Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 1, 145–152

SOLUTION OF BOUNDARY VALUE PROBLEMS OF THE COUPLED THEORY OF ELASTICITY FOR A POROUS BODY

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Abstract. The boundary value problems of the coupled linear theory of elasticity are solved for isotropic one-porous solids of specific shape. Special representations of a general solution of a system of differential equations are constructed by means of elementary functions. With the help of these representations, the solutions to the problems are presented explicitly, in the form of absolutely and uniformly convergent series. The question of the uniqueness of regular solutions to the problems under consideration is investigated.

1. Introduction

Of the theories describing the mechanical properties of single-porous materials, we may single out the Biot theory of consolidation [1] based on the Darcy law concept and the Nunziato–Cowin theory [10,17] based on the concept of volume fractions. In the Biot theory, the independent variables are the displacement vector field and the average fluid pressure in the pore network. Information about the Biot theory, generalizations of this theory and the main results can be found in [6,7,9,11,12,18,20,30]. In the Nunziato–Cowin theory, the independent variables are the displacement vector field and the change in the volume fraction of pores. This theory describes materials with empty pores. The main results in the theories for single-porous materials with voids, as well as the historical development of the concept of volume fractions, can be found in [5,6,8,13,14,18].

When studying many problems of a physical nature in porous media, we often encounter various related processes [4,19]. Therefore, it is natural to consider several related mechanical concepts at the same time. In the works of Svanadze [21–24], a mathematical model is studied that describes the coupled phenomena of the concepts of the Darcy law and the volume fraction of pores. It is shown that this coupled linear model of porous elastic bodies can be established by combining three variables: the displacement vector field, the change in the pore volume fraction, and the average fluid pressure. In this theory, the effect of the relationship between fluid pressure in the pores and the change in the volume fraction of pores is presented. The coupled linear theory of elasticity for isotropic porous materials, in which the Darcy law concept and the volume fraction are related, is considered in [2,3,15,16,22,25,26].

Along with the generalization and development in various directions of the linear theory of elasticity for porous materials, much attention has recently been paid to mathematical research and the construction of solutions to boundary value problems for specific areas. It is important to construct solutions to the problems in an explicit form, which makes it possible to effectively carry out a numerical analysis of the problem under study.

In this article, the Svanadze model [21] is considered in the two-dimensional case, in which the Darcy law concepts and the area fraction of pores are related. The system of general governing equations is expressed in terms of the displacement vector field, changes in the area fraction of pores and fluid pressure in the network of pores. Special representations of the general solution of the system of differential equations of the theory of elastic materials are constructed by using elementary functions. This approach allows us to reduce the original system of equations to equations of a simple structure. Using these representations, one can solve static two-dimensional boundary value problems of the coupled theory of elasticity for a single-porous body.

2020 *Mathematics Subject Classification.* 74F10, 74G05, 74G30.

Key words and phrases. Coupled theory of elasticity; Explicit solutions; Single-porous solid.

Section 2 presents the basic equations of the coupled theory of elasticity and formulates the main boundary value problems of statics for a single-porous body.

In Section 3, we construct a general representation of a solution of the system of equations of coupled elasticity theory by using harmonic, biharmonic and metaharmonic functions.

In Section 4, Green's identities are established and uniqueness theorems are proved for solutions of the formulated problems.

In Section 5, the problems posed are solved for an elastic single-porous disk. Solutions of the problems are obtained in an explicit form, in the form of absolutely and uniformly convergent series.

2. Formulation of Boundary Value Problems

Let a finite isotropic elastic body D, with a closed boundary S, consist of empty pores. Let's designate by $\sum(\mathbf{x})$ the area of a macropoint (areal element) $\mathbf{x}=(x_1,x_2)$, and the area of pores at this point by $\sum_p(\mathbf{x})$. The value of $\sigma(\mathbf{x})$, which is determined by the equality $\sigma(\mathbf{x})=\frac{\Sigma_p(\mathbf{x})}{\sum(\mathbf{x})}$, we call the relative pore area (pore area share). In general, as a result of deformation of the body, the relative area of the pores also changes. This change will be denoted by $\varphi(\mathbf{x})$. Let us formulate the main boundary value problems of the coupled linear theory of elasticity for one-porous media. Find in the domain D a regular solution $\mathbf{U(x)}=(u(\mathbf{x}),\varphi(\mathbf{x}),p(\mathbf{x}))$, where $\mathbf{U(x)}\in C^1(\overline{D})\cap C^2(D), \overline{D}=D\cup S$, satisfying the system of equations of the coupled theory of elasticity for the porous materials [21]:

$$\begin{gathered}\mu\Delta\mathbf{u}+(\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}+\operatorname{grad}(b\varphi-\beta p)=0,\\ (\alpha\Delta-\alpha_1)\varphi-b\operatorname{div}\mathbf{u}+mp=0,\\ k\Delta p=0,\end{gathered}\tag{1}$$

and on the border S one of the conditions

$$\mathbf{u(z)}=f(\mathbf{z}),\quad \varphi(\mathbf{z})=f_3(\mathbf{z}),\quad p(\mathbf{z})=f_4(\mathbf{z})\tag{2}$$

in problem I,

$$\mathbf{P}(\partial_{\mathbf{z}},\mathbf{n})U(\mathbf{z})=\mathbf{f(z)},\quad \frac{\partial\varphi(\mathbf{z})}{\partial\mathbf{n}}=f_3(\mathbf{z}),\quad \frac{\partial p(\mathbf{z})}{\partial\mathbf{n}}=f_4(\mathbf{z})\tag{3}$$

in problem II, where $\mathbf{u}=(u_1,u_2)$ is the displacement vector of the point $\mathbf{x}$, $\mathbf{x}=(x_1,x_2)\in D$; $\varphi(\mathbf{x})$ is the change in the relative pore area, and $p(\mathbf{x})$ is the average pressure of the liquid in the pores; λ and μ are the Lame constants, $\alpha,\alpha_1,\beta,b,m,k$ are the constants characterizing the porosity of the body, $\mathbf{z}=(z_1,z_2)\in S$, $\mathbf{n(z)}=(n_1,n_2)$ is the outer normal to S at the point $\mathbf{z}$; $\mathbf{f}=(f_1,f_2)$, f_1,f_2,f_3 and f_4 are the given functions on S.

$$\mathbf{R}\left(\partial_{\mathbf{x}},\mathbf{n}\right)\mathbf{U(x)}=\left(\mathbf{P}\left(\partial_{\mathbf{x}},\mathbf{n}\right)\mathbf{U(x)},\quad \alpha\frac{\partial\varphi(\mathbf{x})}{\partial\mathbf{n}},\quad \frac{\partial p(\mathbf{x})}{\partial\mathbf{n}}\right)\tag{4}$$

is the stress vector in a porous medium, where

$$\mathbf{P}\left(\partial_{\mathbf{x}},\mathbf{n}\right)\mathbf{U(x)}=\mathbf{T}\left(\partial_{\mathbf{x}},\mathbf{n}\right)\mathbf{u(x)}+b\mathbf{n}\varphi(\mathbf{x})-\beta\mathbf{n}p(\mathbf{x})\tag{5}$$

and

$$\mathbf{T}\left(\partial_{\mathbf{x}},\mathbf{n}\right)\mathbf{u(x)}=\mu\frac{\partial\mathbf{u}}{\partial\mathbf{n}}+\lambda\mathbf{n}\operatorname{div}u+\mu\sum_{i=1}^{2}n_i\operatorname{grad}u_i\tag{6}$$

is the stress vector in the classical theory of elasticity.

3. General Representation of the Solution of the System of Equations

Acting on the first equation of system (1) by the operator div, we obtain a system of equations for the desired values div, φ and p :

$$\begin{gathered}\mu_0\Delta\operatorname{div}\mathbf{u}+b\Delta\varphi-\beta\Delta p=0,\\ -b\operatorname{div}\mathbf{u}+(\alpha\Delta-\alpha_1)\varphi+mp=0,\\ k\Delta p=0,\end{gathered}\tag{7}$$

where $\mu_0 = \lambda + 2\mu$. The determinant of this system has the form $\det = -\mu_0 \alpha k \Delta\Delta(\Delta + \lambda_1^2)$,

$$\lambda_1^2 = -\frac{\mu_0 \alpha_1 - b^2}{\mu_0 \alpha}. \tag{8}$$

Let us assume that

$$\lambda > 0, \quad \mu > 0, \quad k > 0, \quad \alpha > 0, \quad \mu_0 \alpha_1 > b^2. \tag{9}$$

It is clear that

$$\alpha_1 > 0, \quad \lambda_1^2 < 0, \quad \lambda_1 = i\sqrt{\frac{\mu_0 \alpha_1 - b^2}{\mu_0 \alpha}} = i\lambda_0, \quad i = \sqrt{-1}.$$

Since system (7) is homogeneous, we write

$$\Delta\Delta(\Delta + \lambda_1^2) \operatorname{div} \mathbf{u} = 0, \quad \Delta\Delta(\Delta + \lambda_1^2)\varphi = 0, \quad \Delta\Delta(\Delta + \lambda_1^2)p = 0. \tag{10}$$

Taking into account (10), from $(1)_1$, we obtain $\Delta\Delta^2(\Delta + \lambda_1^2)\mathbf{u} = 0$. It follows from this equation that the solution $\mathbf{u(x)}$ contains harmonic, biharmonic and metaharmonic functions. From the second equation (10) we also conclude that the $\varphi(\mathbf{x})$ representation contains harmonic and metaharmonic functions; $p(\mathbf{x})$ is a harmonic function.

By a direct verification, one can make sure that the solutions of equations $(1)_1$ and $(1)_2$ are, accordingly, represented in the following form:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) =& c_0 \mathbf{u}^0(\mathbf{x}) + c_1 \mathbf{u}^1(\mathbf{x}), \\ \varphi(\mathbf{x}) =& \varphi_1(\mathbf{x}) + \varphi_2(\mathbf{x}), \end{aligned} \tag{11}$$

where φ_1 is a harmonic function, $\Delta\varphi_1 = 0$, and φ_2 is a metaharmonic function with the parameter λ_1^2, $(\Delta + \lambda_1^2)\varphi_2 = 0$; c_0 and c_1 are still unknown constants; $\mathbf{u}^0 = (u_1^0, u_2^0)$ is a general solution of the homogeneous equation corresponding to equation $(1)_1$ which can be represented as follows [27]:

$$\mathbf{u}^0(\mathbf{x}) = \operatorname{grad} \big[\Phi_1(\mathbf{x}) + \Phi_2(\mathbf{x})\big] + \operatorname{rot} \Phi_3(\mathbf{x}) + l\Gamma(\mathbf{x}). \tag{12}$$

Here, the functions Φ_2 and Φ_3 are interconnected as follows:

$$\mu_0 \operatorname{grad} \Delta\Phi_2 + \mu \operatorname{rot} \Delta\Phi_3 = 0; \tag{13}$$

$\Delta\Phi_1 = 0$, $\Delta\Delta\Phi_2 = 0$, $\Delta\Delta\Phi_3 = 0$; Φ_1, Φ_2, Φ_3 are scalar functions; $\Gamma = (\Gamma_1, \Gamma_2)$; $\Gamma_1 = x_2$, $\Gamma_2 = -x_1$, $\operatorname{div} \Gamma = 0$; l is a desired constant, $\operatorname{rot} = \big(-\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1}\big)$.

$\mathbf{u}^1 = (u_1^1, u_2^1)$ is one of the particular solutions of the inhomogeneous equation $(1)_1$:

$$\mathbf{u}^1(\mathbf{x}) = -\frac{1}{\mu_0} \operatorname{grad} \Big(-\frac{b}{\lambda_1^2}\varphi_2 + b\varphi_0 - \beta p_0\Big), \tag{14}$$

where we choose φ_0 and p_0 such that $\Delta\varphi_0 = \varphi_1$ and $\Delta p_0 = p$. Obviously, φ_0 and p_0 are biharmonic functions: $\Delta\Delta\varphi_0 = \Delta\varphi_1 = 0$, $\Delta\Delta p_0 = \Delta p = 0$. It is convenient to choose the φ_1 function as follows: $\varphi_1 = \operatorname{div} \mathbf{u}^0 \equiv \Delta\Phi_2$. Then in (14), we can write: $\varphi_0 = \Phi_2$. Now, let us set the values of the coefficients c_0 and c_1. We act with the div operator on the first equality in (11) and the resulting expression is comparable with the $\operatorname{div} \mathbf{u}$ determined from equation $(1)_2$. Taking into account (9), we get

$$c_0 = \frac{\alpha\lambda_1^2}{b}, \quad c_1 = 1. \tag{15}$$

By checking, we make sure that representations (11) satisfy equations (1).

4. Uniqueness Theorems

For a regular solution $\mathbf{U(x)} = (\mathbf{u(x)}, \varphi(\mathbf{x}), p(\mathbf{x}))$, Green's formulas can be written in the following form:

$$\int_D \big[E(\mathbf{u},\mathbf{u}) + (b\varphi - \beta p) \operatorname{div} \mathbf{u}\big] d\mathbf{x} = \int_S \mathbf{u}\big[\mathbf{T}(\partial_y \mathbf{n})\mathbf{u} + (b\varphi - \beta p)\mathbf{n}\big] d_y S, \tag{16}$$

$$\int_D \Big[\alpha|\operatorname{grad}\varphi|^2 + |\operatorname{grad} p|^2 + \Big[\alpha_1 + \frac{\mu_0\alpha_1 - b^2}{\mu_0}\Big]\varphi_2\Big] d\mathbf{x} = \int_S \Big[\alpha\varphi\frac{\partial\varphi}{\partial\mathbf{n}} + p\frac{\partial p}{\partial\mathbf{n}}\Big] d_y S, \tag{17}$$

where under conditions (9), the expression

$$E(\mathbf{u},\mathbf{u}) = (\lambda+\mu)\Big(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_2}{\partial x_2}\Big)^2 + \mu\Big(\frac{\partial u_1}{\partial x_1}-\frac{\partial u_2}{\partial x_2}\Big)^2 + \mu\Big(\frac{\partial u_1}{\partial x_2}+\frac{\partial u_2}{\partial x_1}\Big)^2$$

is of a non-negative quadratic form. Suppose that each problem posed above admits two solutions. For the difference of these solutions, the boundary conditions (2) and (3) will take the form:

$$\mathbf{u}(\mathbf{z}) = 0, \quad \varphi(\mathbf{z}) = 0, \quad p(\mathbf{z}) = 0 \tag{18}$$

– for task I;

$$\mathbf{P}(\partial_z,\mathbf{n})\mathbf{U}(\mathbf{z}) = 0, \quad \frac{\partial\varphi(\mathbf{z})}{\partial\mathbf{n}} = 0 \tag{19}$$

– for task II, $\mathbf{z}\in S$.

Taking into account (9), from (17), we get: $\varphi = 0$, $\operatorname{grad} p = 0$. Therefore

$$\varphi(\mathbf{x}) = \varphi_1+\varphi_2 = k_1, \quad p(\mathbf{x}) = k_2, \quad \varphi_2(\mathbf{x}) = 0, \quad \mathbf{x}\in D, \tag{20}$$

k_1 and k_2 are arbitrary constants. Taking into account (18), from (16), we obtain $\varphi(\mathbf{x})=0, E(\mathbf{u},\mathbf{u})= 0$. The solution of the equation $E(\mathbf{u},\mathbf{u}) = 0$ has the form

$$u_1(\mathbf{x}) = -cx_2+q_1, \quad u_2(\mathbf{x}) = cx_1+q_2, \tag{21}$$

where c, q_1, q_2 are arbitrary constants. Conditions (18) are satisfied if $c = q_1 = q_2 = 0$. So, for the difference of the above solutions, we get: $u_1(\mathbf{x}) = u_2(\mathbf{x}) = \varphi(\mathbf{x}) = p(\mathbf{x}) = 0$, $\mathbf{x}\in D$.

Theorem 1. *Problem I has a unique solution.*

In the case of problem II, according to (20), the functions $\varphi(\mathbf{x})$ and $p(\mathbf{x})$ are constant on D, and according to (19), they are also constant on S. For the difference of the solutions from (16), we obtain

$$\int_D \big[E(\mathbf{u},\mathbf{u}) + (bk_1-\beta k_2)\operatorname{div}\mathbf{u}\big]d\mathbf{x} = 0. \tag{22}$$

Taking into account (19) and (20), from (5), we obtain

$$P(\partial_z,n)U(z) = T(\partial_z,n)u(z) + (bk_1-\beta k_2)n(z) = 0.$$

So, for the difference of the solutions, we arrive at the problem of the classical theory of elasticity

$$\mu\Delta\mathbf{u}(\mathbf{x}) + (\lambda+\mu)\operatorname{grad}\operatorname{div}\mathbf{u}(\mathbf{x}) = 0, \quad \mathbf{x}\in D,$$
$$\mathbf{T}(\partial_z,\mathbf{n})\mathbf{u}(\mathbf{z}) = -(bk_1-\beta k_2)\mathbf{n}(\mathbf{z}), \quad \mathbf{z}\in S.$$

The solution to this problem has the form

$$\mathbf{u}(\mathbf{x}) = a_1\mathbf{x}+b_1, \tag{23}$$

where $a_1 = -\frac{bk_1-\beta k_2}{2(\lambda+\mu)}$, and b_1 is a two-component arbitrary vector. By checking, we make sure that representation (23) satisfies equation (22). So, we have proved

Theorem 2. Two arbitrary solutions of Problem II are the vectors whose components are expressed by formulas (23) and (20).

5. Problem Solving

Let the body D have the shape of a disk bounded by a circumference S of radius R and center coinciding with the origin. Let us rewrite representations (11) in polar coordinates as normal and tangent components:

$$\begin{gathered} u_n = \frac{\partial}{\partial r}(c_0\Phi_1+c_2\Phi_2+c_3\varphi_2+c_4p_4) - c_0\frac{1}{r}\partial_\theta\Phi_3, \\ u_s = \frac{1}{r}\frac{\partial}{\partial\theta}(c_0\Phi_1+c_2\Phi_2+c_3\varphi_2+c_4p_4) + c_0\frac{\partial}{\partial r}\Phi_3 - lr, \\ \varphi = \varphi_1+\varphi_2, \\ c_2 = -\frac{\alpha_1}{b}, \quad c_3 = \frac{b}{\mu_0\lambda_1^2}, \quad c_4 = \frac{m}{b}, \quad r^2 = x_1^2+x_2^2. \end{gathered} \tag{24}$$

Using formula (13), the harmonic, biharmonic and metaharmonic functions contained in (24), in a circular disk D can be represented as the following series [28, 29]:

$$\begin{gathered}
\Phi_1=\sum_{m=1}^{\infty}\Big(\frac{r}{R}\Big)^m\big(\mathbf{X}_{m3}\cdot\nu_m(\theta)\big),\quad \Phi_2=\frac{R^2}{4}\sum_{m=1}^{\infty}\frac{1}{m+1}\Big(\frac{r}{R}\Big)^{m+2}\big(\mathbf{X}_{m1}\cdot\nu_m(\theta)\big),\\
\varphi_1=\sum_{m=1}^{\infty}\Big(\frac{r}{R}\Big)^m\big(\mathbf{X}_{m1}\cdot\nu_m(\theta)\big),\quad \varphi_2=\sum_{m=1}^{\infty}I_m(\lambda_0 r)\big(\mathbf{X}_{m1}\cdot\nu_m(\theta)\big),\\
\Phi_3=\frac{R^2\mu_0}{4\mu}\sum_{m=1}^{\infty}\frac{1}{m+1}\Big(\frac{r}{R}\Big)^{m+2}\big(\mathbf{X}_{m1}\cdot\mathbf{s}_m(\theta)\big),\quad p=\sum_{m=1}^{\infty}\Big(\frac{r}{R}\Big)^m\big(\mathbf{X}_{m4}\cdot\nu_m(\theta)\big),\\
p_0=\frac{R^2}{4}\sum_{m=1}^{\infty}\frac{1}{m+1}\Big(\frac{r}{R}\Big)^{m+2}\big(\mathbf{X}_{m1}\cdot\nu_m(\theta)\big),
\end{gathered}\tag{25}$$

where $\mathbf{X}_{mk}$ is the sought two-component vector, $k=1,2,3,4$, $x=(r,\theta)$, $r^2=x_1^2+x_2^2$, $\nu_m(\theta)=(\cos m\theta,\sin m\theta)$, $\mathbf{s}_m(\theta)=(-\sin m\theta,\cos m\theta)$; $I_m(\lambda_0 r)$ is the Bessel function of the imaginary argument.

Problem I.

The boundary conditions (2) in terms of a normal and a tangent component have the form:

$$u_n(\mathbf{z})=f_n(\mathbf{z}),\quad u_s(\mathbf{z})=f_s(\mathbf{z}),\quad \varphi(\mathbf{z})=f_3(\mathbf{z}),\quad p(\mathbf{z})=f_4(\mathbf{z}).\tag{26}$$

Let the functions f_n, f_s, and f_3, f_4 be expanded into the Fourier series

$$\begin{gathered}
f_n(\mathbf{z})=\frac{\alpha_0}{2}+\sum_{m=1}^{\infty}\big(\alpha_m\cdot\nu_m(\theta)\big),\quad f_s(\mathbf{z})=\frac{\beta_0}{2}+\sum_{m=1}^{\infty}\big(\beta_m\cdot s_m(\theta)\big),\\
f_3(\mathbf{z})=\frac{\gamma_0}{2}+\sum_{m=1}^{\infty}\big(\gamma_m\cdot\nu_m(\theta)\big),\quad f_4(\mathbf{z})=\frac{\delta_0}{2}+\sum_{m=1}^{\infty}\big(\delta_m\cdot\nu_m(\theta)\big),
\end{gathered}$$

where $\alpha_m=(\alpha_{m1},\alpha_{m2})$, $\beta_m=(\beta_{m1},\beta_{m2})$, $\gamma_m=(\gamma_{m1},\gamma_{m2})$ and $\delta_m=(\delta_{m1},\delta_{m2})$ are the Fourier coefficients of the functions f_n, f_s, f_3 and f_4, respectively;

$$\alpha_{m1}=\frac{1}{\pi}\int_0^{2\pi}f_n(\omega)\cos m\omega d\omega,\quad \alpha_{m2}=\frac{1}{\pi}\int_0^{2\pi}f_n(\omega)\sin m\omega d\omega.$$

The components of the remaining vectors β_m,γ_m and δ_m are expressed similarly, $m=0,1,2,\dots$.

Substitute expressions (25) into (24) and pass to the limit as $r\to R$. From (26), for each m, we obtain the system of linear algebraic equations. For $m=0$, we have

$$\begin{gathered}
\frac{c_2}{2}X_{01}+c_3\lambda_0 I_0'(\lambda_0 R)X_{02}+\frac{c_4}{2}X_{04}=\frac{\alpha_0}{2},\quad \frac{c_0\mu_0 R}{2\mu}X_{01}-RX_{03}=\frac{\beta_0}{2},\\
X_{01}+I_0(\lambda_0 R)X_{02}=\frac{\gamma_0}{2},\quad X_{04}=\frac{\delta_0}{2},
\end{gathered}\tag{27}$$

where $X_{03}=l$, $I_m'(\lambda_0 r)=\frac{\partial}{\partial(\lambda_0 r)}I_m(\lambda_0 r)$. For each $m=1,2,\dots$, we obtain

$$\begin{gathered}
\frac{R\left[c_2\mu(m+2)+c_0\mu_0 m\right]}{4\mu(m+1)}\mathbf{X}_{m1}+c_3\lambda_0 I_m'(\lambda_0 R)\mathbf{X}_{m2}+\frac{c_0 m}{R}\mathbf{X}_{m3}+\frac{c_4R(m+2)}{4(m+1)}\mathbf{X}_{m4}=\alpha_m,\\
\frac{R\left[c_2\mu+c_0 m(m+2)\right]}{4\mu(m+1)}\mathbf{X}_{m1}+\frac{c_3 m}{R}I_m(\lambda_0 R)\mathbf{X}_{m2}+\frac{c_0 m}{R}\mathbf{X}_{m3}+\frac{c_4 m}{R}\mathbf{X}_{m4}=\beta_m,\\
\mathbf{X}_{m1}+I_m(\lambda_0 R)\mathbf{X}_{m2}=\gamma_m,\quad \mathbf{X}_{m4}=\delta_m.
\end{gathered}\tag{28}$$

The determinants of systems (27) and (28) are nonzero, since, by Theorem 1, problem I has a unique solution. Let us substitute the solutions of systems (27) and (28) into formulas (25). We substitute the obtained values of the solutions into formulas (12) and (14), and assume that $\varphi_0=\Phi_2$. Formulas (11) and (14) determine the solution of the original problem I, i.e., the values of the functions $\mathbf{u(x)},\varphi(\mathbf{x})$ and $p(\mathbf{x})$.

Problem II.

Using representations (11) and (25), problem II is solved similarly. The boundary conditions (3) in terms of a normal and a tangent component have the form:

$$\begin{gathered}\mathbf{P}(\partial_z,\mathbf{n})\mathbf{U}(\mathbf{z})_n = f_n(\mathbf{z}), \quad \mathbf{P}(\partial_z,\mathbf{n})\mathbf{U}(\mathbf{z})_s = f_s(\mathbf{z}),\\ \alpha\frac{\partial}{\partial r}\varphi(\mathbf{z})_{r=R} = f_3(\mathbf{z}), \quad \frac{\partial}{\partial r}p(\mathbf{z})_{r=R} = f_4(\mathbf{z}),\end{gathered} \tag{29}$$

where

$$\begin{gathered}\mathbf{P}(\partial_x,\mathbf{n})\mathbf{U}(\mathbf{x})_n = \mu_0\frac{\partial}{\partial r}u_n(\mathbf{x}) + \frac{\lambda}{r}\frac{\partial}{\partial\theta}u_s(\mathbf{x}) + b\varphi(\mathbf{x}) - \beta p(\mathbf{x}),\\ \mathbf{P}(\partial_x,\mathbf{n})\mathbf{U}(\mathbf{x})_s = \mu\Big[\frac{\partial}{\partial r}u_s(x) + \frac{1}{r}\frac{\partial}{\partial\theta}u_n(x)\Big], \quad \mathbf{x}\in D.\end{gathered} \tag{30}$$

Let the functions f_n, f_s and f_3, f_4 be expanded into Fourier series, where α_m, β_m, γ_m and δ_m are the Fourier coefficients of the functions f_n, f_s, f_3 and f_4, respectively. We substitute expressions (24) and (25) into (29) and pass to the limit as $r \to R$. For each m, with respect to the sought for values of $\mathbf{X}_{mk}$, we obtain a system of linear algebraic equations, $k = 1,2,3,4$. For $m = 0$, we have

$$\begin{gathered}\Big(\frac{1}{2}c_2\mu_0 + b\Big)X_{01} + \big[c_3\mu_0\lambda_0^2 I_0''(\lambda_0 R) + bI_0(\lambda_0 R)\big]X_{02}\\ \Big(\frac{1}{2}\mu_0 - \beta\Big) + X_{04} = \frac{\alpha_0}{2}, \quad \frac{c_0}{2}X_{01} - \mu X_{03} = \frac{\beta_0}{2},\\ \alpha\lambda_0 I_0'(\lambda_0 R)X_{02} = \frac{\gamma_0}{2}, \quad 0\cdot X_{04} = \frac{\delta_0}{2}\,.\end{gathered} \tag{31}$$

Under the boundary conditions (3), for the harmonic function $p(\mathbf{z})$, we have:

$$\delta_0 = \frac{1}{2\pi}\int_0^{2\pi} f_4(\omega)d\omega = \frac{1}{2\pi R}\int_S \frac{\partial}{\partial n}p\,dl = 0,$$

(dl is the length element of the circumference S, $dl = Rd\omega$). Then from the last equation of system (31), we obtain: $0\cdot X_{04} = 0$, i.e., X_{04} is an arbitrary constant.

For each $m = 1,2,\dots$, we obtain

$$\begin{gathered}\Big[\frac{c_2\mu_0(m+2)}{4} + \frac{c_0\mu_0 Rm}{4\mu} - \frac{c_2\lambda m^2}{4(m+1)} - \frac{c_0\lambda\mu_0 m(m+2)}{4\mu(m+1)} + b\Big]\mathbf{X}_{m1}\\ +\Big[c_3\lambda_0^2 I_m''(\lambda_0 R) - \frac{c_3\lambda m^2}{R^2}I_m(\lambda_0 R) + bI_m(\lambda_0 R)\Big]\mathbf{X}_{m2}\\ +\Big[\frac{c_0\mu_0 m(m-1)}{R^2} - \frac{c_0\lambda m^2}{R^2}\Big]\mathbf{X}_{m3} + \Big[\frac{c_0\mu_0(m+2)}{4} - \frac{c_4\lambda m^2}{4(m+1)} - \beta\Big]\mathbf{X}_{m4} = \alpha_m,\\ \Big[\frac{c_0\mu_0(m+2)}{4\mu} + \frac{c_0\mu_0 m^2}{4\mu(m+1)} + \frac{c_2 m}{4} + \frac{c_2 m(m+2)}{4(m+1)} + b\Big]\mathbf{X}_{m1}\\ +\frac{c_3 m}{R_2}\,[2RI_m'(\lambda_0 R) - mI_m(\lambda_0 R)]\,\mathbf{X}_{m2}\\ +\frac{1}{R^2}\Big[2c_3 mR\lambda_0 I_m'(\mu_0 R) - \frac{c_3 m^2}{R^2}I_m(\lambda_0 R)\Big]\mathbf{X}_{m3} + \frac{c_{4m}}{4}\Big[1 + \frac{m+2}{m+1}\Big]X_{m4} = \frac{\beta_m}{\mu},\\ \frac{m}{R}X_{m1} + \lambda_0 I_m'(\lambda_0 R)\mathbf{X}_{m2} = \frac{\gamma_m}{\alpha}, \quad \frac{m}{R}\mathbf{X}_{m4} = \delta_m.\end{gathered} \tag{32}$$

Let us substitute the solutions of systems (31) and (32) into formulas (25). We substitute the obtained values of the solutions into formulas (12) and (14). Formulas (11) and (14) determine the solution of the original problem II, i.e., the values of the functions $\mathbf{u}(\mathbf{x})$, $\varphi(\mathbf{x})$ and $p(\mathbf{x})$.

In order for the resulting series to converge absolutely and uniformly, it suffices to require: in problem I: $\mathbf{f}\in C^3(S)$, $f\in C^3(S)$; in problem II: $\mathbf{f}\in C^2(S)$, $f\in C^2(S)$.

6. Concluding Remarks

In the present paper, the coupled linear theory of elasticity for isotropic porous solids is considered. The system of general governing equations is expressed in terms of the displacement vector field, changes in the area fraction of pores and fluid pressure in the network of pores. The following results are presented: a) A general representation of the solution of the system of equations of the coupled theory of elasticity is constructed by using elementary functions. b) The boundary value problems of the coupled linear theory of elasticity in the two-dimensional case are solved for isotropic, one-porous solids of specific shape. c) For a regular solution of the system of basic differential equations, Green's formulas are obtained and the uniqueness theorems for solutions to the problems posed are proved. d) The stated problems are solved for an elastic one-porous disk. Solutions to the problems are obtained in an explicit form, in the form of absolutely and uniformly convergent series. e) The application of the method under consideration makes it possible to study a wide class of problems for systems of equations of the coupled theory of elasticity or thermoelasticity for materials with one or double porosity; build explicit solutions of the main boundary value problems not only for a circle, but also for a ring, a plane with a round hole, etc. f) It is expected that the proposed method can be applied primarily to the problems in mechanics, as well as to the problems of computational and applied mathematics.

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(Received 16.03.2023)

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