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OVER-REFLECTION OF ACOUSTIC WAVES IN SHEAR FLOW

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Abstract. Linear dynamics of acoustic waves in a uniform shear flow is studied. It is shown that in the case of very low shear rate the dynamics of perturbations is adiabatic and can be fully described by the Liouville–Green asymptotic solutions. In contrast, in the flow with a moderate and high shear rate the dynamics of perturbations consists of additional phenomenon, acoustic wave over-reflection. Asymptotic analysis is performed and analytical expressions for the transmission and reflection coefficients are derived and analyzed.

1. Introduction

It has long been known that velocity shear induced by linear mechanisms play important role in various physical phenomena [2, 3, 9–11, 15]. Many physical aspects were analysed and interesting applications were proposed in the framework of these studies. However, the lack of quantitative analysis impedes future progress. In the presented paper, we study one of these linear phenomena, the over-reflection of acoustic waves in a uniform shear flow. Dynamics of these perturbations is described by the following equation [6–8]:

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2}+\omega^2(t)u=-\beta(t)I, \tag{1.1}$$

where $I=v-\beta(t)u-S\rho$ is the so-called potential vorticity which physically represents the amplitude of vortical perturbations. In equation (1.1), ρ is the dimensionless density perturbation normalized by background density, u and v are dimensionless perturbations of the parallel and perpendicular velocity components, respectively, S is the dimensionless shear parameter, t is the dimensionless time, $\omega^2(t)=1+\beta^2(t)$ and $\beta(t)=k_y/k_x-St$, where k_x and k_y are parallel and perpendicular wave numbers, respectively.

If u_1 and u_2 are any independent solutions of the homogenous counterpart of equation (1.1), the general solution can be written as:

$$u(t)=C_1u_1(t)+C_2u_2(t)+\frac{I}{W}\int\limits_{-\infty}^{t}\gamma(t_1)[u_1(t)u_2(t_1)-u_1(t_1)u_2(t)]dt_1, \tag{1.2}$$

where W is the Wronskian of the linear solutions. $C_{1,2}$, as well as I, are defined by the initial conditions of the problem.

Definition 1.1. The wave equation is called adiabatic if $\omega^2(t)$ is slowly varying function of time: $\mathrm{d}\omega(t)/\mathrm{d}t\ll\omega^2(t)$ [5, 14].

In the case under consideration, using the definition of $\omega^2(t)$, this condition takes the form

$$S|\beta(t)|\ll[1+\beta^2(t)]^{3/2}. \tag{1.3}$$

Definition 1.2. If the wave equation is adiabatic, its Liouville–Green asymptotic solution is defined as follows [5, 12, 14]:

$$\tilde{u}_{1,2}=\frac{1}{\sqrt{\omega(t)}}e^{\pm i\int\omega(t)dt}. \tag{1.4}$$

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Note that physically these solutions correspond to shear modified acoustic waves that have positive and negative phase velocities along the x-axis, respectively [6, 7]. If $S \ll 1$, condition (1.3) holds for arbitrary $\beta(t)$. In this case, the evolution of perturbations is adiabatic, i.e., the amplitudes $C_{1,2}$ remain constant. Comprehensive study of the adiabatic evolution of perturbations in the uniform shear flow has been performed by several authors [6, 8]. In the next section, we focus on the non-adiabatic evolution of perturbations, i.e., we study the dynamics of perturbations in the flow with relatively high shear parameter S.

2. Over-reflection of Acoustic Waves

In the case of relatively high shear rates the Liouville–Green condition (1.3) fails in the neighborhood of the point $\beta(t) = 0$, but it remains valid for $|\beta(t)| \gg \sqrt{S}$. The problem of asymptotic analysis can be formulated in the usual manner [5, 14]. Assume initially at $t = 0$, $\beta(0) \equiv k_y/k_x \gg \sqrt{S}$ and intensities of acoustic perturbations are $C_{1,2}$, respectively. The problem is to determine intensities of the same perturbations $D_{1,2}$ after passing through the area of non-adiabatic evolution for $\beta(t) \ll -\sqrt{S}$, i.e., for $t > 2k_y/k_x\sqrt{S}$. Assume that initially at $t = 0$, $\beta(0) = k_y/k_x \gg \sqrt{S}$, there exists only the wave with a positive phase velocity along the y-axis (without loss of generality, we also assume that both, k_x and k_y, are positive):

$$u_i = \frac{C_1}{\sqrt{\omega(t)}} e^{-i\int \omega(t)dt}. \tag{2.1}$$

In the general case, where $\beta(t) \ll -\sqrt{S}$, there exist both Liouville–Green solutions (1.4). According to the notations adopted in the theory of acoustic wave propagation in non-uniform flows [13], the Liouville–Green solution with positive phase velocity along the y-axis is treated as a reflected wave (u_r) and another solution is regarded as a transmitted wave (u_t):

$$u_r = \frac{D_1}{\sqrt{\omega(t)}} e^{-i\int \omega(t)dt}, \tag{2.2}$$

$$u_t = \frac{D_2}{\sqrt{\omega(t)}} e^{i\int \omega(t)dt}. \tag{2.3}$$

Definition 2.1. The reflection and transmission coefficients are defined as follows [5, 12, 14]:

$$R = \frac{|D_1|}{|C_1|}, \qquad T = \frac{|D_2|}{|C_1|}. \tag{2.4}$$

Normally, the non-adiabatic evolution changes not only Liouville–Green amplitudes, but also the phases:

$$D_1 = e^{i\phi_1} R C_1, \qquad D_2 = e^{i\phi_2} T C_1. \tag{2.5}$$

Combining equation (1.1) and its complex conjugate, it can be easily shown that if $\omega^2(t)$ is a real function of t, then an arbitrary solution u_x of equation (1.1) satisfies the condition

$$u^* \frac{\mathrm{d}u}{\mathrm{d}t} - u \frac{\mathrm{d}u^*}{\mathrm{d}t} = \text{constant}, \tag{2.6}$$

where asterisk denotes complex conjugation. It is well known [1] that this equation represents the conservation of wave action during the evolution. Substituting equations (2.1)–(2.5) into (2.6), one can obtain

$$R^2 - T^2 = 1. \tag{2.7}$$

So, the amplitude of a reflected wave is always larger than that of an incident wave. This means, that a non-adiabatic evolution of acoustic waves ($T \neq 0$) is always accompanied by the over-reflection phenomenon discovered by Miles [13], who studied the problem of acoustic wave reflection from the surface of tangential discontinuity of velocity. From the derivation of equation (2.7), it is clear that the same conclusion holds for a non-adiabatic evolution of arbitrary wave mode that is governed by equation (1.1) with an arbitrary real function $\omega^2(t)$.

Theorem 2.1. *For the acoustic waves in the uniform shear flow, for any value of the shear rate S, the changes in the reflection and transmission coefficients and phase are given by the following expressions:*

$$R = \sqrt{1 + e^{-\pi/S}}, \tag{2.8}$$

$$T = e^{-\pi/2S}, \tag{2.9}$$

$$\phi_1 = \frac{\ln(2S)}{2S} - \arg\Big[\Gamma\Big(\frac{1}{2} + \frac{i}{2S}\Big)\Big], \tag{2.10}$$

$$\phi_2 = \frac{\pi}{2}. \tag{2.11}$$

Proof. Exact formal analytical solutions of the homogenous part of equation (1.1) can be presented by the parabolic cylinder functions [14]

$$\bar{u}_1 = E\Big(-\frac{1}{2S}, \sqrt{\frac{2}{S}}\beta(t)\Big), \tag{2.12}$$

$$\bar{u}_2 = E^*\Big(-\frac{1}{2S}, \sqrt{\frac{2}{S}}\beta(t)\Big). \tag{2.13}$$

Taking into account the asymptotic expansion of parabolic cylinder function [1],

$$E(a,\eta) \approx \sqrt{\frac{2}{\eta}} \exp\Big[i\Big(\frac{\eta^2}{4} - a\ln\eta + \Phi + \frac{\pi}{4}\Big)\Big] \tag{2.14}$$

$$\text{for } \eta \gg |a|,, \tag{2.15}$$

where $\Phi = \arg\Gamma\left(\frac{1}{2} + ia\right)$, and using the connection formula [5]

$$iE(a,-\eta) = e^{\pi a}E(a,\eta) - \sqrt{1 + e^{2\pi a}}E^*(a,\eta), \tag{2.16}$$

we conclude that for $\beta(t) \gg S^{1/2}$, the exact solution (2.12) coincides with the incident wave (2.1), accurate to the constant multiplier. Whereas for $\beta(t) \ll -S^{1/2}$, exact solutions (2.12) and (2.13) agree with reflected (2.2) and transmitted (2.3) waves, respectively. Bearing this in mind and combining equations (2.1)–(2.5) and (2.14)–(2.16) one can readily derive equations (2.8)–(2.11). $\square$

Noteworthy is the fact that equations (2.8)–(2.11) represent exact asymptotic solution of the problem, i.e., they are valid for an arbitrary value of normalized shear parameter S. Calculating the phase integral [5, 14]

$$\delta = \frac{1}{S}\Big|\int_{-i}^{i} \sqrt{1+\beta^2}d\beta\Big| = \frac{\pi}{2S}, \tag{2.17}$$

and noting that $T_{ph} = e^{-\delta}$ in the framework of the method, we see that the method of phase integrals gives the same results for T and R as in equations (2.8) and (2.9). However, the validity of the method is bounded by the condition $S \ll 1$.

3. Discussion and Conclusions

In this section, we discuss mathematical corollaries and physical consequences of the obtained results.

Corollary. *If the incident wave has a negative phase velocity with respect to the x-axis,*

$$\bar{u}_i = \frac{C_2}{\sqrt{\omega(t)}} e^{i\int \omega(t)dt}, \tag{3.1}$$

then the expressions for reflection and transmission coefficients remain the same as in equations (2.8)–(2.9)*, whereas the expressions* (2.10)–(2.11) *for $\phi_{1,2}$ change the sign.*

Corollary. *It follows from equation* (2.9) *that if $S \ll 1$, then the intensity of transmitted wave is exponentially small with respect to the large parameter $1/S$. In its turn, this implies that if $S \ll 1$, then the adiabatic invariants $C_{1,2}$ have exponential* (*with respect to* $1/S$) *accuracy of conservation* [5,12,14].

Corollary. *It follows from equation* (2.8) *that the intensity of a transmitted wave never exceeds that of an incident wave* ($T < 1$)*, and therefore* $R < \sqrt{2}$.

The last point we discuss is an approximate physical estimation of the time scale of transmitted wave generation process. It follows from the condition of validity of asymptotic expansion (2.14) that the time scale Δt_b of the "birth" [4] of transmitted wave is of order

$$\Delta t_b \sim S^{-1/2}. \tag{3.2}$$

Note that this estimation is valid for arbitrary S. From equation (3.2), it follows that for relatively small shear rates, creation of the transmitted wave is a slow process compared to the wave period (which in our dimensionless notations is of order 1), and is a quick process compared to the adiabatic changes (having characteristic timescale $1/S$). We have to note that the estimation similar to (3.2) was first obtained by Berry [4] for a wide class of $\omega^2(t)$.

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