Transactions of A. Razmadze
Mathematical Institute
Vol. **178** (2024), issue 1, 87–96

PRODUCT OF g-VOLTERRA SPACES IN GENERALIZED TOPOLOGICAL SPACES

PON JEYANTHI[1], GOPAL GEETHA[2] AND ENNIS ROSAS[3]

Abstract. In this paper, we introduce the concept of generalized pseudo-base in generalized topological spaces, and using this concept, we study the product of g-Volterra space and weakly g-Volterra space in generalized topological spaces. In addition, we study the product of g-Baire space with g-Volterra space in generalized topological spaces.

1. Introduction

In 1881, Volterra [18] proved several theorems on Volterra spaces. After a century, in 1993, Gauld et al. [9] restated the concept of Volterra spaces and proved that a space X is Volterra if for each $f, g\colon \mathbb{X} \to \mathbb{R}$ for which $C(f)$ and $C(g)$ are dense in X, the set $C(f) \cap D(g)$ is also dense in X. Also, the same authors [7] proved that X is strongly Volterra (respectively, Volterra) if $C(f) \cap C(g)$ is dense in X (respectively, non-empty) whenever $f, g\colon X \to \mathbb{R}$ are two functions for which $C(f)$ and $C(g)$ are dense in X. In [8], the authors proved that a topological space X is Volterra (respectively, weakly Volterra) if for every pair G and H of dense G_δ subsets of X, the set $G \cap H$ is dense (respectively, non-empty). In 2005, Cao et al. [1] revisited the papers of Gauld et al. [7,8] and studied Volterraness in homogeneous spaces. Spadaro [17] established the relation between P-spaces and the Volterra property. Milan Matejdes [14] studied the basic properties of weak ε-Volterra and ε-Volterra spaces which correspond to the known results of the Volterra and weakly Volterra spaces. Gauld et al. [8] and Cao et al. [2] studied the relation between Baire space and Volterra space in topological spaces. Csaszar [4], initiated the discussion on the theory of generalized topological spaces and studied various basic operators related to generalized topological spaces. Also, he discovered many important families of sets in generalized topological spaces (see [3,5]). Later, Li and Lin [13] defined g-dense, g-nowhere dense and g-residual in X and also studied Baireness on generalized topological spaces. Recently, in [11], we introduced the concepts of g-Volterra space and g-first countable space in generalized topological spaces and studied the relation between g-Volterra space and g-first countable space in generalized topological spaces. Also, in [12], we introduced the concept of weakly g-Volterra space in generalized topological spaces and studied the relation between g-Volterra space and weakly g-Volterra space. In addition, we studied the mapping theorems on g-Volterra space and weakly g-Volterra space in generalized topological spaces.

Oxtoby [15], Cao et al. [1] and Thangamariappan and Renukadevi [16] studied Cartesian product of Baire spaces, Volterra space in homogeneous space and the product of Volterra spaces in topological spaces, respectively. Followed by the results in [1,15] and [16], in this paper, we introduce the concept of generalized pseudo-base in generalized topological spaces and, using this concept, we study the product of g-Volterra space and weakly g-Volterra space in generalized topological spaces. Further, we study the product of g-Baire space with g-Volterra space in generalized topological spaces.

2. Notion and Definitions

In this section, we recall some basic concepts and the known results to prove our main results.

Let X be a non-empty set and $\exp(X)$ be the power set of X. A family of sets $g \subset \exp(X)$ is said to be a generalized topology (briefly, GT) on X, if (i) $\emptyset \in g$ and (ii) union of elements of g belongs

2020 *Mathematics Subject Classification.* 54A05.
Key words and phrases. g-Volterra space; Weakly g-Volterra space; g-pseudo-base; g-Baire space; Homogeneous.

to g. The pair (X, g) is called a generalized topological space (briefly, GTS). The elements of g are called g-open subsets of X and the complements are called g-closed subsets of X.

Let (X, g) be a GTS and $A \subset X$. The closure of A and the interior of A in (X, g) are defined as $c_g(A) = \bigcap\{F\colon X\backslash F \in g \text{ and } A \subset F\}$ and $i_g(A) = \bigcup\{U\colon U \in g \text{ and } U \subset A\}$.

We recall the following definitions which are useful for the present study.

Definition 2.1 ([13]). A subset A of X is called

i) g-dense in X if $c_g(A) = X$.
ii) g-nowhere dense in X if $i_g c_g(A) = \emptyset$.

We denote the family of all g-dense (resp., g-nowhere dense) subsets of (X, g) by $D(g)$ (resp., $N(g)$).

Definition 2.2. Let (X, g) be a generalized topological space. Then $A \subset X$ is called g-somewhere dense in X if $i_g(c_g(A)) \neq \emptyset$. The family of all g-somewhere dense subsets of (X, g) is denoted by $S(g)$ or $S(g_X)$.

Lemma 2.3 ([13]). *Let (X, g) be a GTS and $A \subset X$. Then*

(i) $c_g(A) = X \setminus i_g(X \setminus A)$.
(ii) $i_g(A) = X \setminus c_g(X \setminus A)$.

Definition 2.4. Let (X, g) be a GTS and $A \subset X$. Then $A \in N(g)$ if and only if $c_g(A)$ is g-codense in X, that is, $X \setminus c_g(A) \in D(g)$. A GTS (X, g) is called strong [13], if $X \in g$. Clearly, (X, g) is strong if and only if $c_g(\phi) = \phi$. Therefore $X \in g$.

Definition 2.5 ([13]). Let (X, g) be a GTS. Then $A \subset X$ is called

(i) g-first category in X if there exists a sequence $\{A_n\}$ consisting of g-nowhere dense subsets of X such that $A = \bigcup\limits_{n \in N} A_n$;
(ii) g-second category in X, if A is not g-first category in X.

We denote the family of all g-first category subsets of (X, g) by $M(g)$.

Definition 2.6 ([13]). Let (X, g) be a GTS and $A \subset S \subset X$. Then (S, g_S), where $g_S = \{U \cap S : U \in g\}$ is the relative GT on S, is a GTS which is called a subspace of (X, g). Denote the closure of A and the interior of A in the subspace (S, g_S) by $c_{g_S}(A)$ and $i_{g_S}(A)$, respectively.

Definition 2.7 ([11]). A GTS (X, g) is said to be g-Volterra space if each pair of G and H of g-dense G_δ subsets of X, then $G \cap H$ is g-dense in X.

The following example shows that (X, g) is g-Volterra.

Example 2.8. Let X be the set of all natural numbers. Consider $A = \{1, 3, 5, \dots\}$ and $g = \{\phi\} \cup \{A \cup B : \phi \neq B \subset \{2, 4, 6, \dots\}\}$. Then (X, g) is a strong GTS and g is not a topology on X. Here, A is g-dense in X and $U \cap A \neq \phi$ for any $U \in g - \{\phi\}$. By Proposition 3.3 [13], we have $A \in D(g)$. Therefore $g - \{\phi\} \subset D(g)$. Let G and H be g-dense G_δ subsets of X. Thus $A \subset G \cap H$, $G \cap H \in D(g)$ and, clearly, (X, g) is g-Volterra.

Definition 2.9 ([12]). Let (X, g) be a GTS. For every pair of g-dense G_δ subsets G and H of X, $G \cap H$ is non-empty, then X is a weakly g-Volterra space.

The following example shows that (X, g) is weakly g-Volterra.

Example 2.10. Let X be the set of all natural numbers and $g = \{\emptyset, N - \{a\}\}$, where $a \in N$. Then (X, g) is a GTS and (X, g) is not a topology on X. Let G and $H \in g - \{\emptyset\}$. If every pair of G and H of g-dense G_δ subsets of X, then trivially $G \cap H$ is non-empty. Hence (X, g) is weakly g-Volterra.

Definition 2.11 ([11]). Let (X, g) be a GTS and $x \in X$. A g-local base of x, denoted by $\mathbf{C}_x$, is a collection of g-open neighbourhoods of x such that for all $U \in g$ with $x \in U$ there exists a $C \in \mathbf{C}_x$ such that $x \in C \subset U$.

Lemma 2.12. *Let (X, g) be a GTS and let $A \subset S \subset X$. Then $c_{g_S}(A) = c_g(A) \cap S$.*

Proof. Let B be the closure of A in S. Clearly, $c_g(A) \cap S$ is closed in S. Since B is the smallest closed set in S containing A, we have $B \subset c_g(A) \cap S$. On the otherhand, $B = C \cap S$, where C is closed in X. Since $c_g(A)$ is the smallest closed set containing A, therefore $c_g(A) \subset C$. Thus $c_g(A) \cap S \subset C \cap S = B$. □

3. g-Volterra Space and Weakly g-Volterra Space in Generalized Topological Spaces

In this section, we study g-Volterra space and weakly g-Volterra spaces in generalized topological spaces and obtain some characterizations. Also, we obtain an equivalent condition between g-Volterra and weakly g-Volterra spaces.

Lemma 3.1. *Let (X, g) be a generalized topological space and $A \subset X$. Suppose that X is a g-Volterra space and S is a G_δ subset containing A such that $i_g(c_g(A))$ is g-dense in S. Then S is a g-Volterra space.*

Proof. Suppose that $i_g(c_g(A))$ is g-dense in S, then $c_g(S) = c_g(S \cap i_g(c_g(A)))$. Since $A \subset S$, we have $c_g(A) \subset c_g(S) \subset c_g(S \cap i_g(c_g(A))) \subset c_g(A)$. Thus $c_g(A) = c_g(S)$. Let M and N be g-dense G_δ subsets of S and we define $A^1 = A \cup (X \setminus c_g(S))$, $M^1 = M \cup (X \setminus c_g(S))$ and $N^1 = N \cup (X \setminus c_g(S))$. Then A^1, M^1 and N^1 are g-dense G_δ sets in X. Since X is a g-Volterra space, therefore $A^1 \cap M^1 \cap N^1$ is g-dense. Now, we have $c_g(A) = c_g(S)$ which implies that $i_g(c_g(A)) \subset c_g(A \cap M \cap N)$. Therefore $c_g(S) \subset c_g(S \cap i_g(c_g(A))) \subset i_g(c_g(A)) \subset c_g(S \cap M \cap N)$. Thus $M \cap N$ is g-dense in S and hence S is a g-Volterra space. □

Theorem 3.2. *Let a generalized topological space X be a weakly g-Volterra space and S be a g-dense G_δ subset of X, then S is a weakly g-Volterra space.*

Proof. Assume that X is a weakly g-Volterra space and S is a g-dense G_δ subset of X. By Definition 2.9, S is a weakly g-Volterra space. □

Theorem 3.3. *A generalized topological space (X, g) is a g-Volterra space if and only if every non-empty g-open subspace of X is a weakly g-Volterra space.*

Proof. Let X be a g-Volterra space. By Theorem 3.2, every non-empty g-open subspace is a weakly g-Volterra space. Conversely, we show that every non-empty g-open subspace is a weakly g-Volterra space, then X is a g-Volterra space. Let M and N be two g-dense G_δ subsets of X. Let A be a non-empty g-open subset of X. Then $A \cap M$ and $A \cap N$ are the g-dense subsets of A which are g-dense in A. Now, we have $A \cap (M \cap N) = (A \cap M) \cap (A \cap N)$. Since A is a weakly g-Volterra space, we have $A \cap (M \cap N) \neq \emptyset$. Thus $M \cap N$ is g-dense in X and hence X is a g-Volterra space. □

Lemma 3.4. *Let (X, g) be a generalized topological space. If X contains a non-empty weakly g-Volterra g-open subspace Y, then X is a weakly g-Volterra space.*

Proof. Let M and N be two g-dense G_δ subsets of X. Then $M \cap Y$ and $N \cap Y$ are two g-dense G_δ subsets of Y. Since Y is weakly g-Volterra and $(M \cap N) \cap Y \subset M \cap N$, therefore $M \cap N \neq \emptyset$. Hence X is a weakly g-Volterra space. □

Lemma 3.5. *Let (X, g) be a generalized topological space. If X contains a g-dense G_δ subspace which is not a weakly g-Volterra space, then X is not a weakly g-Volterra space.*

Proof. Let M and N be two g-dense G_δ subsets of X. Then $M \cap Y$ and $N \cap Y$ are two g-dense G_δ subsets of Y. Since Y is weakly g-Volterra and $(M \cap N) \cap Y \subset M \cap N$, so, $M \cap N \neq \emptyset$. Hence X is a weakly g-Volterra space. □

Lemma 3.6. *Let (X, g) be a generalized topological space. Then the union of any family of non-empty g-open non-weakly g-Volterra subspace is not a weakly g-Volterra space.*

Proof. Let $\mathcal{G}$ be a family of non-empty g-open subspaces of X such that every member of $\mathcal{G}$ is not a weakly g-Volterra space in X. Let $\mathcal{T}_{NV}$ be the set of all collections of non-empty g-open subsets of X with the following properties:

(i) Every collection of $\psi \in \mathcal{T}_{NV}$ is pairwise disjoint.
(ii) For every collection of $\psi \in \mathcal{T}_{NV}$ and each member of $V \in \psi$, there exists some $U \in \mathcal{G}$ such that $V \subset U$.

Let $V = \{V_\alpha : \alpha \in A\}$. Then $\bigcup\{U : U \in \mathcal{G}\} \subset c_g(V)$. It follows from the condition (ii) and Lemma 3.4 that for each $\alpha \in A$, V_α is not a weakly g-Volterra space. Thus there are two families $\{G_\alpha : \alpha \in A\}$ and $\{H_\alpha : \alpha \in A\}$ of G_δ sets of X such that

(iii) $G_\alpha \cap H_\alpha = \emptyset$ for all $\alpha \in A$ and
(iv) $G_\alpha \subset V_\alpha \subset c_g(G_\alpha)$ and $H_\alpha \subset V_\alpha \subset c_g(H_\alpha)$ for all $\alpha \in A$.

Let $G = \{G_\alpha : \alpha \in A\}$ and $H = \{H_\alpha : \alpha \in A\}$. Using conditions (ii) and (iii), we obtain $G \cap H = \emptyset$. Let $G_\alpha = \bigcap\{{G_\alpha}^n : n \geq 1\}$ and $H_\alpha = \bigcap\{{H_\alpha}^n : n \geq 1\}$, where ${G_\alpha}^n$ and ${H_\alpha}^n$ are non-empty g-open subsets of X contained in V_α such that ${G_\alpha}^{n+1} \subset {G_\alpha}^n$ and ${H_\alpha}^{n+1} \subset {H_\alpha}^n$ for all $\alpha \in A$ and $n \in N$. Now, we choose $G_\alpha = \bigcup\{{G_\alpha}^n : \alpha \in A\}$ and $H_\alpha = \bigcup\{{H_\alpha}^n : \alpha \in A\}$ for all $n \in N$. Thus we obtain $G = \bigcap\{G_n : n \geq 1\}$ and $H = \bigcap\{H_n : n \geq 1\}$. Since $\{G_\alpha : \alpha \in A\}$ and $\{H_\alpha : \alpha \in A\}$ are the two families in the subspace V of X, we have $V \subset \bigcup\{c_g(G_\alpha)^V : \alpha \in A\} = (c_g(G))^V$ and $V \subset \bigcup\{c_g(H_\alpha)^V : \alpha \in A\} = (c_g(H))^V$. Thus G and H are the two disjoint g-dense G_δ sets in the subspace V of X. It follows from Lemma 3.5 that $c_g(V)$ is not a weakly g-Volterra subspace of X. Since $\bigcup\{U : U \in \mathcal{G}\} \subset c_g(V)$, by Lemma 3.4, $\bigcup\{U : U \in \mathcal{G}\}$ is not a weakly g-Volterra subspace of $c_g(V)$. Therefore $\bigcup\{U : U \in \mathcal{G}\}$ is not a weakly g-Volterra subspace of X. □

Using Lemma 3.4 and Theorem 3.3, we prove the following theorem. In our previous paper [11], we have proved that every g-Volterra space is weakly g-Volterra and presented an example that the converse implication is not true. We prove the theorem by using the following

Definition 3.7. A generalized topological space (X, g) is said to be homogeneous if for any two distinct points $x, y \in X$ there exists a homeomorphism $f : X \to X$ such that $f(x) = y$.

Theorem 3.8. *Let (X, g) be a generalized topological space and homogeneous space. Then X is a g-Volterra space if and only if X is a weakly g-Volterra space.*

Proof. The necessary condition is trivial. To prove the sufficiency part, assume that X is a weakly g-Volterra space and X_V is a non-empty g-open g-Volterra subspace of X. Let U be any non-empty g-open subspace of X. Then there exists a point $x \in X_V$ and a homeomorphism $f : X \to X$ such that $f(x) \in U$. Since $f(X_V)$ is g-Volterra, so, $U \cap f(X_V)$, a non-empty g-open subspace of $f(X_V)$, is g-Volterra. By Lemma 3.4, U is a weakly g-Volterra subspace of X. By Theorem 3.3, the space X is a g-Volterra space. □

Theorem 3.9. *Let (X, g) be a generalized topological space and A be a non-empty g-open subspace of a generalized topological space X, then A is not a weakly g-Volterra space in X if and only if for any g-open subset U of X with $U \cap A \neq \emptyset$, then there exists a non-empty g-open subset V of X contained in U such that $V \cap A$ is not a weakly g-Volterra space in X.*

Proof. By Lemma 3.4, the necessary condition arises directly. Therefore, first, we prove the sufficiency part. Suppose that A is a g-nowhere dense subset of X. Let U and V be any two g-dense G_δ sets in A. If A is a weakly g-Volterra space, then $U \cap V$ is g-somewhere dense in the subspace A. We prove this by contradiction. Let G be any non-empty g-open subset of A and H be a g-open subset of X with $G = H \cap A$. Also, $i_g(A)$ is g-dense in A, then $H \cap i_g(A) \neq \emptyset$. Since A is g-nowhere dense set of X, then $U \cap V$ is a g-nowhere dense subset of X. Thus there exists a non-empty g-open subset O of X contained in $H \cap i_g(A)$ such that $O \cap (U \cap V) = \emptyset$. This shows that $U \cap V$ is a g-nowhere dense set in the subspace A, which is a contradiction. Hence A is not weakly g-Volterra in X. Next, we consider that A is a g-somewhere dense subset of X. Let $U = i_g(c_g(A))$. Then U is a non-empty g-open subset of X. Let $\mathcal{G} = \{U_\beta : \beta \in B\}$ be the family of all non-empty g-open subsets of X such that for each $\beta \in B$, $U_\beta \subset U$ and $U_\beta \cap A$ is not a weakly g-Volterra subspace of X. Now, for each $\beta \in B$, $U_\beta \cap A$ is not a weakly g-Volterra space. By Lemma 3.5, $\bigcup\{U_\beta \cap A : \beta \in B\}$ is not a weakly g-Volterra space in the subspace A. Then $\bigcup\{U_\beta \cap A : \beta \in B\}$ is a g-dense g-open subspace of $U \cap A$. Hence by Lemma 3.5, $U \cap A$ is not a weakly g-Volterra space. Since $i_g(A) \subset U \cap A$, by Lemma 3.4,

$i_g(A)$ is not a weakly g-Volterra space in X. Finally, $i_g(A)$ is g-dense and g-open in the subspace A. Also, by Lemma 3.5, A is not a weakly g-Volterra subspace of X. □

4. Product of g-Volterra Spaces in Generalized Topological Space

In this section, we introduce a generalized pseudo-base in generalized topological spaces and, using this concept, we study the product of g-Volterra spaces in generalized topological spaces.

Definition 4.1. A family $\mathcal{B}$ of non-empty g-open sets in a generalized topological space is called a generalized pseudo- base (g-pseudo-base) if every non-empty g-open set contains at least one member of $\mathcal{B}$.

Example 4.2. Consider R with a usual topology. Now, $\mathcal{B} = \{(-1/n, 1/n) \cup (n, n+1) : n \in R\}$, then $\mathcal{B}$ has a g-pseudo-base at 0.

Definition 4.3 ([16])**.** Let $E \subset X \times Y$, the section of E corresponding to any $x \in X$ is defined as $E(x) = \{y\colon (x, y) \in E\}$.

Definition 4.4. Let $J \neq \emptyset$ be an index set, $X_\alpha \neq \emptyset$ for $\alpha \in J$ and let $X = \prod_{\alpha \in J} X_\alpha$ be the Cartesian product of the sets X_α. We denote the projection mapping $p_\alpha : X \to X_\alpha$ by p_α.

Let $\{X_\alpha\}_{\alpha \in J}$ be an indexed family of sets. We define the product in generalized topologies as follows:

Suppose that g_α is a given generalized topology on X_α for $\alpha \in J$ and M_α is the union of all elements of g_α; that is, $M_\alpha = \bigcup_{\alpha \in J} g_\alpha$. Consider the collection of all sets of the form $\prod_{\alpha \in J} U_\alpha$, where U_α is g_α-open in X_α and for each α and U_α equals M_α, except for finitely many indices α. Thus $\mathcal{B} = \{\prod_{\alpha \in J} U\alpha | U_\alpha \in \mathcal{B}_\alpha\ for\ each\ \alpha\ and\ U_\alpha = M_\alpha\, except\ for\ finitely\ many\ indices\ \alpha\}$, where $\mathcal{B}_\alpha \subset g_\alpha$ for each α is the g-pseudo base for g_α. Clearly, $\emptyset \in \mathcal{B}$. We define a generalized topology $g = g(\mathcal{B})$ having $\mathcal{B}$ for a g-pseudo base. We call the product of the generalized topologies g_α by g, denoted by $g = \prod_{\alpha \in J} g_\alpha$.

Consider $A_\alpha \subset X_\alpha$, $A = \prod_{\alpha \in J} A_\alpha$, $x \in X$ and $x_\alpha = p_\alpha(x)$.

Example 4.5. Let $J = N$, $X_\alpha = R$ be the Euclidean topology for every α. We choose $U_\alpha = (-1/\alpha, 1/\alpha)$ such that $U_\alpha \in g_\alpha$ for $\alpha \in J$. We define $X = \prod_{\alpha \in J} X_\alpha$ and $g = \prod_{\alpha \in J} g_\alpha$. Consider $\mathcal{B} = \{\prod_{\alpha \in J} U_\alpha \mid U_\alpha \in \mathcal{B}_\alpha \subset g_\alpha$ for each α and $U_\alpha = M_\alpha$ except for finitely many indices $\alpha\}$, where $\mathcal{B}_\alpha \subset g_\alpha$ and each $\mathcal{B}_\alpha$ has a g_α-pseudo-base. Therefore $\mathcal{B}$ has a g-pseudo-base.

Csaszar [6] defined the following proposition in generalized topological spaces.

Proposition 4.6 ([6])**.** *Let (X, g) be a generalized topological space. Then the following conditions hold:*

(i) *For $\alpha \in J$, $i_g(A) \subset \prod_{\alpha \in J} i_g(A_\alpha)$.*
(ii) *If J is finite, then $i_g(A) = \prod_{\alpha \in J} i_g(A_\alpha)$.*
(iii) $c_g(A) = \prod_{\alpha \in J} c_g(A_\alpha)$.

Proposition 4.7. *The projection mapping p_α is (g, g_α)-open and p_α need not be (g, g_α)-continuous.*

The following example shows that the projection mapping need not be continuous in generalized topological spaces.

Example 4.8. Let $X = X_1 \times X_2$, $g = g_1 \times g_2$, $X_1 = \{1, 2\}$, $g_1 = \{\emptyset, X_1\}$, $X_2 = \{3, 4\}$, $g_2 = \{\emptyset, \{3\}\}$. $X_1 \times X_2 = \{(1, 3), (1, 4), (2, 3), (2, 4)\}$ and $g_1 \times g_2 = \{\emptyset \times \emptyset, \emptyset \times \{3\}, X_1 \times \emptyset, X_1 \times \{3\}\}$ by g. Also, $\mathcal{B} = \{\emptyset, X_1 \times \{3\}\}$, then $X_1 \in g_1$ and ${p_1}^{-1}(X_1) = X \notin \mathcal{B}$. Therefore p_α is not a (g, g_α) continuous.

Lemma 4.9 ([6])**.** *Let (X, g) be a generalized topological space. Then $X = M_g = \prod_{\alpha \in J} U_\alpha$.*

Proof. If $U \in g$, then $p_\alpha(U) \in g_\alpha$. Let $\prod_{\alpha\in J} U_\alpha \in \mathcal{B} \subset g$, then $U \subset \prod_{\alpha\in J} p_\alpha(U) \subset \prod_{\alpha\in J} U_\alpha \in g$. Therefore $\prod_{\alpha\in J} U_\alpha$ is the largest g-open set in g. □

Proposition 4.10. *If every g_α is strong, then g is strong and p_α is (g, g_α)-continuous for each $\alpha \in J$.*

Proof. By Lemma 4.9, $M_g = \prod_{\alpha\in J} U_\alpha = \prod_{\alpha\in J} X_\alpha = X$ such that $X \in g$. Then $U_\alpha = X_\alpha$ for each α. Also, $U_\alpha \in g_\alpha$ which implies that ${p_\alpha}^{-1}(M_\alpha) = \prod_{i\in J} N_i$, where $N_\alpha = M_\alpha$ and $N_i = X_i$ for $i \neq \alpha$. Hence $\prod_{i\in J} N_i \in \mathcal{B} \subset g$. □

Lemma 4.11 ([13])**.** *Let $\{(X_\alpha, g_\alpha) : \alpha \in J\}$ be a family of generalized topological spaces and $(\prod_{\alpha\in J} X_\alpha, g)$ be its product. Let $p_\alpha : \prod_{\alpha\in J} X_\alpha \to X_\alpha$ be the projection, then the following properties hold:*

(i) *For each $\alpha \in J$, p_α is (g, g_α)-open.*
(ii) *For each $\alpha \in J$, if g_α is strong, then g is strong and p_α is (g, g_α)-continuous.*

Definition 4.12. For any $0 \leq m < n$, we define $X^{(n)} = \prod_{i=1}{}^{n} X_i$, $X^{(m,n)} = \prod_{i=m+1}{}^{n} X_i$ and $Y^{(n)} = \prod_{i=n+1}{}^{\infty} X_i$.

Lemma 4.13. *If each X_i has a countable generalized pseudo-base, then the product space $\prod_{\alpha\in J} X_\alpha$ has a countable g-pseudo-base.*

Proof. Consider a sequence $\{X_i\}$ of spaces which has a countable g-pseudo-base. In the following case we prove the result by induction on n.

Case (i): When $n = 2$, $\mathcal{B} = \{U \times V : U \in \mathcal{B}_1$ and $V \in \mathcal{B}_2\}$. Since $\mathcal{B}_1$ and $\mathcal{B}_2$ are countable, then $\mathcal{B}$ is countable. If $(x, y) \in G$, then there exist g-open sets $G_1 \subset X_1$ and $G_2 \subset X_2$ such that $(x, y) \in G_1 \times G_2 \subset G$. Since $\mathcal{B}_1$ is a generalized pseudo-base for X_1, there exists $U \in \mathcal{B}_1$ such that $x \in U \subset G_1$. Also, $\mathcal{B}_2$ is a generalized pseudo-base for X_2, there exists $V \in \mathcal{B}_2$ such that $y \in V \subset G_2$. Thus $(x, y) \in U \times V \subset G_1 \times G_2 \subset G$ and $X_1 \times X_2$ has a countable generalized pseudo-base. Assume that the result is true for $n-1$. Therefore $X_1 \times X_2 \times \cdots \times X_{n-1}$ have a countable g-pseudo-base. By Case(i), $X_1 \times X_2 \times \cdots \times X_n$ have a countable g-pseudo-base. Hence $X^{(n)}$ has a countable g-pseudo-base $\{G(n, i)\}$. Thus $G(n, i) \times Y^{(n)}$, where n and i are the positive integers which constitute a countable g-pseudo-base in $\prod_{\alpha\in J} X_\alpha$ which has a countable g-pseudo-base. □

The following example shows that the product of two g-Volterra spaces need not be a g-Volterra space in the generalized topological spaces.

Example 4.14. Let (X_1, g_1) and (X_2, g_2) be two generalized topological spaces, where $X_1 = \{1, 2\}$, $X_2 = \{3, 4\}$, $g_1 = \{\emptyset, X_1\}$, $g_2 = \{\emptyset, \{3\}\}$. Observe that (X_1, g_1) and (X_2, g_2) are g-Volterra spaces. Now, consider $X = X_1 \times X_2$ and $g = g_1 \times g_2$. It is easy to see that (X, g) is a generalized topological space. Now, if we take $G = \{(1, 3)\}$ and $H = \{(2, 4)\}$, then both G and H are g-dense G_δ subsets of X. Clearly, $G \cap H = \emptyset$ and $G \cap H$ is not a g-dense subset of X. Therefore X is not a g-Volterra space.

In the following example, we show that the product of two g-Volterra spaces is a g-Volterra space in generalized topological spaces even if one of them has a g-pseudo-base.

Example 4.15. Let (X_1, g_1) be a strong generalized topological space and (X_2, g_2) be a generalized topological space, where $X_1 = X_2 = N$, $g_1 = \{\emptyset, N - \{a\}, N\}$, where $a \in N$ and $g_2 = \{\emptyset, N - \{b\}\}$, where $b \in N$ and a distinct of b. We take $\mathcal{B} = \{\emptyset, N - \{a\}, N\}$. Note that $\mathcal{B}$ is a g_1-pseudo-base for the generalized topological space (X_1, g_1). Observe that (X_1, g_1) and (X_2, g_2) are g-Volterra spaces and X_1 has g_1-pseudo-base. Now, consider $X = X_1 \times X_2 = N \times N$, $g = g_1 \times g_2$. It is easy to see that (X, g) is a generalized topological space. The collection of g-open sets in $N \times N$ are $\{\emptyset, N - \{a\} \times N - \{b\}, N \times N - \{b\}\}$ and the collection of g-closed sets in $N \times N$ are $\{N \times N, \{a\} \times N \cup N \times \{b\}, N \times N - \{b\}\}$. Now, G and H are g-dense and the non-empty G_δ subsets in the generalized topological space $(N \times N, g)$

are $N-\{a\}\times N-\{b\}$ and $N\times N-\{b\}$. The possibility for G and H is $N-\{a\}\times N-\{b\}$ or $N\times N-\{b\}$.

When two of them are intersected, we obtain:

(i) $N-\{a\}\times N-\{b\}$ intersected with $N-\{a\}\times N-\{b\} = N-\{a\}\times N-\{b\}$ is g-dense in the GTS $(N\times N, g)$.

(ii)$N\times N-\{b\}$ intersected with $N\times N-\{b\} = N\times N-\{b\}$ is g-dense in the GTS $(N\times N, g)$.

(iii) $N-\{a\}\times N-\{b\}$ intersected with $N\times N-\{b\} = N-\{a\}\times N-\{b\}$ is g-dense in the GTS $(N\times N, g)$.

From the above three cases, we find that the generalized topological space $(N\times N, g)$ is g-Volterra.

Example 4.16. Consider $X_1 = X_2 = R$ with the discrete topology. Then X_1 and X_2 are g-Volterra spaces and the product space $X_1\times X_2$ is a g-Volterra space. Observe that if we take the basis for the product topology on $R\times R$ given by $\mathcal{B} = \{\{x_1\}\times\{x_2\}\colon x_1, x_2\in R\}$, it has a g-pseudo-base.

Theorem 4.17. *Let X and Y be g-Volterra spaces and X be a g-second category space with a g-pseudo-base $\mathcal{B}$ such that no two members of $\mathcal{B}$ are pairwise disjoint and at least one of them has a countable g-pseudo-base, then their product $X\times Y$ is a g-Volterra space.*

Proof. Suppose that $X\times Y$ is not a g-Volterra space. Then by Theorem 3.3, there exists g-open set $U\times V$ in $X\times Y$ such that $U\times V$ is not a weakly g-Volterra space. By Definition 2.9, there exists two g-dense gG_δ sets M and N in $U\times V$ such that $M\cap N=\emptyset$. Since M and N are G_δ sets, we have $M = \bigcap\limits_{n=1}^{\infty} M_n$ and $N = \bigcap\limits_{n=1}^{\infty} N_n$, where M_n and N_n are g-open sets. Since M and N are g-dense, each M_n and N_n is g-dense. Let $\{V_\alpha\}$ be a countable g-pseudo-base for Y. For each n, α, we define $h_{n,\alpha} = M_n\cap(U\times V_\alpha)$ and $g_{n,\alpha} = N_n\cap(U\times V_\alpha)$. Also, we define $H_{n,\alpha} = p_X(h_{n,\alpha})$ and $G_{n,\alpha} = p_X(g_{n,\alpha})$ such that Hn,k and $G_{n,k}$ are g-open. Since M_n is g-dense in $U\times V$ which implies that $M_n\cap(U\times V_\alpha)$ is g-dense in $U\times V_\alpha$. Thus $h_{n,\alpha}$ is g-dense in $U\times V_k$. Let U_1 be any g-open set in U. Then $U_1\times V_\alpha$ is a g-open set in $U\times V_\alpha$. Therefore $(U_1\times V_\alpha)\cap h_{n,\alpha}\neq\emptyset$. Thus each $H_{n,\alpha}$ is g-dense in U. Similarly, $G_{n,\alpha}$ is g-dense in U. Since X is of g-second category, by [13, Theorem 5.3], there exists a g-open subset D in X such that D is a g-Baire space.

Thus $D\cap U\neq\emptyset$ and $D\cap U$ is a g-open set in D. Since D is a g-Baire space, this implies that $D\cap U$ is a g-Baire space. Therefore $D\cap U$ intersects each $G_{n,\alpha}$ and $H_{n,\alpha}$. Hence $D\cap U\cap G_{n,\alpha}$ and $D\cap U\cap H_{n,\alpha}$ are g-dense g-open sets in $D\cap U$. Since $D\cap U$ is a g-Baire space, $\bigcap\limits_{n,\alpha}(B\cap U\cap G_{n,\alpha})$ and $\bigcap\limits_{n,\alpha}(D\cap U\cap H_{n,\alpha})$ are g-dense g-open sets in $D\cap U$. By Theorem 3.3, D and U are g-open sets in X and X is a g-Volterra space, then $D\cap U$ is a weakly g-Volterra space. Thus $\bigcap\limits_{n,\alpha}[D\cap U\cap G_{n,\alpha}]\cap\bigcap\limits_{n,\alpha}[D\cap U\cap G_{n,\alpha}]$ is non-empty. Then we choose $x\in U$ such that $x\in\bigcap\limits_{n,\alpha}[D\cap U\cap G_{n,\alpha}]\cap\bigcap\limits_{n,\alpha}[D\cap U\cap H_{n,\alpha}]$.

$$\begin{aligned}&\implies x\in\bigcap_{n,\alpha}[B\cap U\cap G_{n,\alpha}\cap H_{n,\alpha}].\\&\implies x\in G_{n,\alpha}\cap H_{n,\alpha}\text{ for every } n,\alpha.\end{aligned}$$

There exists y such that $(x,y)\in h_{n,\alpha}$, $(x,y)\in g_{n,\alpha}$. Also, $(x,y)\in M_n\cap(U\times V_\alpha)$, $(x,y)\in N_n\cap(U\times V_\alpha)$ for all n, α. We define $M(x) = \{y\in V\colon (x,y)\in M\}$ and $N(x) = \{y\in V\colon (x,y)\in N\}$. Now, we have $M(x) = (\bigcap\limits_n M_n)(x) = \bigcap\limits_n[M_n(x)]$.

Suppose

$$\begin{aligned}y\in(\bigcap_n M_n)(x) &\iff (x,y)\in\bigcap_n M_n.\\&\iff (x,y)\in M_n\text{ for all } n.\\&\iff y\in M_n(x)\text{ for all } n.\\&\iff y\in\bigcap_n(M_n)(x).\end{aligned}$$

Therefore $M(x)$ is a G_δ set. Also $(x,y) \in M_n \cap (U \times V_\alpha)$ for each V_α and for all n, k.

$$\implies (x,y) \in \Big(\bigcap_n M_n \cap (U \times V_\alpha)\Big).$$
$$\implies \text{ that } (x,y) \in M_n \cap (U \times V_\alpha).$$

Therefore there exists $y \in V_\alpha$ such that $(x,y) \in M$ and so, $y \in V$ such that $y \in M(x)$. Thus $M(x) \cap V_\alpha \neq \emptyset$. Hence $M(x)$ is g-dense in V. Similarly, $N(x)$ is also g-dense in V. Since Y is a g-Volterra space and V is a weakly g-Volterra space, therefore $M(x) \cap N(x) \neq \emptyset$.

Thus there exists $z \in U$ such that $z \in M(x) \cap N(x)$.

$$\implies z \in M(x) \text{ and } z \in N(x).$$
$$\implies (x,z) \in M \text{ and } (x,z) \in N.$$
$$\implies (x,z) \in M \cap N, \text{ which is a contradiction.}$$

Therefore $M \cap N$ is non-empty and $U \times V$ is a weakly g-Volterra space. Hence $X \times Y$ is a g-Volterra space. □

Theorem 4.18. *Suppose that X_i is a g-second category space that has a countable g-pseudo-base $\mathcal{B}$ such that no two members of $\mathcal{B}$ are pairwise disjoint. If every X_i is a g-Volterra space for finite n, then $X^{(m,n)}$ is a g-Volterra space.*

Proof. We prove the result by induction on n. If $n = 2$, then $0 \leq m < 2$. Suppose $m = 1$, then $X^{(m,2)} = X^2$ is g-Volterra. Suppose $m = 0$, then $X^{(m,2)} = \prod_{i=1}^{n} X_i = X_1 \times X_2$. By Theorem 4.17, $X_1 \times X_2$ is a g-Volterra space. Assume that $X^{(m,n-1)}$ is a g-Volterra space, then we have $\prod_{i=m+1}^{n-1} X_i = X_{m+1} \times X_{m+2} \times \cdots \times X_{n-1}$ is a g-Volterra space. We obtain $X^{(m,n)} = \prod_{i=m+1}^{n} X_i = (\prod_{i=m+1}^{n-1} X_i) \times X_n = X^{(m,n-1)} \times X_n$. Since $X^{(m,n-1)}$ and X_n are g-Volterra space, therefore $X^{(m,n)}$ is g-Volterra space. □

Theorem 4.19. *Suppose that X_i is a g-second category space which has a countable g-pseudo-base $\mathcal{B}$ such that no two members of $\mathcal{B}$ are pairwise disjoint. If every X_i is a g-Volterra space, then $\prod_{\alpha\in J} X_\alpha$ is a g-Volterra space.*

Proof. Consider $X = \prod_{\alpha\in J} X_\alpha$. Let M and N be two g-dense G_δ sets in X. Since M and N are g-dense, therefore M_n and N_n are g-dense. Assume that M_n and N_n are the decreasing sequences of g-dense g-open sets in X. Let A be a g-open set in X. Since M_1 is a g-dense g-open set in X, $A \cap M_1$ is a non-empty g-open set in X. By Lemma 4.13, every X_i has a countable g-pseudo-base, then $\prod_{\alpha\in J} X_\alpha$ has a countable g-pseudo base. Then there exists a basic g-open set $U^{(n_1)} \times Y^{(n_1)}$ such that $U^{(n_1)} \times Y^{(n_1)} \subset A \cap M_1 \cap N_1$, where $U^{(n_1)} \times Y^{(n_1)}$ is a g-open set in $X^{(n_1)} \times Y^{(n_1)}$. Now, we define $M_n^{(n_1)} = M_n \cap (U^{(n_1)} \times Y^{(n_1)})$ and $N_n^{(n_1)} = N_n \cap (U^{(n_1)} \times Y^{(n_1)})$ for all n. Then $M_n^{(n_1)}$ and $N_n^{(n_1)}$ form a sequence of g-dense g-open sets $U^{(n_1)} \times Y^{(n_1)}$. Let $V_\alpha^{(n_1)}$ be a countable generalized pseudo-base for $Y^{(n_1)}$. Also, we define $h_{n,\alpha}^{(n_1)} = M_n^{(n_1)} \cap (U^{(n_1)} \times V_\alpha^{(n_1)})$ and $g_{n,\alpha}^{(n_1)} = N_n^{(n_1)} \cap (U^{(n_1)} \times V_\alpha^{(n_1)})$. Let $H_{n,\alpha}^{(n_1)} = p_{X^{(n_1)}}(h_{n,\alpha}^{(n_1)})$ and $G_{n,\alpha}^{(n_1)} = p_{X^{(n_1)}}(g_{n,\alpha}^{(n_1)})$. Since the projection mapping p_α is (g, g_α)-open and $h_{n,\alpha}^{(n_1)}$ is (g, g_α)-open and so, $H_{n,\alpha}^{(n_1)}$ is also (g, g_α)-open. Then $M_n^{(n_1)}$ is g-dense in $U^{(n_1)} \times Y^{(n_1)}$.

$$\implies M_n^{(n_1)} \cap U^{(n_1)} \times V_\alpha^{(n_1)} \text{ is } g\text{-dense in } U^{(n_1)} \times V_\alpha^{(n_1)}.$$
$$\implies h_{n,\alpha}^{(n_1)} \text{ is } g\text{-dense in } U^{(n_1)} \times V_\alpha^{(n_1)}.$$
$$\implies (U_1^{(n_1)} \times V_\alpha^{(n_1)}) \cap h_{n,\alpha}^{(n_1)} \neq \emptyset \text{ for any } g\text{-open set } U_1^{(n_1)} \text{ in } U^{(n_1)}.$$

Then $U_1^{(n_1)} \cap p_{X^{(n_1)}}(h_{n,\alpha}^{(n_1)}) \neq \emptyset$ and thus $U_1^{(n_1)} \cap H_{n,\alpha}^{(n_1)} \neq \emptyset$. Therefore $H_{n,\alpha}^{(n_1)}$ is g-dense in $U^{(n_1)}$. Since every X_i is a g-second category space, there exists a g-open subset D_i in X_i such that D_i is a g-Baire space. Suppose that $D^{(n_1)}$ is a g-open g-Baire subspace of $X^{(n_1)}$. Then $D^{(n_1)} \cap U^{(n_1)} \neq \emptyset$. Also, $D^{(n_1)} \cap U^{(n_1)}$ is a g-open subset of $D^{(n_1)}$ and $D^{(n_1)}$ is a g-Baire space which implies that $D^{(n_1)} \cap U^{(n_1)}$

is a g-Baire subspace. Since $U^{(n_1)}$ is g-open, the sets $D^{(n_1)} \cap U^{(n_1)} \cap H_{n,\alpha}^{(n_1)}$ and $D^{(n_1)} \cap U^{(n_1)} \cap G_{n,\alpha}^{(n_1)}$ are g-dense g-open in $D^{(n_1)} \cap U^{(n_1)}$. Since $D^{(n_1)} \cap U^{(n_1)}$ is g-Baire, $\bigcap_{n,\alpha}[D^{(n_1)} \cap U^{(n_1)} \cap G_{n,\alpha}^{(n_1)}]$ $\bigcap_{n,\alpha}(D^{(n_1)} \cap U^{(n_1)} \cap H_{n,\alpha}^{(n_1)})$ will be g-dense G_δ sets in $D^{(n_1)} \cap U^{(n_1)}$. By Theorem 3.3, $D^{(n_1)}$ and $U^{(n_1)}$ are g-open in $X^{(n_1)}$ and $X^{(n_1)}$ is a g-Volterra space, then $D^{(n_1)} \cap U^{(n_1)}$ is a weakly g-Volterra space. Therefore $\{\bigcap_{n,\alpha}(D^{(n_1)} \cap U^{(n_1)} \cap G_{n,\alpha}^{(n_1)})\} \cap \{\bigcap_{n,\alpha}(D^{(n_1)} \cap U^{(n_1)} \cap H_{n,\alpha}^{(n_1)})\}$ is non-empty. Then there exists $b_1 \in U^{(n_1)}$ such that $b_1 \in \{\bigcap_{n,\alpha}(D^{(n_1)} \cap U^{(n_1)} \cap G_{n,\alpha}^{(n_1)})\} \cap \{\bigcap_{n,\alpha}(D^{(n_1)} \cap U^{(n_1)} \cap H_{n,\alpha}^{(n_1)})\}$ which implies that $b_1 \times Y^{(n_1)} \subset U^{(n_1)} \times Y^{(n_1)} \subset A \cap M_1 \cap N_1$. Thus $M_n(b_1)$ and $N_n(b_1)$ are g-dense g-open sets in $Y^{(n_1)}$.

Next, we prove the result by induction on k.

Suppose that $n_1 < n_2 < n_3 \cdots n_k$ and $b_i \in X^{(n_{i-1}, n_i)}$, $i = 1, 2, 3, \dots k$ such that

(i) $(b_1, b_2, \dots, b_k) \times Y^{(n_k)} \subset A \cap M_k \cap N_k$,

(ii) $M_n(b_1, b_2, \dots, b_k)$ and $N_n(b_1, b_2, \dots, b_k)$ are g-dense g-open in $Y^{(n_k)}$ for $n = 1, 2, 3, \dots$.

Since X has a countable g-pseudo-base, there is a basic g-open set $U^{(n_{k+1})} \times Y^{(n_{k+1})}$ such that $U^{(n_{k+1})} \times Y^{(n_{k+1})} \subset A \cap M_{k+1}(b_1, b_2, \dots, b_k) \cap N_{k+1}(b_1, b_2, \dots, b_k)$. We define $H_{n,\alpha}^{(n_{k+1})}$ and $G_{n,\alpha}^{(n_{k+1})}$, the projections of $X^{(n_k, n_{k+1})}$, then $b_{k+1} \in \bigcap_{n,\alpha}[D^{(n_{k+1})} \cap U^{(n_{k+1})} \cap H_{n,\alpha}^{(n_{k+1})} \cap G_{n,\alpha}^{(n_{k+1})}]$. Also, $M_n(b_1, b_2, \dots, b_k)$ and $N_n(b_1, b_2, \dots, b_k)$ are g-dense g-open in $Y^{(n_{k+1})}$. We choose $b_{k+1} \times Y^{(n_{k+1})} \subset M_{k+1}(b_1, b_2, \dots, b_k)$, then $b_1, b_2, \dots, b_k, b_{k+1} \times Y^{(n_{k+1})} \subset M_{k+1}$. Thus $(b_1, b_2, \dots, b_k) \times Y^{(n_k)} \subset A \cap M_k \cap N_k$ and $(b_1, b_2, \dots, b_k) \times Y^{(n_k)} \subset G$ which implies that $(b_1, b_2, \dots, b_k) \times Y^{(n_{k+1})} \subset G \cap M_{k+1} \cap N_{k+1}$. Hence the conditions are satisfied for $k+1$. Therefore the sequences $\{n_k\}$ and $\{b_k\}$ are satisfied with every positive integer k. Let $x \in X$ and there exists a sequence $\{b_k\}$ such that $x \in G \cap M_k \cap N_k$ for all k which implies that $x \in G \cap (\cap M_k) \cap (\cap N_k)$. Therefore $x \in G \cap M \cap N$ which implies that $M \cap N$ is non-empty. Hence $M \cap N$ is g-dense in X. Thus $\prod_{\alpha \in J} X_\alpha$ is a g-Volterra space. □

5. Product of g-Baire Space and g-Volterra Space in Generalized Topological Spaces

In this section, we show that the product of a g-Baire space and g-Volterra space with a countable g-pseudo-base is g-Volterra.

Theorem 5.1. *Let X and Y be generalized topological spaces. Let X be a g-Baire space and Y be a g-Volterra space with a countable g-pseudo-base, then $X \times Y$ is a g-Volterra space.*

Proof. Let X be a g-Baire space and Y be a g-Volterra space with a countable generalized pseudo-base $\{V_\alpha\}$. Suppose that $X \times Y$ is not a g-Volterra space. Then there exists a g-open set $U \times V$ in $X \times Y$ such that $U \times V$ is not a weakly g-Volterra space. Let M and N be two g-dense G_δ sets in $U \times V$ such that $M \cap N = \emptyset$. Since M and N are G_δ sets, hence $M = \bigcap_{n=1}^{\infty} M_n$ and $N = \bigcap_{n=1}^{\infty} N_n$, where M_n and N_n are g-open sets. Again, M and N are g-dense, then M_n and N_n are g-dense. For each n, α, we define $h_{n,\alpha} = M_n \cap (U \times V_\alpha)$, $g_{n,\alpha} = N_n \cap (U \times V_\alpha)$, $H_{n,\alpha} = p_X(h_{n,\alpha})$ and $G_{n,\alpha} = p_X(g_{n,\alpha})$ such that Hn, α and $G_{n,\alpha}$ are g-dense g-open sets in U. Since M_n is g-dense in $U \times V$, this implies that $M_n \cap (U \times V_\alpha)$ is g-dense in $U \times V_\alpha$. Thus $h_{n,\alpha}$ is g-dense in $U \times V_\alpha$. Let U_1 be any g-open set in U. Then $U_1 \times V_\alpha$ is a g-open set in $U \times V_\alpha$. Therefore $(U_1 \times V_\alpha) \cap h_{n,\alpha} \neq \emptyset$ which implies that $U_1 \cap p_X(h_{n,\alpha}) \neq \emptyset$ which further implies that $U_1 \cap H_{n,\alpha} \neq \emptyset$. Thus each $H_{n,\alpha}$ is g-dense in U, $U \cap H_{n,\alpha}$ and $U \cap G_{n,\alpha}$ are g-dense g-open sets in U.

Since U is g-Baire, $\bigcap_{n,\alpha}[U \cap H_{n,\alpha}]$ and $\bigcap_{n,\alpha}[U \cap G_{n,\alpha}]$ are g-dense G_δ sets in U and $\bigcap_{n,\alpha}[U \cap H_{n,\alpha}] \cap [_{n,\alpha}\ U \cap G_{n,\alpha}]$ is non-empty. Therefore there exists $x \in U$ such that $\bigcap_{n,\alpha}[U \cap H_{n,\alpha} \cap G_{n,\alpha}]$ which implies that $x \in H_{n,\alpha} \cap G_{n,\alpha}$ for every n, α. We define $M(x) = \{y \in V \text{ such that } (x, y) \in M\}$ and $N(x) = \{y \in V \text{ such that } (x, y) \in N\}$. Then $M(x)$ and $N(x)$ are g-dense G_δ sets in V. Since Y is a g-Volterra space and also V is a weakly g-Volterra space, this implies that $M(x) \cap N(x) \neq \emptyset$. Thus

$M \cap N \neq \emptyset$ which is a contradiction. Therefore $M \cap N$ is nonempty and $U \times V$ is a weakly g-Volterra space. Clearly, $X \times Y$ is a g-Volterra space. □

Acknowledgement

We are very much grateful to the referee for the valuable comments to improve the paper to a larger extent.

References

1. J. Cao, D. Gauld, Volterra spaces revisited. *J. Aust. Math. Soc.* **79** (2005), no. 1, 61–76.
2. J. Cao, H. J. K. Junnila, When is a Volterra space Baire? *Topology Appl.* **154** (2007), no. 2, 527–532.
3. A. Császár, Generalized open sets. *Acta Math. Hungar.* **75** (1997), no. 1-2, 65–87.
4. A. Császár, Generalized topology, generalized continuity. *Acta Math. Hungar.* **96** (2002), no. 4, 351–357.
5. A. Császár, Generalized open sets in generalized topologies. *Acta Math. Hungar.* **106** (2005), no. 1-2, 53–66.
6. A. Császár, Product of generalized topologies. *Acta Math. Hungar.* **123** (2009), no. 1-2, 127–132.
7. D. B. Gauld, S. Greenwood, Z. Piotrowski, On Volterra spaces. II. In: *Papers on general topology and applications (Gorham, ME, 1995)*, 169–173, Ann. New York Acad. Sci., 806, New York Acad. Sci., New York, 1996.
8. D. B. Gauld, S. Greenwood, Z. Piotrowski, On Volterra spaces. III. *Topology Proc.* **23** (1998), Spring, 167–182.
9. D. B. Gauld, Z. Piotrowski, On Volterra spaces. *Far East J. Math. Sci.* **1** (1993), no. 2, 209–214.
10. G. Gruenhage, D. Lutzer, Baire and Volterra spaces. *Proc. Amer. Math. Soc.* **128** (2000), no. 10, 3115–3124.
11. P. Jeyanthi, G. Geetha, Generalized Volterra spaces. *An.Univ. Oradea, Fasc. Mat.* **27** (2020), no. 1, 43–46.
12. P. Jeyanthi, G. Geetha, Weakly g-Volterra spaces in generalized topological spaces. *An. Univ. Oradea Fasc. Mat.* **29** (2022), no. 2, 81–87.
13. Z. Li, F. Lin, Baireness on generalized topological spaces. *Acta Math. Hungar.* **139** (2013), no. 4, 320–336.
14. M. Matejdes, Generalized Volterra spaces. *Int. J. Pure Appl. Math.* **85** (2013), no. 5, 955–963.
15. J. C. Oxtoby, Cartesian product of Baire spaces. *Fund. Math.* **49** (1961), no. 2, 157–166.
16. V. Renukadevi, R. Thangamariappan, On product of Volterra spaces. *J. Adv. Math. Stud.* **8** (2015), no. 2, 286–290.
17. S. Spadaro, P-spaces and the Volterra property. *Bull. Aust. Math. Soc.* **87** (2013), no. 2, 339–345.
18. V. Volterra, Alcune Osservasioni sulle Funzioni Punteggiate Discontinue. *Giornale di Matematiche* **19** (1881), 76–86.

(Received 29.07.2022)

[1]Department of Mathematics, Research Centre, Govindammal Aditanar College for Women, Tiruchendur-628215, Tamilnadu, India

[2]Govindammal Aditanar College for Women, Affiliated to Manonmaniam Sundaranar University, Tiruchendur-628215, Tamilnadu, India

[3]Departmento De Ciencias Naturales Y Exactas Universidad De La Costa, Barranquilla-Colombia
Email address: jeyajeyanthi@rediffmail.com
Email address: geetha0063@gmail.com
Email address: ennisrafael@gmail.com