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WEAKLY $g(x)$-QUASI INVO-CLEAN RINGS

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Abstract. Let R be an associative ring with identity and $g(x)$ be a fixed polynomial in $C(R)[x]$. In this paper, we introduce the notion of weakly $g(x)$-quasi invo-clean rings where every element r can be written as $r = v + s$ or $r = v - s$, where $v \in \mathrm{Qinv}(R)$ and s is a root of $g(x)$. We study various properties of weakly $g(x)$-quasi invo-clean rings. It is proved that if $R = \prod_{i=1}^{n} R_i$, $g(x) \in \mathbb{Z}[x]$ and there exist $1 \le l \le n$ such that R_l is weakly $g(x)$-quasi invo-clean and R_j is $g(x)$-quasi invo-clean for all $j \ne l$, then R is weakly $g(x)$-quasi invo-clean.

1. Introduction

Let R be an associative ring with identity. An element v of R is said to be an involution if $v^2 = 1$ and a quasi-involution if either v or $1-v$ is an involution [11]. Let $U(R)$, $\mathrm{Id}(R)$, $\mathrm{Nil}(R)$, $C(R)$, $\mathrm{Inv}(R)$ and $\mathrm{Qinv}(R)$ denote respectively the set of units, the set of idempotents, the set of nilpotents, the set of centrals, the set of involutions and the set of quasi involutions of R. The ring R is said to be clean if each $r \in R$ can be expressed as $r = u + e$, where $u \in U(R)$ and $e \in \mathrm{Id}(R)$ [2,12]. The ring R is said to be weakly clean if each $r \in R$ can be expressed as $r = u + e$ or $r = u - e$, where $u \in U(R)$ and $e \in \mathrm{Id}(R)$ [1,4]. Let R be a ring and $g(x)$ be a polynomial in $C(R)[x]$. An element in R is said to be $g(x)$-clean if it can be written as the sum of a unit and a root of $g(x)$. The ring R is said to be $g(x)$-clean if each element in R is $g(x)$-clean [10]. An element in R is said to be weakly $g(x)$-clean if it can be written as the sum or difference of a unit and a root of $g(x)$. The ring R is said to be weakly $g(x)$-clean if each element in R is weakly $g(x)$-clean [3]. The ring R is said to be invo-clean if for each $r \in R$, there exist $v \in \mathrm{Inv}(R)$ and $e \in \mathrm{Id}(R)$ such that $r = v + e$ [5,7,8]. In [7, Theorem 2.2], it is shown that if R is an invo-clean ring and e is an idempotent, then the corner subring eRe is also invo-clean. In particular, if for any $n \in N$ the full matrix $n \times n$ ring $M_n(R)$ is invo-clean, then so is R. In [5, Corollary 2.16], it is shown that if R is an invo-clean ring, then $J(R)$ is nil with the index of nilpotence, not exceeding 3. In [6, Corollary 3.2], it is proved that a ring R of characteristic 2 is strongly invo-clean if and only if R is strongly nil-clean with the index of nilpotence at most 2. Here, we introduce the notion of an invo k-clean ring as a new generalization of an invo-clean ring. Let $2 \le k \in \mathbb{N}$. The ring$R$ is said to be weakly invo-clean if for each $r \in R$, there exist $v \in \mathrm{Inv}(R)$ and $e \in \mathrm{Id}(R)$ such that $r = v + e$ or $r = v - e$ [6]. The ring R is said to be $g(x)$-invo-clean if for each $r \in R$, there exist $v \in \mathrm{Inv}(R)$ and a root s of $g(x)$ such that $r = v + s$ [9]. The ring R is said to be weakly $g(x)$-invo-clean if for each $r \in R$, there exist $v \in \mathrm{Inv}(R)$ and a root s of $g(x)$ such that $r = v + e$ or $r = v - e$ [15]. The ring R is said to be quasi invo-clean if for each $r \in R$, there exist $v \in \mathrm{Qinv}(R)$ and $e \in \mathrm{Id}(R)$ such that $r = v + e$ [8]. The ring R is said to be $g(x)$-quasi invo-clean if for each $r \in R$, there exist $v \in \mathrm{Qinv}(R)$ and a root s of $g(x)$ such that $r = v + s$ [13]. Here, we introduce the notion of a weakly $g(x)$-quasi invo-clean ring. Let R be a ring and $g(x)$ be a polynomial in $C(R)[x]$. Then an element $r \in R$ is called weakly $g(x)$-quasi invo-clean if there exist $v \in \mathrm{Qinv}(R)$ and a root s of $g(x)$ such that $r = v + s$ or $r = v - s$. A ring R is called weakly $g(x)$-quasi invo-clean if every element of R is weakly $g(x)$-quasi invo-clean. We study various properties of weakly $g(x)$-quasi invo-clean rings as a proper generalization of quasi invo-clean rings and a proper subclass of $g(x)$-quasi invo-clean rings. It is shown that if $R = \prod_{i=1}^{n} R_i$, $g(x) \in \mathbb{Z}[x]$ and there exist $1 \le l \le n$ such that R_l is weakly $g(x)$-quasi invo-clean and R_j is $g(x)$-quasi invo-clean for all $j \ne l$, then R is weakly $g(x)$-quasi invo-clean (Theorem 2.1). Also, it is proved that if R is a ring, k is an even positive integer

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and $a, b \in R$, then R is weakly $(ax^k - bx)$-quasi invo-clean if and only if R is weakly $(ax^k + bx)$-quasi invo-clean (Theorem 2.2).

2. Main Results

Along with [5,6] and [8], in this section, we start our work with the following basic notion.

Definition 2.1. An element $r \in R$ is said to be an invo-clean element if there exist $v \in \mathrm{Inv}(R)$ and $e \in \mathrm{Id}(R)$ such that $r = v + e$. A ring R is said to be invo-clean if each element in R is invo-clean [5]. Simple examples of invo-clean rings that could be plainly verified are these: $\mathbb{Z}_2$, $\mathbb{Z}_3$ and $\mathbb{Z}_4$. Conversely, $\mathbb{Z}_5$ is not invo-clean but, however, it is clean being finite [5].

Definition 2.2. An element $r \in R$ is said to be a weakly invo-clean element if there exist $v \in \mathrm{Inv}(R)$ and $e \in \mathrm{Id}(R)$ such that $r = v + e$ or $r = v - e$. A ring R is said to be weakly invo-clean if each element in R is weakly invo-clean [6].

Definition 2.3. Let R be a ring and $g(x)$ be a polynomial in $C(R)[x]$. An element in R is said to be $g(x)$-invo-clean if it can be written as the sum of an involution and a root of $g(x)$. The ring R is said to be $g(x)$-invo-clean if each element in R is $g(x)$-invo-clean [9].

Definition 2.4. Let R be a ring and $g(x)$ be a polynomial in $C(R)[x]$. Then an element $r \in R$ is said to be weakly $g(x)$-invo-clean if there exist $v \in \mathrm{Inv}(R)$ and the root s of $g(x)$ such that $r = v + s$ or $r = v - s$. The ring R is said to be weakly $g(x)$-invo-clean if every element of R is weakly $g(x)$-invo-clean [15].

Example. Let $R = \mathbb{Z}_5$ and $g(x) = x^2 - x \in C(R)[x]$. Since R is weakly invo-clean but not invo-clean, R is weakly $g(x)$-invo-clean, but not $g(x)$-invo-clean.

Definition 2.5. An element $r \in R$ is said to be a quasi invo-clean element if there exist $v \in \mathrm{Qinv}(R)$ and $e \in \mathrm{Id}(R)$ such that $r = v + e$. A ring R is said to be quasi invo-clean if each element in R is quasi invo-clean [8].

Definition 2.6. An element $r \in R$ is said to be a weakly quasi invo-clean element if there exist $v \in \mathrm{Qinv}(R)$ and $e \in \mathrm{Id}(R)$ such that $r = v + e$ or $r = v - e$. A ring R is said to be weakly quasi invo-clean if each element in R is weakly quasi invo-clean [14].

It is evident that invo-clean rings are both weakly invo-clean and (weakly) quasi invo-clean as this implication is extremely non-reversible by looking quickly at the field $\mathbb{Z}_5$.

Definition 2.7. Let R be a ring and $g(x)$ be a polynomial in $C(R)[x]$. An element in R is said to be $g(x)$-quasi invo-clean if it can be written as the sum of a quasi involution and a root of $g(x)$. The ring R is said to be $g(x)$-quasi invo-clean if each element in R is $g(x)$-quasi invo-clean [13].

Example. Let $R = \mathbb{Z}_5$ and $g(x) = x^5 + 4x \in C(R)[x]$. Then R is $g(x)$-quasi invo-clean.

In what follows, we define the weakly $g(x)$-quasi invo-clean rings and then study some of the basic properties of weakly $g(x)$-quasi invo-clean rings. Moreover, we give some necessarily examples.

Definition 2.8. Let R be a ring and $g(x)$ be a polynomial in $C(R)[x]$. Then an element $r \in R$ is called weakly $g(x)$-quasi invo-clean if there exist $v \in \mathrm{Qinv}(R)$ and the root s of $g(x)$ such that $r = v + s$ or $r = v - s$. The ring R is called weakly $g(x)$-quasi invo-clean if every element of R is weakly $g(x)$-quasi invo-clean.

It is clear that the weakly $(x^2 - x)$-quasi invo-clean rings are precisely the weakly quasi invo-clean rings. Obviously, $g(x)$-quasi invo-clean rings are weakly $g(x)$-quasi invo-clean and also if $g(-x) = -g(x)$ or $g(-x) = g(x)$, then the concepts $g(x)$-quasi invo-clean and weakly $g(x)$-quasi invo-clean coincide. So, the case in which $g(x)$ is neither an even nor an odd polynomial is of interest. Every quasi invo-clean or $g(x)$-quasi invo-clean ring is weakly $g(x)$-quasi invo-clean. The following example shows that every weakly $g(x)$-quasi invo-clean ring is neither a $g(x)$-quasi invo-clean nor a quasi invo-clean ring, in general.

Example. (i) Let $R = \mathbb{Z}_5 \times \mathbb{Z}_5$. Then R is neither a weakly invo-clean nor a quasi invo-clean ring [6, Example 4.16]. Since $\mathrm{Qinv}(\mathbb{Z}_5) = \{0, 1, 2, 4\}$ and $\mathrm{Id}(\mathbb{Z}_5) = \{0, 1\}$, R is a weakly $(x^2 - x)$-quasi invo-clean ring.

(ii) Let $R = \mathbb{Z}_7$ and $g(x) = x^7 + 6x \in C(R)[x]$. Then $\mathrm{Qinv}(R) = \{0, 1, 2, 6\}$ and $\mathrm{Id}(R) = \{0, 1\}$. Hence R is a weakly $g(x)$-quasi invo-clean ring which is not (weakly) quasi invo-clean.

(iii) Let $R = \mathbb{Z}_8$. Then $\mathrm{Qinv}(R) = \{0, 1, 2, 5, 6, 7\}$ and $\mathrm{Id}(R) = \{0, 1\}$. Hence R is a weakly $(x^2 - x)$-quasi invo-clean ring which is not $(x^2 - x)$-quasi invo-clean.

Proposition 2.1. *Let R be a Boolean ring with the number of elements $|R| > 2$, $c \in R \setminus \{0, 1\}$ and $g(x) = (x + 1)(x + c)$. Then R is not weakly $g(x)$-quasi invo-clean.*

Proof. Suppose that R is weakly $g(x)$-quasi invo-clean. Hence $c = v + s$ or $c = v - s$ such that $v \in \mathrm{Qinv}(R)$ and $g(s) = 0$. Since $v \in \mathrm{Qinv}(R)$ and R is a Boolean ring, $v = 1$ or $v = 0$. If $v = 1$, then $s = c - 1$ or $s = 1 - c$. But $g(c - 1) \neq 0$ and $g(1 - c) \neq 0$, which is a contradiction. If $v = 0$, then $s = c$ or $s = -c$. But $g(c) \neq 0$ and $g(-c) \neq 0$, a contradiction.Then R is not weakly $g(x)$-quasi invo clean. □

Theorem 2.1. *Let $\{R_i\}_{i=1}^n$ be rings, $R = \prod_{i=1}^n R_i$ and $g(x) \in \mathbb{Z}[x]$. If there exist $1 \leq l \leq n$ such that R_l is weakly $g(x)$-quasi invo-clean and R_j is $g(x)$-quasi invo-clean for all $j \neq l$, then R is weakly $g(x)$-quasi invo-clean.*

Proof. Suppose that there exist $1 \leq l \leq n$ such that R_l is weakly $g(x)$-quasi invo-clean and R_j is $g(x)$-quasi invo-clean for all $j \neq l$. Let $r = (r_i) \in R$. Then there exist $v_l \in \mathrm{Qinv}(R)$ and a root s_l of $g(x)$ such that $r_l = v_l + s_l$ or $r_l = v_l - s_l$. If $r_l = v_l + s_l$, then for each $i \neq l$, $r_i = v_i + s_i$ such that $v_i \in \mathrm{Qinv}(R)$ and $g(s_i) = 0$. Then $r = (v_i) + (s_i)$ such that $(v_i) \in \mathrm{Qinv}(R)$ and $g(s_i) = 0$. If $r_l = v_l - s_l$, then for each $i \neq l$, $r_i = v_i - s_i$ such that $v_i \in \mathrm{Qinv}(R)$ and $g(s_i) = 0$. Then $r = (v_i) - (s_i)$ such that $(v_i) \in \mathrm{Qinv}(R)$ and $g(s_i) = 0$. Therefore R is weakly $g(x)$-quasi invo-clean. □

Theorem 2.2. *Let R be a ring, k be an even positive integer and $a, b \in R$. Then R is weakly $(ax^k - bx)$-quasi invo-clean if and only if R is weakly $(ax^k + bx)$-quasi invo-clean.*

Proof. Suppose that R is weakly $(ax^{2n} - bx)$-quasi invo-clean and $r \in R$. Hence $1 - r = v \pm s$ where $v \in \mathrm{Qinv}(R)$ and $as^{2n} - bs = 0$. Then $r = (1 - v) \pm (-s)$ such that $1 - v \in \mathrm{Qinv}(R)$ and $a(-s)^{2n} + b(-s) = 0$. Therefore R is weakly $(ax^{2n} + bx)$-quasi invo-clean.

Conversely, assume that R is weakly $(ax^{2n} + bx)$-quasi invo-clean and $r \in R$. Hence $1 - r = v \pm s$ where $v \in \mathrm{Qinv}(R)$ and $as^{2n} + bs = 0$. Then $r = (1 - v) \pm (-s)$ such that $1 - v \in \mathrm{Qinv}(R)$ and $as^{2n} - bs = 0$. Therefore R is weakly $(ax^{2n} - bx)$-quasi invo-clean. □

The follwoing shows that Theorem 2.2 does not hold for odd powers.

Example. The ring $\mathbb{Z}_7$ is a weakly $(x^7 + 6x)$-quasi invo-clean ring which is not weakly $(x^7 - 6x)$-quasi invo-clean.

Corollary. *Let R be a ring. Then R is weakly quasi invo-clean if and only if R is weakly $(x^2 + x)$-quasi invo-clean.*

Proof. It follows from Theorem 2.2. □

Lemma 2.1. *Let R be a commutative ring and $h = \sum_{i=0}^n r_i x^i \in \mathrm{Qinv}(R[x])$. Then $r_0 \in \mathrm{Qinv}(R)$ and $r_i \in \mathrm{Nil}(R)$ for each $1 \leq i \leq n$.*

Proof. Since $h = \sum_{i=0}^n r_i x^i \in \mathrm{Qinv}(R[x])$, $h^2 = 1$ or $(1 - h)^2 = 1$. Hence $r_0^2 = 1$ or $(1 - r_0)^2 = 1$, and so, $r_0 \in \mathrm{Qinv}(R)$. Suppose that P is a prime ideal of R. Hence $(R/P)[x]$ is an integral domain. Let $\psi : R[x] \longrightarrow (R/P)[x]$ by $\psi(\sum_{i=0}^n r_i x^i) = \sum_{i=0}^n (r_i + P)x^i$. Then ψ is a ring epimorphism. Since $\psi(h)\psi(h) = 1$ or $\psi(1 - h)\psi(1 - h) = 1$, $\deg(\psi(h)\psi(h)) = \deg(\psi(1))$ or $\deg(\psi(1 - h)\psi(1 - h)) = \deg(\psi(1))$. Then $r_1 + P = r_2 + P = \cdots = r_n + P = P$. Therefore $r_i \in \mathrm{Nil}(R)$ for each $1 \leq i \leq n$. □

Theorem 2.3. *Let R be a commutative ring. Then $R[x]$ is not weakly $(x^2 - x)$-quasi invo-clean.*

Proof. Suppose that $R[x]$ is weakly $(x^2 - x)$-quasi invo-clean. Hence $x = v \pm s$, where $v \in \mathrm{Qinv}(R[x])$ and s is a root of $x^2 - x$. Then $x - s \in \mathrm{Qinv}(R[x])$ or $x + s \in \mathrm{Qinv}(R[x])$. So, $1 \in \mathrm{Nil}(R)$ by Lemma 2.1, which is a contradiction. □

We finish our article with the following three problems.

Problem 2.1. Let R be a weakly $g(x)$-quasi invo-clean ring. Is each homomorphic image of R is weakly $g(x)$-quasi invo-clean?

Problem 2.2. What is the behaviour of the matrix rings over weakly $g(x)$-quasi invo-clean rings?

Problem 2.3. Let R be a weakly $g(x)$-quasi invo-clean ring and $e \in \mathrm{Id}(R)$. What is the behaviour of the corner ring eRe?

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