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EXTRAPOLATION IN GRAND BANACH FUNCTION SPACES AND APPLICATIONS

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Abstract. In this note extrapolation results in grand Banach function spaces $E^{p),\varphi(\cdot)}$ are presented. The boundedness of martingale transform in these spaces is also established. We deal with diagonal and off-diagonal cases. Banach function spaces are defined on quasi-metric measure spaces but the results are new even for domains in $\mathbb{R}^n$.

Let (X,d,μ) be a quasi-metric measure space with a quasi-metric d and measure μ. We will assume that μ is a finite measure, and it satisfies the doubling condition, i.e., there is a positive constant D_μ such that for all $x \in X$ and $r>0$, $\mu(B(x,2r)) \le D_\mu \mu(B(x,r))$, where $B(x,r) := \{y \in X : d(x,y) < r\}$ is the ball with center x and radius r. In this case we say that (X,d,μ) is a space of homogeneous type (SHT briefly). Throughout the paper we will assume that (X,d,μ) is an SHT.

Let $L^0(\mu) = L^0(X,\mu)$ be the space of (equivalence classes of) μ-measurable real-valued functions. A Banach space E is said to be a Banach function space (BFS briefly) on X if the following properties are satisfied (see [1]):

(i) $\|f\|_E = 0$ if and only if $f=0$ $\mu - a.e.$;

(ii) $|g| \le |f|$ $\mu - a.e.$ implies that $\|g\|_X \le \|f\|_X$;

(iii) if $0 \le f_j \uparrow f$ $\mu - a.e.$, the, $\|f_j\|_E \uparrow \|f\|_E$;

(iv) if $\chi_F \in L^0(\mu)$ is such that $\mu(F) < \infty$, then $\chi_F \in E$;

(v) if $\chi_F \in L^0(\mu)$ is such that $\mu(F)<\infty$, then $\int_F f d\mu \le C_F \|f\|_E$ for all $f \in E$ and with some positive constant C_F.

For a BFS E it is defined Köthe dual (or associated) space E' consists of all $f \in L^0(\mu)$

$$\|f\|_{E'} = \sup \left\{ \int\limits_X fg d\mu : \|g\|_E \le 1 \right\} < \infty.$$

It is known that the space E' is a Banach function space (see e.g., [1, Theorem 2.2]).

For instance, (weighted) Lebesgue, Lorentz, Orlicz, variable exponent Lebesgue spaces are examples of a BFS.

For a Banach space E and $0<p<\infty$, the p-convexification of E is defined as follows:

$$E^p = \{f : |f|^p \in E\}.$$

E^p can be equipped with the quasi-norm $\|f\|_{E^p} = \||f|^p\|_E^{1/p}$. It can be observed that if $1 \le p < \infty$, then E^p is a Banach space as well.

In the papers [14] and [26] the authors introduced grand Banach function space $E^{p),\varphi(\cdot)}$. In this space the norm is defined as follows:

$$\|f\|_{E^{p),\varphi(\cdot)}} := \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{E^{p-\varepsilon}},$$

where $1<p<\infty$ and φ is positive function on $(0,p-1)$ such that it is non-decreasing on $(0,\sigma)$ for some small positive σ, and moreover, $\lim_{t\to 0+} \varphi(x) = 0$. In this case we write that $\varphi \in \Phi_p$. If $\varphi(\varepsilon) \equiv \varepsilon^\theta$, where $\theta>0$, then we denote $E^{p),\varphi(\cdot)}$ by $E^{p),\theta}$.

The following properties hold for $E^{p),\varphi(\cdot)}$ (see [26] for $\varphi(t) \equiv t$, but the proof is the same for all $\varphi(\cdot) \in \Phi_p$):

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(i) $E^{p),\varphi(\cdot)}$ is a BFS;
(ii) $E^{p,\varphi(\cdot)}$ is contained in E;
(iii) for all $0<\varepsilon<p-1$, the following embeddings hold:

$$E^{p}\hookrightarrow E^{p),\varphi(\cdot)}\hookrightarrow E^{p-\varepsilon}.$$

(iv) if f belongs to the the closure of L^{∞} in $E^{p),\varphi(\cdot)}$ denoted by $E_b^{p),\varphi(\cdot)}$, then $\lim\limits_{\varepsilon\to 0}\varphi(\varepsilon)^{\frac{1}{p-\varepsilon}}E^{p-\varepsilon}=0$.

Small E^p space was characterized and studied in [26] (see [5] for the classical small Lebesgue spaces).

Let $1<p<\infty$. A weight function w defined on X belongs to the Muckenhoupt class $A_p(X)$ if

$$[w]_{A_p(X)}:=\sup_{B}\left(\frac{1}{\mu(B)}\int\limits_{B}w(x)\,d\mu(x)\right)\left(\frac{1}{\mu(B)}\int\limits_{B}w^{1-p'}(x)d\mu(x)\right)^{p-1}<\infty,$$

where the supremum is taken over all balls $B\subset X$.

Further, we say that $w\in A_1(X)$ if

$$(Mw)(x)\le Cw(x),\quad \text{for } \ \mu-\text{a.e. } x\in X, \tag{1}$$

where M is the Hardy–Littlewood maximal operator defined on X, i.e.,

$$Mg(x)=\sup_{B\ni x}\frac{1}{\mu(B)}\int\limits_{B}|g(y)|\,d\mu(y).$$

We denote by $[w]_{A_1}$ the best possible constant in (1).

We say that a BFS E belongs to $\mathbb{M}$ if the operator M is bounded in E.

It is known that (see, e.g., [6, 7, 27]) that important operators of Harmonic Analysis are bounded in weighted Lebesgue spaces under the Muckenhoupt condition on weights.

Let us now recall the basic concepts of the martingale transform. Let $(\Omega,\mathcal{F},\mathbb{P})$ be a complete probability space and let $\mathcal{M}_0$ be the class of all measurable functions on Ω. Suppose that $\mathcal{F}=(\mathcal{F})_{n\ge 0}$ is a filtration on $(\Omega,\mathcal{F},\mathbb{P})$, where $\mathcal{F}=(\mathcal{F})_{n\ge 0}$ is a non-decreasing sequence of sub-σ- algebras of Ω. Let $\mathcal{F}_{-1}=\mathcal{F}_0$. For any martingale $f=(f_n)_{n\ge 0}$ on Ω, we set $d_if=f_i-f_{i-1}$, $i>0$ and $d_0f=f_0$. Let $f=(f_n)_{n\ge 0}$ be a uniformly integrable martingale. We identify the martingale f with its pointwise limit f_∞, where the existence of the limit is guaranteed by the uniform integrability. For any integrable function f, the martingale generated by f is given by $f_n=E_nf$, where E_n is the expectation operator associated with $\mathcal{F}_n$, $n\ge 0$.

The maximal function and the truncated maximal function of the martingale f is defined by

$$\mathcal{M}f=\sup_{i\ge 0}|f_i| \quad\text{and}\quad \mathcal{M}_nf=\sup_{0\le i\le n}|f_i|,\ \ n\ge 0,$$

respectively.

For any predictable sequence $v=(v_n)_{n\ge 0}$ and martingale f, the martingale transform T_v is defined as

$$(T_vf)_n=\sum_{k=1}^{n}v_kd_kf,\quad (T_vf)_0=0.$$

Moreover, whenever $\|v\|_{L^\infty}:=\sup_{n\ge 0}\|v_n\|_{L^\infty}<\infty$, for any $f\in E$, the martingale transform $T_vf=(T_vf))_{n\ge 0}$ converges a.e. on Ω. Thus, we are allowed to identify the martingale transform T_vf with its pointwise limit $(T_vf)_\infty$.

We will assume that every $\sigma-$ algebra $\mathcal{F}_n$ is regular and generated by finitely or countably many atoms, where $B\in\mathcal{F}_n$ is called an atom it it satisfies the nested property. That is, any $A\subseteq B$ with $A\in\mathcal{F}_n$ satisfying $P(A)=P(B)$ *or* $P(A)=0$.

Denote the set of atoms by $\mathcal{A}=\cup_{n\ge 0}\mathcal{A}(\mathcal{F})_{n\ge 0}$ satisfy the above condition, we say that $\mathcal{F}$ is generated by atoms.

The well-known result on the convergence of the martingale transform states that whenever f is a bounded L^1 martingale, then $T_vf=((T_vf)_n)_{n\ge 0}$ converges almost everywhere on $\{x\in\Omega:\mathcal{M}v(x)<\ \infty\}$.

Let $\mathcal{F}$ is generated by atoms. Then for any measurable function f,

$$\mathcal{M}f(x) = \sup_{A \ni x} \frac{1}{\mathbb{P}(A)} \int\limits_A |f| d\mathbb{P},$$

where the supremum is taken over all $A \in \mathcal{A}$ containing x.

Let us recall the definition of the Muckenhoupt weight functions on probability space. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with $\mathcal{F}$ generated by atoms. We say that a.e. positive integrable on Ω function w (weight) elongs to the Muckenhoupt class A_p if

$$[w]_{A_p(\Omega)} := \sup_{A \in \mathcal{A}} \left(\frac{1}{\mathbb{P}(A)} \int\limits_B w \, d\mathbb{P} \right) \left(\frac{1}{\mathbb{P}(A)} \int\limits_B w^{1-p'} d\mathbb{P} \right)^{p-1} < \infty, \quad p' := \frac{p}{p-1}.$$

Further, we say that a weight w belongs to the class $A_1(\Omega)$ if there is a positive constant C such that

$$\frac{1}{\mathbb{P}(A)} \int\limits_A w d\mathbb{P} \leq C w, \quad \text{a.e. on } A.$$

The best possible constant C in the previous inequality is called A_1- characteristic of w and is denoted, as before, by $[w]_{A_1(\Omega)}$.

Let E be a BFS on $(\Omega, \mathcal{F}, \mathbb{P})$. When we deal with martingale transform we are also interested in grand weak BFS, denoted by $E_w^{p),\varphi(\cdot)}$, $p > 1$, and defined with respect to the norm

$$\|f\|_{E^{p),\varphi(\cdot)}} = \sup_{0<\varepsilon<p-1} \varphi(\varepsilon)^{\frac{1}{p-\varepsilon}} \|f\|_{E_w^{p-\varepsilon}},$$

where E_w is a weak BFS defined by

$$E_w = \left\{ f : \Omega \to \mathbb{R} : \|f\|_{E_w} = \sup_{\lambda > 0} \|\chi_{x \in \Omega : |f(x)| > \lambda}\|_E < \infty \right\}.$$

Grand weak Lebesgue spaces were introduced in [16] (see also [21, p. 743]).

Finally, we write that a BFS E defined on $(\Omega, \mathcal{F}, \mathbb{P})$ belongs to $\mathbb{M}$ if the operator $\mathcal{M}$ is bounded in E.

1. Main Results

1.1. **Extrapolation Statements.** In the papers [10, 14, 20, 24] Rubio de Francía's extrapolation results were studied in general BFSs. In [3] the same problem was studied in rearrangement-invariant Banach function spaces. We refer also to [18–20] for extrapolation results in grand Lebesgue and Lorentz spaces with constant exponents (see [2] and [17] for extrapolation in variable exponent and grand variable exponent Lebesgue spaces, respectively).

Our main results read as follows:

Theorem 1 (Diagonal Case)**.** *Let $\mathcal{E}$ be a family of pairs (f, g) of measurable non-negative functions f, g defined on X. Suppose for some $1 \leq p_0 < \infty$, for every $w \in A_{p_0}(X)$ and all $(f, g) \in \mathcal{E}$, the one-weight inequality holds*

$$\left(\int\limits_X g^{p_0}(x) w(x) \, d\mu(x) \right)^{\frac{1}{p_0}} \leq C N\big([w]_{A_{p_0}}\big) \left(\int\limits_X f^{p_0}(x) w(x) \, d\mu(x) \right)^{\frac{1}{p_0}},$$

where C and $N(\big([w]_{A_{p_0}}\big)$ are positive constants such that C is independent of (f, g) and w, and $N(\big([w]_{A_{p_0}}\big)$ is independent of (f, g) and depends on $[w]_{A_{p_0}}$ so that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Let E be a BFS and let there exist $1 < q_0 < \infty$ such that E^{1/q_0} is again a BFS.

Then for any $p > 1$, $\varphi(\cdot) \in \Phi_p$, there exists a positive constant C such that for all $(f, g) \in \mathcal{E}$, the inequality

$$\|g\|_{E^{p),\varphi(\cdot)}} \leq C \|f\|_{E^{p),\varphi(\cdot)}}, \quad (f, g) \in \mathcal{E},$$

holds provided that $(E^{(p-\varepsilon)/q_0})' \in \mathbb{M}$, $\varepsilon \in (0, \sigma)$, and that $\sup_{0<\varepsilon<\sigma} \|M\|_{(E^{(p-\varepsilon)/q_0})'} < \infty$, where σ is some small positive constant.

Theorem 2 (Off-diagonal Case)**.** *Let $\mathcal{E}$ be a family of pairs (f,g) of measurable non-negative functions f,g on X. Suppose that for some $1 \le p_0, q_0 < \infty$ and for every $w \in A_{1+q_0/(p_0)'}(X)$ and $(f,g) \in \mathcal{E}$, the one-weight inequality holds*

$$\bigg(\int\limits_X g^{q_0}(x)w(x)\,d\mu(x)\bigg)^{\frac{1}{q_0}} \le CN\big([w]_{A_{1+\frac{q_0}{(p_0)'}}}(X)\big)\bigg(\int\limits_X f^{p_0}(x)w^{\frac{p_0}{q_0}}(x)\,d\mu(x)\bigg)^{\frac{1}{p_0}},$$

where C and $N\big([w]_{A_{1+\frac{q_0}{(p_0)'}}}(X)\big)$ are positive constants such that C is independent of (f,g) and w, and $N\big([w]_{A_{1+\frac{q_0}{(p_0)'}}}\big)$ is independent of (f,g) and depends on $[w]_{A_{1+\frac{q_0}{(p_0)'}}}(X)$ so that the mapping $\cdot \mapsto N(\cdot)$ is non-decreasing. Assume that E and $\overline{E}$ are BFSs such that there exist $1 < \widetilde{p}_0 < \infty$, $1 < \widetilde{q}_0 < \infty$ satisfying the conditions

$$\frac{1}{\widetilde{p}_0} - \frac{1}{\widetilde{q}_0} = \frac{1}{p_0} - \frac{1}{q_0},$$
$$\overline{E}^{1/\widetilde{q}_0}, \quad E^{1/\widetilde{p_0}} \quad \text{are } BFSs.$$

Then for every pair (p,q), $1 < p < q < \infty$, satisfying the condition: for every sufficiently small $\eta > 0$, there is $\varepsilon > 0$ such that $\frac{1}{p-\varepsilon} - \frac{1}{q-\eta} = \frac{1}{p_0} - \frac{1}{q_0}$ and

$$\big(\overline{E}^{(q-\eta)/\widetilde{q}_0}\big)' = \Big[\big(E^{(p-\varepsilon)/\widetilde{p}_0}\big)'\Big]^{\widetilde{p}_0/\widetilde{q}_0},$$

and for every $\theta > 0$, there exists a positive constant C such that for all $(f,g) \in \mathcal{E}$ the inequality

$$\|g\|_{\overline{E}^{q),q\theta/p}} \le C\|f\|_{E^{p),\theta}}$$

holds provided that $\big(\overline{E}^{(q-\eta)/\widetilde{q}_0}\big)' \in \mathbb{M}$, $\eta \in (0,\delta)$, and that $\sup_{0<\eta<\delta}\|M\|_{(\overline{E}^{(q-\eta)/\widetilde{q}_0})'} < \infty$ for some small positive constant δ.

Remark. Taking $g = Tf$ in Theorems 1, 2, as a particular case, we can formulate appropriate extrapolation statements for T, where T is one of the operators of Harmonic analysis such that it is bounded in $L^{p_0}_w(X)$ for $p_0 > 1$ and and all $w \in A_{p_0}(X)$. Such operators are, for example, Hardy–Littlewood maximal and Calderón–Zygmund singular integral operators, commutators of singular integrals, Riesz potential operators and their commutators, etc.

2. Martingale Transform

For the martingale transform we have the following statements.

Theorem 3. *Let E be a BFS on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$. Let $\mathcal{F}$ be generated by atoms and predictable sequence $v = (v_n)_{n\ge 0}$ satisfies the condition $\|v\|_{L^\infty} < \infty$.*

(i) Suppose that there is a constant $p_0 > 1$ such that E^{1/p_0} is again a BFS. Then for $p \in (1,\infty)$, there is a positive constant C such that for all $f \in E^{p),\varphi(\cdot)}$,

$$\|T_v f\|_{E^{p),\varphi(\cdot)}} \le C\|f\|_{E^{p),\varphi(\cdot)}}$$

holds, provided that there is a positive constant $\sigma \in (0,p-1)$ such that for all $\varepsilon \in (0,\sigma)$, $(E^{(p-\varepsilon)/p_0})'$ $\in$ $\mathbb{M}$ and moreover, $\sup_{0<\varepsilon<\sigma}\|\mathcal{M}\|_{(E^{(p-\varepsilon)/p_0})'} < \infty$.

(ii) Then for every $p \in (1,\infty)$, there is a positive constant C such that for all $f \in E^{p),\varphi(\cdot)}$,

$$\|T_v f\|_{E^{p),\varphi(\cdot)}_w} \le C\|f\|_{E^{p),\varphi(\cdot)}},$$

holds, provided that there is a positive constant $\sigma \in (0,p-1)$ such that for all $\varepsilon \in (0,\sigma)$, $(E^{p-\varepsilon})' \in \mathbb{M}$ and moreover, $\sup_{0<\varepsilon<\sigma}\|\mathcal{M}\|_{(E^{p-\varepsilon})'} < \infty$.

Finally we mention that the boundedness of the martingale transform in BFSs defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ was established in [11].

References

1. C. Bennett, R. Sharpley, *Interpolation of Operators*. Pure and Applied Mathematics, 129. Academic Press, Inc., Boston, MA, 1988.
2. D. Cruz-Uribe, A. Fiorenza, J. M. Martell, C. Pérez, The boundedness of classical operators on variable L^p spaces. *Ann. Acad. Sci. Fenn. Math.* **31** (2006), no. 1, 239–264.
3. G. P. Curbera, J. García-Cuerva, J. M. Martell, C. Pérez, Extrapolation with weights, rearrangement-invariant function spaces, modular inequalities and applications to singular integrals. *Adv. Math.* **203** (2006), no. 1, 256–318.
4. J. Duoandikoetxea, Extrapolation of weights revisited: new proofs and sharp bounds. *J. Funct. Anal.* **260** (2011), no. 6, 1886–1901.
5. A. Fiorenza, Duality and reflexivity in grand Lebesgue spaces. *Collect. Math.* **51** (2000), no. 2, 131–148.
6. J. García–Cuerva, J. L. Rubio de Francia, *Weighted Norm Inequalities and Related Topics*. North-Holland Mathematics Studies, 116. North-Holland Publishing Co., Amsterdam, 1985.
7. L. Grafakos, *Classical Fourier Analysis*. Third edition. Graduate Texts in Mathematics, 249. Springer, New York, 2014.
8. L. Greco, T. Iwaniec, C. Sbordone, Inverting the p-harmonic operator. *Manuscripta Math.* **92** (1997), no. 2, 249–258.
9. E. Harboure, R. Macías, C. Segovia, Extrapolation results for classes of weights. *Amer. J. Math.* **110** (1988), no. 3, 383–397.
10. K. -P. Ho, Extrapolation to mixed norm spaces and applications. *Acta Comment. Univ. Tartu. Math.* **25** (2021), no. 2, 281–296.
11. K. -P. Ho, Martingale transforms on Banach function spaces. *Electron. Res. Arch.* **30** (2022), no. 6, 2247–2262. `DOI:10.3934/era.2022114`.
12. T. Iwaniec, C. Sbordone, On the integrability of the Jacobian under minimal hypotheses. *Arch. Rational Mech. Anal.* **119** (1992), no. 2, 129–143.
13. M. Izumisawa, N. Kazamaki, Weighted norm inequalities for martingales. *Tohoku Math. J. (2)* **29** (1977), no. 1, 115–124. `https://doi.org/10.2748/tmj/1178240700`.
14. V. Kokilashvili, M. Mastyło, A. Meskhi, Calderón–Zygmund singular operators in extrapolation spaces. *J. Funct. Anal.* **279** (2020), no. 10, 108735, 21 pp.
15. V. Kokilashvili, A. Meskhi, A note on the boundedness of the Hilbert transform in weighted grand Lebesgue spaces. *Georgian Math. J.* **16** (2009), no. 3, 547–551.
16. V. Kokilashvili, A. Meskhi, One-weight weak-type estimates for fractional and singular integrals in grand Lebesgue spaces. *Banach Center Publications.* **102** (2014), 131–142.
17. V. Kokilashvili, A. Meskhi, Maximal and Calderón-Zygmund operators in grand variable exponent Lebesgue spaces. *Georgian Math. J.* **21** (2014), no. 4, 447–461.
18. V. Kokilashvili, A. Meskhi, Weighted extrapolation in Iwaniec-Sbordone spaces. Applications to integral operators and approximation theory. (Russian) *translated from Tr. Mat. Inst. Steklova* **293** (2016). Funktsional. Prostranstva, Teor. Priblizh., Smezhnye Razdely Mat. Anal., 167–192; *Proc. Steklov Inst. Math.* **293** (2016), no. 1, 161–185.
19. V. Kokilashvili, A. Meskhi, Extrapolation in grand Lebesgue spaces with A_∞ weights. (Russian) *English version published in Math. Notes* **104** (2018), no. 3–4, 518–529. *Mat. Zametki* **104** (2018), no. 4, 539–551.
20. V. Kokilashvili, A. Meskhi, Extrapolation in weighted classical and grand Lorentz spaces. Application to the boundedness of integral operators. *Banach J. Math. Anal.* **14** (2020), no. 3, 1111–1142. `DOI:10.1007/s43037-020-00054-1`.
21. V. Kokilashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral Operators in Non-standard Function Spaces*. vol. 2. Variable exponent Hölder, Morrey-Campanato and grand spaces. Operator Theory: Advances and Applications, 249. Birkhäuser/Springer, 2016.
22. R. A. Macías, C. Segovia, Lipschitz functions on spaces of homogeneous type. *Adv. in Math.* **33** (1979), no. 3, 257–270.
23. A. Meskhi, Criteria for the boundedness of potential operators in grand Lebesgue spaces. *Proc. A. Razmadze Math. Inst.* **169** (2015), 119–132.
24. Z. Nieraeth, Extrapolation in general quasi-Banach function spaces. *J. Funct. Anal.* **285** (2023), no. 10, Paper no. 110130, 109 pp.
25. J. L. Rubio, de Francia, Factorization and extrapolation of weights. *Bull. Amer. Math. Soc. (N.S.)* **7** (1982), no. 2, 393–395.
26. M. Singh, Grand and small X^p spaces and general duality. *Positivity*, **25** (2021), no. 4, 1469–1488. `https://doi.org/10.1007/s11117-021-00819-4`.
27. J. O. Strömberg, A. Torchinsky, *Weighted Hardy Spaces*. Lecture Notes in Mathematics, 1381. Springer-Verlag, Berlin, 1989.

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