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THE EXISTENCE AND UNIQUENESS OF THE CAUCHY PROBLEM FOR THE BOLTERRA DIFFERENTIAL EQUATIONS

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Abstract. In the present paper, the evolutionary differential equations are investigated; the conditions for the solvability and uniqueness of the Cauchy problem for evolutionary differential equations are proved.

Assume that I^n is an n-dimensional segment

$$I^n = \underbrace{[0,1] \times \cdots \times [0,1]}_{n},$$

$C(I^n;\mathbb{R}^k)$ is a set of continuous mappings from I^n into $\mathbb{R}^k$.

Definition 1. The operator

$$g : C(I^n;\mathbb{R}^k) \to C(I^n;\mathbb{R}^m)$$

is said to be Volterra (evolutionary) if for any $(t_1,\dots,t_n) \to I^n$ and $u, v \in C(I^n;\mathbb{R}^k)$, from the equality

$$u(s_1,\dots,s_n) = v(s_1,\dots,s_n) \ \text{ for } \ 0 \le s_i \le t_i \ \ (i=1,\dots,n)$$

it follows that

$$g(u)(t_1,\dots,t_n) = g(v)(t_1,\dots,t_n).$$

The Volterra differential equations are considered in the papers [1–5].

Let us consider the system of differential equations

$$\frac{\partial u_i(t_1,\dots,t_n)}{\partial t_i} = f_i(u_1,\dots,u_n)(t_1,\dots,t_n) \ \ (i=1,\dots,n), \tag{1}$$

where

$$f_i : C(I^n;\mathbb{R}^n) \to C(I^n;\mathbb{R}) \ \ (i=1,\dots,n)$$

are continuous Volterra (evolutionary) operators. System (1) is called Volterra (evolutionary) differential system.

For system (1), consider the Cauchy problem

$$u_i(t_1,\dots,t_n)\big|_{t_i=0} = \varphi_i(t_1,\dots,t_{i-1},t_{i+1},\dots,n) \ \ (i=1,\dots,n), \tag{2}$$

where

$$\varphi_i \in C(I^{n-1};\mathbb{R}) \ \ (i=1,\dots,n).$$

For every $t \in [0,1]$ and every $v \in V(I^n;\mathbb{R})$, we introduce the notation

$$\|v\|_t = \max\Big\{|v(t_1,\dots,t_n)| : \ \ 0 \le t_1 \le t, \dots, 0 \le t_n \le t\Big\}.$$

Theorem 1. *Suppose that for any $u_i, \overline{u}_i \in C(I^n;\mathbb{R})$ and $t \in]0,1]$, the inequality*

$$\begin{gathered}\big|f_i(u_1,\dots,u_n)(t_1,\dots,t_n) - f_i(\overline{u}_1,\dots,\overline{u}_n)(t_1,\dots,t_n)\big| \\ \le \ell\big(\|u_1\|_1,\dots,\|u_n\|_1,\|\overline{u}_1\|_1,\dots,\|\overline{u}_n\|_1\big)t_i^{-\varepsilon}\sum_{k=1}^{n}\sum_{j=1}^{n}\|u_k-\overline{u}_k\|_{t^{1/\alpha_{jk}}}^{\alpha_{jk}} \ \ (i=1,\dots,n), \\ \textit{for} \ \ 0 \le t_i \le t, \dots, 0 \le t_n \le t,\end{gathered}$$

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holds, where $\ell : \mathbb{R}_+^{2n} \to \mathbb{R}_+$ *is a continuous function,* $\varepsilon \in [0,1[\,,\ \alpha_{jk} \in]0,1]$ $(i = 1,\dots,n)$. *Then problem* (1), (2) *has a unique solution.*

Here, we formulate some corollaries of Theorem 1.

Consider the case in which (1) has the form

$$\frac{\partial u_i(t_1,\dots,t_n)}{\partial t_i} = g_i\Big(t_1,\dots,t_n,u_1\big(\tau_{11}(t_1,\dots,t_n),\dots,\tau_{1n}(t_1,\dots,t_n)\big),\dots, u_n\big(\tau_{n1}(t_1,\dots,t_n),\dots,\tau_{nn}(t_1,\dots,t_n)\big)\Big) \ (i=1,\dots,n), \tag{3}$$

where $\tau_{kj} : I^n \to [0,1]$ are continuous functions $(k,j = 1,\dots,n)$.

Corollary 1. *Suppose that the inequality*

$$\Big|g_i(t_1,\dots,t_n,x_1,\dots,x_n) - g_i(t_1,\dots,t_n,y_1,\dots,y_n)\Big| \le \ell(x_1,\dots,x_n,y_1,\dots,y_n)t_i^{-\varepsilon}|x_k-y_k|^{\alpha_k} \ \ (i=1,\dots,n),$$

$$\tau_{kj}(t_1,\dots,t_n) \le \max\big\{t_1^{1/\alpha_k},\dots,t_n^{1/\alpha_k}\big\}$$

is satisfied on $I^n \times \mathbb{R}^n$, *where* $\ell : \mathbb{R}^{2n} \to \mathbb{R}_+$ *is a continuous function,* $\varepsilon \in [0,1[\,,\ \alpha_k \in]0,1]$, $\tau_{kj} : I^n \to [0,1]$ *are continuous functions* $(k = 1,\dots,n)$ $(j = 1,\dots,n)$. *Then problem* (3), (2) *has a unique solution.*

Let us consider the following Goursat problem:

$$\frac{\partial^2 u(t_1,t_2)}{\partial t_1 \partial t_2} = g\Big(t_1,t_2,u\big(\tau_{11}(t_1,t_2),\tau_{12}(t_1,t_2)\big), \frac{\partial u(\tau_{21}(t_1,t_2),\tau_{22}(t_1,t_2)}{\partial \tau_{21}}, \frac{\partial u(\tau_{31}(t_1,t_2)\tau_{32}(t_1,t_2))}{\partial \tau_{32}}\Big), \tag{4}$$

$$u(t_1,0) = \psi(t_1), \quad \frac{\partial u(0,t_2)}{\partial t_2} = \psi_2(t_2). \tag{5}$$

Corollary 2. *Suppose that the inequality*

$$\Big|g(t_1,t_2,x,y,z) - g(t_1,t_2,\overline{x},\overline{y},\overline{z})\Big| \le \ell(x,y,z,\overline{x},\overline{y},\overline{z})(t_1,+t_2)^{-\varepsilon}\Big[\frac{|x-\overline{x}|^{\alpha_1}}{t_1+t_2} + |y-\overline{y}|^{\alpha_2} + |z-\overline{z}|^{\alpha_3}\Big],$$

$$\tau_{kj}(t_1,t_2) \le \max\big\{t_1^{1/\alpha_k}, t_2^{1/\alpha_k}\big\} \ \ (j=1,2;\ \ k=1,2,3)$$

is satisfied on $I^2 \times \mathbb{R}^3$, *where* $\ell : \mathbb{R}^6 \to \mathbb{R}_+$ *is a continuous function,* $\varepsilon \in [0,1[\,,\ \alpha_k \in]0,1]$, $\tau_{kj} : I^2 \to [0,1]$ *are continuous functions. Then problem* (4), (5) *has a unique solution.*

Consider the problem

$$\frac{dx(t)}{dt} = \ell\big(t, x(\tau(t))\big), \tag{6}$$

$$x(0) = 0, \tag{7}$$

where $\tau(t) : [0,1] \to [0,1]$ is a continuous function.

Corollary 3. *Suppose that on* $I \to \mathbb{R}$ *the inequality*

$$\Big|\ell\big(t,x(\tau(t))\big) - \ell\big(t,\overline{x}(\tau(t))\big)\Big| \le \eta\big(\|x\|,\|\overline{x}\|\big)t^{-\varepsilon}\big|x(\tau(t)) - \overline{x}(\tau(t))\big|^{\alpha},$$

$$\tau(t) < t^{1/\alpha},$$

is satisfied, where $\eta : \mathbb{R}^2 \to \mathbb{R}_+$ *is a continuous function,* $\varepsilon \in [0,1[\,,\ \alpha \in]0,1]$, $\tau : [0,1] \to [0,1]$ *are continuous functions. Then problem* (6), (7) *has a unique solution.*

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