psi(\{x\}) = X - (X - \{x\})_\psi = \emptyset$ if and only if $X = (X - \{x\})_\psi$. □

**Proposition 4.6.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space. $A \nsubseteq \mathrm{Cl}_\psi[f_\psi(A)]$ if and only if there exist $x \in A$ and a $\psi$-open set $V_x$ of $x$ for which $X - A$ is relatively with $\mathscr{I}_\psi$-dense in $V_x$.*

*Proof.* Let $A \nsubseteq \mathrm{Cl}_\psi[f_\psi(A)]$. There exists $x \in X$ such that $x \in A$, but $x \notin \mathrm{Cl}_\psi[f_\psi(A)]$. Hence there exists a $\psi$-open set $V_x$ of $x$ such that $V_x \cap f_\psi(A) = \emptyset$. This implies that $V_x \cap [X - (X - A)_\psi] = \emptyset$ and so, $V_x \subseteq (X - A)_\psi$. Let $U$ be any nonempty $\psi$-open set in $V_x$. Since $V_x \subseteq (X - A)_\psi$, therefore $U \cap (X - A) \notin \mathscr{I}$. This implies that $X - A$ is relatively with $\mathscr{I}_\psi$-dense in $V_x$. The converse part is obvious by reversing process. □

**Proposition 4.7.** *Let $(X, m, \mathscr{I})$ be an ideal $m$-space with $\mathscr{I}$ is $\psi$-codense. Then $f_\psi(A) \neq \emptyset$ if and only if $A$ contains a nonempty $\Psi^*_\psi$-interior.*

*Proof.* Let $f_\psi(A) \neq \emptyset$. By Theorem 2.2 (1), $f_\psi(A) = \cup\{U \in \Psi_m : U - A \in \mathscr{I}\}$ and there exists a nonempty set $U \in \Psi_m$ such that $U - A \in \mathscr{I}$. Let $U - A = P$, where $P \in \mathscr{I}$. Now, $U - P \subseteq A$. By Theorem 1.3, $U - P \in \Psi^*_\psi$ and $A$ contains a nonempty $\Psi^*_\psi$-interior.

Conversely, suppose that $A$ contains a nonempty $\Psi^*_\psi$-interior. Hence there exist $U \in \Psi_m$ and $P \in \mathscr{I}$ such that $U - P \subseteq A$. So, $U - A \subseteq P$. Let $H = U - A \subseteq P$, then $H \in \mathscr{I}$. Hence $\cup\{U \in \Psi_m : U - A \in \mathscr{I}\} = f_\psi(A) \neq \emptyset$. □

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