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ON ONE PROBLEM OF THE PLANE THEORY OF VISCOELASTICITY FOR A CIRCULAR PLATE WITH POLYGONAL HOLE

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The problem of the plane theory of viscoelasticity for a circular plate with a polygonal hole is considered according to the Kelvin–Voigt model. The external boundary of the plate is assumed to be subjected to the normal contractive force (pressure), and a rigid smooth washer of a somewhat larger size is embedded into the hole in such a way that normal displacements of the boundary points take constant values, in the absence of friction.

Using the methods of conformal mappings and boundary value problems of analytic functions, the unknown complex potentials are constructed efficiently (in an analytic form). The estimates of these potentials in the vicinity of angular points are given.

Introduction

As is known (see [5, 6]), the methods of conformal mappings and boundary value problems of analytic functions are successfully applied to simply connected domains mapped onto a circle by rational functions, but they are less applicable to multiconnected (including doubly-connected) domains. Formulas analogous to those of Christophel–Schwartz for doubly-connected domains, obtained in [2], allow one to solve the mentioned problems efficiently (in an analytic form) in the case of doubly-connected domains and their modifications which may appear when passing to the limit. Of interest is the extension of these results to the problems of the plane theory of viscoelasticity.

The present paper considers one of such problems for a circular domain with a polygonal hole for a viscoelastic plate according to the Kelvin–Voigt model [1, 7].

Statement of the problem. Let a viscoelastic plate on the plane z of complex variable occupy a doubly-connected domain S which is bounded by a circumference $L_0 = \{|z| = R_0\}$ and a convex polygon (A) with vertices at the points A_j $(j = \overline{1,n})$. By L_1 we denote the polygonal boundary (i.e., $L_1 = \bigcup\limits_{k=1}^{n} L_1^{(k)}$, $L_1^{(k)} = A_kA_{k+1}$ $(k = \overline{1,n}, A_{n+1} = A_1)$) and by $\pi\alpha_j^0$ the inner with respect to S angles at vertices A_j. The angle lying between the Ox-axis and exterior normal to the contour L_1 at the point $\sigma \in L_1$ we denote by $\alpha(\sigma)$, i.e., $\alpha(\sigma) = \alpha_1^{(j)} = \text{const}$, $\sigma \in L_1^{(j)}$, $(j = \overline{1,n})$.

Assume that the boundary L_0 is under the action of uniformly distributed normal pressure P_0, and a rigid smooth disc of somewhat larger size is embedded into the polygon (A) so that on the segments of $L_1^{(j)}$ we have the values of normal displacement $v_n(\sigma) = v_1^{(j)} = \text{const}$ $(j = \overline{1,n})$ and the friction is absent. The given boundary conditions simplify the process of solving, but do not change the essence of the problem.

The aim of the present paper is to determine complex potentials characterizing stress and displacement distribution in the plate by the Kelvin–Voigt model.

Solution of the problem. Here, we present some results following from the works [5, 8] and [3].

1. Conformal mapping of the domain S onto a circular ring $D = \{1 < |\zeta| < R\}$ is realized by the function $z = \omega(\zeta)$ whose derivative is a solution of the Riemann–Hilbert boundary value problem for

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a circular ring D

$$\mathrm{Re}\,[i\xi e^{-i\alpha(\xi)}\omega'(\xi)]=0,\ \ \xi\in l_1;\qquad \mathrm{Re}\,[i\omega'(\xi)]=0,\ \ \xi\in l_0,$$

where l_1 and l_0 are the preimages of contours L_1 and L_0 under the mapping $z=\omega(\zeta)$, i.e., $l_1=\{|\zeta|=1\}$; $l_0=\{|\zeta|=R\}$, and for the condition $\prod\limits_{k=1}^{n}(a_k)^{a_k^0-1}=1$, it has the form

$$\omega'(\zeta)=K^0\prod_{k=1}^{n}\Big(1-\frac{a_k}{\zeta}\Big)^{a_k^0-1}\prod_{k=1}^{n}\prod_{j=1}^{\infty}\Big(1-\frac{\zeta}{R^{2j}a_k}\Big)^{a_k^0-1}\Big(1-\frac{a_k}{R^{2j}\zeta}\Big)^{a_k^0-1},\tag{1}$$

where K^0 is the real constant, $a_k=\omega^{-1}(A_k)$ $(k=\overline{1,n})$.

2. The boundary conditions of the first and second basic problems for a viscoelastic plate S have by the Kelvin–Voigt model the form

$$\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}=i\int_0^{\sigma}(X_n+iY_n)ds+c_1+ic_2,\tag{2}$$

$$\Gamma\varphi(\sigma,t)-M\Big[\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}\Big]=2\mu^*(u+iv),\ \ \sigma\in L=L_0\cup L_1,\tag{3}$$

where t denotes here and in the sequel the parameter of time; Γ and M are the operators of time t,

$$\begin{gathered}\Gamma\varphi(\sigma,t)=\int_0^t\big[\varkappa^*e^{k(\tau-t)}+2e^{m(\tau-t)}\big]\varphi(\sigma,\tau)d\tau,\\ M\Big[\varphi(\sigma,t)+\sigma\overline{\varphi'(\sigma,t)}+\overline{\psi(\sigma,t)}\Big]=\int_0^t e^{m(\tau-t)}\Big[\varphi(\sigma,\tau)+\sigma\overline{\varphi'(\sigma,\tau)}+\overline{\psi(\sigma,\tau)}\Big]d\tau,\ \ \sigma\in L.\end{gathered}\tag{4}$$

Since the given domain is doubly-connected, it is advisable to use the functions $\Phi(z,t)=\varphi'(z,t)$ and $\Psi(z,t)=\psi'(z,t)$ which are single-valued in the case of a multiconnected domain, as well.

Taking into account that

$$X_n+iY_n=(N+iT)e^{iv(\sigma)}=-i(N+iT)\frac{d\sigma}{ds},$$

from (2), by differentiation with respect to σ, we obtain [5]

$$\Phi(\sigma,t)+\overline{\Phi(\sigma,t)}+\sigma_s'^2\Big[\overline{\sigma}\Phi'(\sigma,t)+\Psi(\sigma,t)\Big]=N-iT,\quad \sigma\in L.\tag{5}$$

Analogously, in view of the equlities

$$u+iv=(v_n+iv_\tau)e^{i\alpha};\quad v_\tau=0,\ \ v_n=v_n^{(j)}=\text{const},\quad T(\sigma)=0,\ \ \sigma\in L_1^{(j)}\ \ (j=\overline{1,n});$$
$$v_n=v_n^{(0)}=\text{const},\ \ v_\tau=0,\ \ T(\sigma)=0,\ \ N(\sigma)=P_0;\ \ e^{i\alpha}=\frac{\sigma}{R_0},\ \ \sigma\in L_0,$$

from (3), by differentiation with respect to σ, we get

$$\Gamma[\Phi(\sigma,t)]-M[N-iT]=\begin{cases}2\mu^*v_n^{(0)}R_0^{-1}, & \sigma\in L_0\\ 0, & \sigma\in L_1.\end{cases}\tag{6}$$

Since $N(\sigma)=P_0$, $\sigma\in L_0$; $T(\sigma)=0$, $\sigma\in L$ from (6) follows the boundary value problem

$$\mathrm{Re}\,\Gamma[\Phi(\sigma,t)]=P(t),\ \ \sigma\in L_0;\qquad \mathrm{Im}\,\Gamma[\Phi(\sigma,t)]=0,\ \ \sigma\in L_1,\tag{7}$$

where

$$P(t)=P_0F(t)+2\mu^*R_0^{-1}v_n^{(0)};\qquad F(t)=\frac{1}{m}[1-e^{-mt}].$$

Mapping the domain S onto a circular ring D (see p. 1) and introducing the notation $\Phi[\omega(\zeta),t]=\Phi_0(\zeta,t)$, from (7) we obtain the Riemann–Hilbert boundary value problem for a circular ring $D=\{1<|\zeta|<R\}$,

$$\mathrm{Re}\,[\Omega(\eta,t)-P(t)]=0,\ \ \eta\in l_0;\qquad \mathrm{Im}\,[\Omega(\eta,t)-P(t)]=0,\ \ \eta\in l_1\tag{8}$$

where

$$\Omega(\zeta,t)=\Gamma\Phi_0(\zeta,t).$$

Since problem (8) has only a trivial solution, we will have $\Omega(\zeta,t)=P(t)$, $\zeta\in D$ and, consequently, to determine the function $\Phi_0(\zeta,t)$, we obtain the integral equation $\Gamma\Phi_0(\zeta,t)=P(t)$, or taking into account (4), we have

$$\int_0^t\big[\varkappa^* e^{k(\tau-t)}+2e^{m(\tau-t)}\big]\Phi_0(\zeta,\tau)d\tau=P(t). \tag{9}$$

Differentiating (9) with respect to t, and then summing the obtained equality with (9) multipled by m, we get

$$(m-k)\varkappa^*\int_0^t e^{k\tau}\Phi_0(\zeta,\tau)d\tau+(\varkappa^*+2)e^{kt}\Phi_0(\zeta,t)=(P_0+2\mu^* mR_0^{-1}v_n^0)e^{kt}. \tag{10}$$

From (10) by differentiation with respect to t we obtain the differential equation

$$\dot{\Phi}_0(\zeta,t)+a\Phi_0(\zeta,t)=b, \tag{11}$$

where

$$a=\frac{m\varkappa^*+2k}{\varkappa^*+2};\qquad b=\frac{k(P_0+2\mu^* mR_0^{-1}v_n^{(0)})}{\varkappa^*+2} \tag{12}$$

($\dot{\Phi}(\zeta,t)$ denotes the derivative with respect to t).

From (10) follows the initial condition

$$\Phi_0(\zeta,0)=\frac{b}{k}. \tag{13}$$

The solution of equation (11) for the initial condition (13) has the form

$$\Phi_0(\zeta,t)=b\Big[\frac{1}{a}+\Big(\frac{1}{k}-\frac{1}{a}\Big)e^{-at}\Big], \tag{14}$$

where a and b are defined by formula (12). Further, to determine the function $\Psi_0(\zeta,t)=\Psi[\omega(\zeta),t]$, we make use of equality (5) which after the conformal mapping and by passing to a complex-conjugate value can be written in the form (see [5])

$$\Phi_0(\eta,t)+\overline{\Phi_0(\eta,t)}-\frac{\eta^2}{\rho^2\overline{\omega'(\eta)}}\Big[\overline{\omega(\eta)}\Phi_0'(\eta,t)+\omega'(\eta)\Psi_0(\eta,t)\Big]=N-iT,\ \ \eta\in l=l_0\cup l_1, \tag{15}$$

$$(\rho=R,\ \ \eta\in l_0\ \text{ and }\ \rho=1,\ \ \eta\in l_1).$$

In view of (15), from (6), we obtain

$$\Gamma\overline{\Phi_0(\eta,t)}-M\Big\{\Phi_0(\eta,t)+\overline{\Phi_0(\eta,t)}-\frac{\eta^2}{\rho^2\overline{\omega'(\eta)}}\Big[\overline{\omega(\eta)}\Phi_0'(\eta,t)+\omega'(\eta)\Psi_0(\eta,t)\Big]\Big\}$$

$$=\begin{cases}2\mu^* R_0^{-1}v_n^{(0)}, & \eta\in l_0,\\ 0, & \eta\in l_1.\end{cases} \tag{16}$$

By virtue of (14) and (16), we arrive at the boundary value problem

$$\operatorname{Im}\left[\Omega_1(\eta,t)\right]=0,\qquad \eta\in l=l_0\cup l_1, \tag{17}$$

where

$$\Omega_1(\zeta,t)=M\big[\zeta^2\omega'^2(\zeta)\Psi_0(\zeta,t)\big]. \tag{18}$$

The solution of problem (17) has the form

$$\Omega_1(\zeta,t)=K_1(\zeta,t), \tag{19}$$

where $K_1(\zeta,t)$ is the real function.

It follows from (4), (14) and (16) that

$$\operatorname{Re}\,\Omega_1(\eta,t)=F(\eta,t),\qquad \eta\in l_0, \tag{20}$$

where

$$F(\eta,t)=R^2\big[-\Gamma^*[\Phi_0(\eta,t)]+2\mu R_0^{-1}v_n^{(0)}\big]\omega'^2(\eta),\qquad \eta\in l_0,$$

$$\Gamma^*[\Phi_0(\eta,t)]=\int\limits_0^t \varkappa^* e^{k(\tau-t)}\Phi_0(\eta,\tau)d\tau.$$

From (19) and (20), we obtain

$$K_1(\zeta,t)=F(\zeta,t), \tag{21}$$

and thus, on the basis of (4), (18) and (21), for the function $\Psi_0(\zeta,t)$, we get the equation

$$M[\Psi_0(\zeta,t)]=\frac{f(t)}{\zeta^2}, \tag{22}$$

where

$$f(t)=R^2\,\big[-\Gamma^*[\Phi_0(\eta,t)]+2\mu^* R_0^{-1}v_n^{(0)}\big]. \tag{23}$$

Relying on (4) and (22), it is not difficult to obtain

$$\Psi_0(\zeta,t)=\frac{1}{\zeta^2}\big[mf(t)+\dot f(t)\big].$$

Taking into account (14) and (23), performing the corresponding calculations, we have

$$\Psi_0(\zeta,t)=\frac{R^2}{\zeta^2}\Big[\frac{\varkappa^* b}{k}\Big(-\frac{m}{a}+\Big(\frac{m}{a}-1\Big)e^{-at}\Big)+2\mu^* mR_0^{-1}v_n^{(0)}\Big].$$

Introducing here the value a from formula (12), we finally get

$$\Psi_0(\zeta,t)=\frac{R^2}{\zeta^2}\Big[2m\mu^* R_0^{-1}v_n^{(0)}-\frac{b\varkappa^*}{k(m\varkappa^*+2k)}\big[m(\varkappa^*+2)-2(m-k)e^{-mt}\big]\Big].$$

Let us now investigate behaviour of the function $\varphi'(z,t)$ in the vicinity of angular points A_j $(j=\overline{1,n})$. Taking into account the fact that the conformally mapping function in the vicinity of the point A_j has the form (see [4])

$$\omega(\zeta)=A_j+(\zeta-a_j)^{\alpha_j^0}\Omega_j(\zeta),$$

where $\Omega_j(a_j)\neq 0$, and bearing in mind (1), for the function $\varphi'(z,t)=\frac{\Phi_0(\zeta,t)}{\omega'(\zeta)}$ in the vicinity of the point A (A is one of the points A_j), we obtain the estimate

$$|\varphi'(z,t)|<M(t)|z-A|^{\frac{1}{\alpha_0}-1}.$$

In particular, for the rectangle we have $|\varphi'(z,t)|<M(t)|z-A|^{-\frac{1}{3}}$ and for a rectangular cut $|\varphi'(z,t)|<M(t)|z-A|^{-\frac{1}{2}}$.

Note that in the statement of the problem there appears the value P_0, while in the expression of the function $\Phi_0(\zeta,t)$, there appear, as is seen from (12) and (14), both values P_0 and $v_n^{(0)}$. Consequently, we have to find the dependence between these values. If we suppose that at the initial moment we have on L_0 the value of P_0, then

$$X_x=Y_y=P_0,\quad X_x+Y_y=4\,\mathrm{Re}\,[\Phi(\sigma,t)],\quad \mathrm{Re}\,[\Phi(\sigma,t)]=\frac{P_0}{2},\ \ \sigma\in L_0,$$

and condition (13) due to (12) can be written in the form

$$\frac{P_0}{2}=\frac{P_0+2\mu^* m v_n^{(0)}R_0^{-1}}{\varkappa^*+2},$$

which implies that

$$v_n^{(0)}=\frac{\varkappa^* P_0 R_0}{4\mu^* m}.$$

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