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ON UNIFORM SETS IN EUCLIDEAN SPACE $\mathbf{R}^m$ AND G-VOLUMES

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Abstract. For the class of invariant volumes on the Euclidean space $\mathbf{R}^m$, the notions of absolutely negligible sets and absolutely nonmeasurable sets in $\mathbf{R}^m$ are considered. Some properties of these sets are applied to maximal (by inclusion) invariant volumes on the same space.

The main purpose of this short communication is to examine certain properties of uniform subsets of the Euclidean space $\mathbf{R}^m$ in the context of maximal (by inclusion) invariant volumes on this space.

Let G denote some group of isometries of the space $\mathbf{R}^m$ and let $\mathcal{S}$ be a G-invariant ring of bounded subsets of $\mathbf{R}^m$ such that the unit cube $[0,1]^m$ belongs to $\mathcal{S}$.

By definition, a G-invariant volume on $\mathcal{S}$ is any functional $v : \mathcal{S} \to \mathbf{R}$ satisfying the following four conditions:

(a) $v(X) \geq 0$ for each set $X \in \mathcal{S}$;
(b) $v(X \cup Y) = v(X) + v(Y)$ for any two disjoint sets X and Y from $\mathcal{S}$;
(c) $v([0,1])^m = 1$;
(d) $v(g(X)) = v(X)$ for every $X \in \mathcal{S}$ and for every $g \in G$.

If G is large enough (e.g., if G contains the group T_m of all translations of $\mathbf{R}^m$), then $\mathcal{S}$ contains the family of all so-called elementary figures in $\mathbf{R}^m$ and the completion of any G-volume on $\mathcal{S}$ extends the classical Jordan volume (cf. [4]). One of such extensions is the m-dimensional Lebesgue measure λ_m on $\mathbf{R}^m$. At the same time, there are many volume-type functionals which properly extend the Jordan volume and essentially differ from λ_m.

Example 1. Let K be a bounded nowhere dense closed subset of $\mathbf{R}^m$ of strictly positive m-dimensional Lebesgue measure and let Is_m denote the group of all isometries of $\mathbf{R}^m$. It is known that there exists an Is_m-volume v on $\mathbf{R}^m$ which extends the Jordan volume and satisfies the equality $v(K) = 0$. So, $v \neq \lambda_m$.

Every G-volume can be extended to a maximal (by the standard inclusion relation) G-volume. This fact is a trivial corollary of the Kuratowski–Zorn lemma.

Example 2. If G is a solvable group of isometries of the space $\mathbf{R}^m$, then the domain of any maximal G-volume on $\mathbf{R}^m$ coincides with the family of all bounded subsets of $\mathbf{R}^m$. This is a consequence of the famous Banach theorem (see, e.g., [6,12]). Since for $m = 1, 2$ the group Is_m is solvable, there are universal Is_1-volumes and Is_2-volumes on $\mathbf{R} = \mathbf{R}^1$ and on $\mathbf{R}^2$, respectively. Also, since the group T_m is commutative for any natural number m, there are universal T_m-volumes on $\mathbf{R}^m$.

In connection with the above results, there naturally arises (see, for instance, [5]) the following

Problem. Give a characterization of all maximal G-volumes in the case when G is not a solvable group of isometries of $\mathbf{R}^m$.

As far as we know, this problem remains still unsolved.

It is not difficult to show that no maximal G-volume v can be countably additive (as a functional defined on the ring of v-measurable sets). A more delicate argument is needed for establishing the fact that in the case of a paradoxical group G, the domain of such a v is not a countably additive family of sets. In other words, if a G-volume v is maximal, then there always exists a countable family $\{X_i : i \in I\} \subset \mathrm{dom}(v)$ such that the set $\cup\{X_i : i \in I\}$ is bounded, but does not belong to $\mathrm{dom}(v)$.

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To demonstrate this circumstance, several auxiliary notions should be introduced and some statements connected with those notions should be proved.

A subset X of $\mathbf{R}^m$ will be called G-absolutely negligible (with respect to the class of all G-volumes) if for every G-volume v on $\mathbf{R}^m$, there exists a G-volume v' on $\mathbf{R}^m$ extending v and such that $X \in \mathrm{dom}(v')$ and $v'(X) = 0$ (cf. [6], where the analogous definition is introduced for the class of all G-invariant measures extending λ_m).

It is not hard to verify that the family of all G-absolutely negligible subsets of $\mathbf{R}^m$ is a G-invariant ideal denoted by $\mathcal{J}_G$. For any set $X \in \mathcal{J}_G$ and for any G-volume v on $\mathbf{R}^m$, the inner v-volume of X is equal to zero. This observation enables one to infer the following result.

Theorem 1. *If v is an arbitrary G-volume on $\mathbf{R}^m$, then there exists a G-volume v' on $\mathbf{R}^m$ which extends v and satisfies the inclusion $\mathcal{J}_G \subset \mathrm{dom}(v')$.*

It would be interesting to get a characterization of all G-absolutely negligible subsets of $\mathbf{R}^m$ in terms of G. Below, we describe a certain subfamily of $\mathcal{J}_G$.

Let e be a nonzero vector in the space $\mathbf{R}^m$.

A subset Z of $\mathbf{R}^m$ will be called uniform in direction e if for any straight line $l \subset \mathbf{R}^m$, parallel to e, the set $l \cap Z$ is either empty or singleton (cf. [10, 11]).

Example 3. Using the method of transfinite induction, it can be proved that if $m \geq 2$, then there exists a uniform in direction e subset Z of $\mathbf{R}^m$ such that for any straight line $l \subset \mathbf{R}^m$, not parallel to e, the intersection of Z and l is of cardinality continuum (denoted by $\mathbf{c}$). Moreover, such a Z may be a λ_m-thick subset of $\mathbf{R}^m$.

Example 4. Let e_1 and e_2 be any two nonzero non-collinear vectors in $\mathbf{R}^2$. There exist two sets Z_1 and Z_2 in $\mathbf{R}^2$ such that:

(1) Z_1 is uniform in direction e_1;
(2) Z_2 is uniform in direction e_2;
(3) for every straight line $l \subset \mathbf{R}^2$, the equality

$$\mathrm{card}(l \cap (Z_1 \cup Z_2)) = \mathbf{c}$$

holds true.

It follows from (3) that the union $Z_1 \cup Z_2$ contains some Mazurkiewicz set (for the definition of a Mazurkiewicz set and transfinite construction of it see, e.g., [3, 11]; cf. also [8]).

A bounded subset A of $\mathbf{R}^m$ will be called admissible if there exist a natural number $n > 0$ and nonzero vectors $e_1, e_2, \dots, e_n$ in $\mathbf{R}^m$ such that

$$A = \cup\{A_i : i = 1, 2, \dots, n\},$$

where each set A_i is uniform in direction e_i.

Observe that the family of all admissible sets in $\mathbf{R}^m$ is an Is_m-invariant ideal of subsets of $\mathbf{R}^m$.

Example 5. Let A be any admissible subset of $\mathbf{R}^m$ and let v be any T_m-volume on $\mathbf{R}^m$ such that $A \in \mathrm{dom}(v)$. Then, using the Banach theorem mentioned above, it can be shown that $v(A) = 0$.

More precisely, we have the following statement.

Theorem 2. *Let G be a group of isometries of the space $\mathbf{R}^m$ containing the group T_m.*

Then every admissible subset of $\mathbf{R}^m$ is G-absolutely negligible.

Theorems 1 and 2 imply the next statement.

Theorem 3. *Let G be a group of isometries of $\mathbf{R}^m$ containing T_m and let v be an arbitrary G-volume on $\mathbf{R}^m$.*

Then there exists a G-volume v' on $\mathbf{R}^m$ extending v and such that all admissible subsets of $\mathbf{R}^m$ belong to $\mathrm{dom}(v')$.

Consequently, if v is a maximal (by the inclusion relation) G-volume on $\mathbf{R}^m$, then all admissible subsets of $\mathbf{R}^m$ belong to $\mathrm{dom}(v)$.

The next deep statement was obtained by R. O. Davies (see [1, 2]; cf. also [10], where a slightly stronger result is obtained by using the Continuum Hypothesis).

Theorem 4. *If $m \geq 2$ is a natural number, then there exists a disjoint countable family $\{X_i : i \in I\}$ of uniform subsets of $\mathbf{R}^m$ such that $\cup\{X_i : i \in I\}$ coincides with the entire $\mathbf{R}^m$.*

A subset Y of the space $\mathbf{R}^m$ will be called G-absolutely nonmeasurable (with respect to the class of all G-volumes) if for every G-volume v on $\mathbf{R}^m$, the relation $Y \notin \mathrm{dom}(v)$ is valid.

Notice that if $m = 1, 2$, then there are no G-absolutely nonmeasurable sets in $\mathbf{R}^m$ (see Example 2).

Also, the well-known results of Hausdorff and von Neumann (see, e.g., [6, 9, 12]) imply the following statement.

Theorem 5. *Let G be a group of isometries of the space $\mathbf{R}^m$ containing two independent (in the group-theoretical sense) rotations of $\mathbf{R}^m$ about its origin.*

Then there exists a bounded G-absolutely nonmeasurable subset of $\mathbf{R}^m$.

Taking into account the facts presented above, one readily obtains the following result.

Theorem 6. *Let G be a group of isometries of the space $\mathbf{R}^m$ having these two properties:*

(1) *G contains the group T_m;*

(2) *there are two independent rotations of $\mathbf{R}^m$ about the origin of $\mathbf{R}^m$.*

Let X be a bounded G-absolutely nonmeasurable subset of $\mathbf{R}^m$ (the existence of which is guaranteed by Theorem 5).

Then for any maximal (by inclusion) G-volume v on $\mathbf{R}^m$, there exists a countable family $\{X_i : i \in I\}$ of sets of v-volume zero such that the union of all X_i $(i \in I)$ coincides with X. So, the relation

$$\cup\{X_i : i \in I\} \notin \mathrm{dom}(v)$$

holds true.

Remark 1. Notice that the analogue of Theorem 6 is valid for nonzero σ-finite countably additive measures on $\mathbf{R}^m$ even in the case of an uncountable solvable group G of isometries of $\mathbf{R}^m$ (cf. [6, 7]). Recall that such G does not admit any paradoxical decompositions.

Remark 2. Although the main result of papers [1] and [2] by Davies may be considered as a paradoxical decomposition of the space $\mathbf{R}^m$ for $m \geq 2$, these works are not cited in the Bibliography of the widely known monograph [12].

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