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ON THE AM-GM INEQUALITY AND THE GENERAL PROBLEMS OF MAXIMIZATION OF PRODUCTS

GEORGE CHELIDZE[1,2], MIKHEIL NIKOLEISHVILI[1,3] AND VAJA TARIELADZE[1]

Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. General problems on maximizing the products of several positive real, rational and integer numbers whose sum is given are investigated.

1. Introduction

The problem is formulated as follows:

Let $n > 1$ be a natural number, $L > 0$ and $x_i > 0$, $i = 1, \dots, n$; maximize $\prod_{i=1}^n x_i$ when $\sum_{i=1}^n x_i = L$.

A solution of this *extremum problem* is contained in the following version of the arithmetic mean–geometric mean inequality, for short, the *AM-GM inequality.*

Theorem 1.1. *Let $n > 1$ be a natural number, $L > 0$ and $x_i > 0$, $i = 1, \dots, n$ be real numbers with $\sum_{i=1}^n x_i = L$. Then*

$$\prod_{i=1}^n x_i \le \left(\frac{L}{n}\right)^n, \tag{1.1}$$

we get equality in (1.1) *if and only if*

$$x_i = \frac{L}{n}, \quad i = 1, \dots, n.$$

There are many different proofs of this statement; e.g., 12 proofs are presented in [1], while, according to [9], one can find 52 proofs in [2] and 74 proofs in [3]. In [13], it has been shown that *the first part* of Theorem 1.1 is equivalent to the following *Bernoulli inequality*:

$$x^n \ge 1 + n(x-1) \quad \text{for any} \quad x > 0 \ \text{ and } \ n \in \mathbb{N}.$$

It seems that the one of the shortest ways for a proof of the AM-GM inequality is its derivation from the following assertion.

Proposition 1.2. *Let $n > 1$ be a natural number. If the numbers*

$$x_i > 0, \quad i = 1, \dots, n$$

satisfy

$$\prod_{i=1}^n x_i = 1,$$

then

$$\sum_{i=1}^n x_i \ge n,$$

the above inequality turns into equality if and only if

$$x_i = 1, \quad i = 1, \dots, n.$$

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A simple inductive proof of Proposition 1.2 can be seen on pages 17-19 of nicely written work [12]; see also the proof in [1, §11, pp. 9-10], attributed to G. Ehlers (1954)). The same opinion is supported in [18]; in this connection, the names of Dörrie (1921; see [7]), H. Kreis (1946) and Korovkin (1952) (see [10, p. 7]) are mentioned in [3, p. 90].

Theorem 1.1, as well as Proposition 1.2 can be formulated for rational numbers, the proof of Proposition 1.2 presented in [3, p. 90] (but not the proof in [11]!) uses only the tools of arithmetic, however, a derivation of Theorem 1.1 from Proposition 1.2 is impossible without the use of irrational numbers. A really simple direct arithmetic inductive proof of Theorem 1.1 can be found in [19] (see also [6] and [14, p. 30]).

A direct arithmetic, *not inductive* proof of the following reformulation of Theorem 1.1, which belongs to Johan Frederik Steffensen (1930–1931; see [17]) and *does not use even the division*, is presented in [3, pp. 90-91].

Theorem 1.3. *Let $n > 1$ be a natural number, $x_i > 0$, $i = 1, \dots, n$. Then*

$$n^n \prod_{i=1}^n x_i \leq \left(\sum_{i=1}^n x_i \right)^n$$

and we have the equality if and only if

$$x_i = x_1, \quad i = 1, \dots, n.$$

In his book [8] (not mentioned in [3]), Heinrich Dörrie gave a direct, not inductive, arithmetic proof of Theorem 1.1. In Section 2, we give two similar proofs of Theorem 2.3 (a reformulation of Theorem 1.1), *one of which is based on the idea of* [8, Section 10].

Seemingly, the second author of the present text is the first who considers the above-mentioned maximization problem for natural numbers; he observed that for them the bound $\left(\frac{L}{n}\right)^n$ was not the best possible and he found the correct bound (see [5]). In Section 3, the result of [5] is included and it is shown that, in general, *the sharper bound* $\left[\left(\frac{L}{n}\right)^n\right]$ *may not likewise be the best possible* (see Proposition 3.1).

The given paper is dedicated to the ways of solution of the following *general extremum problem* (see Subsection 2.1).

Let $n > 1$ be a natural number, $\mathbb{X}$ be an infinite set of non-negative real numbers, $L \in \mathbb{X}$, $L > 0$ and $x_i, k_i, s_i \in \mathbb{X}$, $i = 1, \dots, n$; maximize

$$\prod_{i=1}^n (x_i + s_i)$$

when $\sum_{i=1}^n x_i = L$ and $x_i \geq k_i$, $i = 1, \dots, n$.

In Section 2, we treat the cases when $\mathbb{X} = \mathbb{R}_+$ or $\mathbb{X} = \mathbb{Q}_+$ (see Theorem 2.5) and Section 3 deals with the case $\mathbb{X} = \mathbb{Z}_+$ (see Theorem 3.3).

2. Problem for the Case $\mathbb{X} \in \{\mathbb{R}_+, \mathbb{Q}_+\}$

2.1. **Formulation of the problem.** In this subsection, $\mathbb{X} \in \{\mathbb{R}_+, \mathbb{Q}_+, \mathbb{Z}_+\}$, where, as usual, $\mathbb{R}_+ := [0, +\infty[$ is the set of non-negative real numbers, $\mathbb{Q}_+$ is the set of non-negative rational numbers and $\mathbb{Z}_+$ is the set of non-negative integers.

Moreover, we fix a natural number n, the number $L \in \mathbb{X}$, $L > 0$, the numbers $k_i \in \mathbb{X}$, $i = 1, \dots, n$, assume that

$$L > \sum_{i=1}^n k_i$$

and introduce the set

$$\mathbb{X}(L, n; \mathbf{k}_n) := \left\{ (x_1, \dots, x_n) \in \mathbb{X}^n \,:\, \sum_{i=1}^n x_i = L, \;\; x_i \geq k_i, \;\; i = 1, \dots, n \right\},$$

where $\mathbf{k}_n := (k_1, \dots, k_n)$.

We write

$$\mathbb{X}(L,n) := \mathbb{X}(L,n;0_n),$$

where $\mathbf{0}_n := (0,\dots,0)$.

Clearly,

$$\mathbb{Z}_+(L,n;\mathbf{k}_n) \subset \mathbb{Q}_+(L,n;\mathbf{k}_n) \subset \mathbb{R}_+(L,n;\mathbf{k}_n),$$

the set $\mathbb{Z}_+(L,n;\mathbf{k}_n)$ is finite; if $n>1$, then $\mathbb{Q}_+(L,n;\mathbf{k}_n)$ is an infinite closed subset of $\mathbb{Q}^n$, while $\mathbb{R}_+(L,n;\mathbf{k}_n)$ is an infinite compact convex subset of $\mathbb{R}^n$.

The set $\mathbb{R}_+(1,n)$ in a convex analysis is called *standard* $(n-1)$-*simplex* (or *unit* $(n-1)$-*simplex*).

Let us now fix

$$\mathbf{s}_n = (s_1,\dots,s_n) \in \mathbb{X}^n$$

and introduce the quantity

$$b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{X}) = \sup\left\{\prod_{i=1}^{n}(x_i+s_i) : (x_1,\dots,x_n)\in\mathbb{X}(L,n;\mathbf{k}_n)\right\}. \tag{2.1}$$

Observe that the following equality:

$$b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{X}) = b(L+\sum_{i=1}^{n}s_i,n;\mathbf{k}_n+\mathbf{s}_n;\mathbf{0}_n;\mathbb{X}). \tag{2.2}$$

(*pointed out to us by our Referee in June* 4, 2023) holds.

Clearly,

$$b(L,1;\mathbf{k}_1,\mathbf{s}_1;\mathbb{X}) = L+s_1,$$

and, in general,

$$b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{X}) \le \prod_{i=1}^{n}(L+s_i).$$

Let us introduce the following set:

$$\mathbb{X}_{\text{ext}}(L,n;\mathbf{k}_n;\mathbf{s}_n) :$$

$$\left\{(x_1,\dots,x_n)\in\mathbb{X}(L,n;\mathbf{k}_n)\ :\ b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{X}) = \prod_{i=1}^{n}(x_i+s_i)\right\}.$$

Note that if $\mathbb{X}=\mathbb{R}_+$, as the set $\mathbb{R}_+(L,n;\mathbf{k}_n)$ is compact and the function

$$(x_1,\dots,x_n)\ \mapsto\ \prod_{i=1}^{n}(x_i+s_i)$$

is continuous, we have

$$(\mathbb{R}_+)_{\text{ext}}(L,n;\mathbf{k}_n;\mathbf{s}_n) \neq \emptyset$$

and so, in the Definition 2.1 instead of sup, we can write max.

If $n>1$, then we will see in what follows (see Remark 2.6), although this is not clear in advance, that we also have

$$(\mathbb{Q}_+)_{\text{ext}}(L,n;\mathbf{k}_n;\mathbf{s}_n) \neq \emptyset.$$

Taking into account the introduced notation, our main optimization problem can be formulated as follows.

Problem 2.1. *Let $\mathbb{X}\in\{\mathbb{R}_+,\mathbb{Q}_+,\mathbb{Z}_+\}$. For a fixed natural number n, the number $L\in\mathbb{X}$ with $L>0$ and n-tuples of numbers*

$$\mathbf{k}_n = (k_1,\dots,k_n)\in\mathbb{X}^n,\quad \mathbf{s}_n=(s_1,\dots,s_n)\in\mathbb{X}^n,$$

we calculate

$$b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{X})$$

and describe the set

$$\mathbb{X}_{\text{ext}}(L,n;\mathbf{k}_n;\mathbf{s}_n).$$

In view of equality (2.2), this problem can be reformulated as follows.

Problem 2.2. *Let $\mathbb{X} \in \{\mathbb{R}_+, \mathbb{Q}_+, \mathbb{Z}_+\}$. For a fixed natural number n, the number $L \in \mathbb{X}$ with $L > 0$ and n-tuples of numbers*

$$\mathbf{k}_n = (k_1, \dots, k_n) \in \mathbb{X}^n, \quad \mathbf{s}_n = (s_1, \dots, s_n) \in \mathbb{X}^n,$$

we calculate

$$b(L + \sum_{i=1}^{n} s_i, n; \mathbf{k}_n + \mathbf{s}_n; \mathbf{0}_n; \mathbb{X})$$

and describe the set

$$\mathbb{X}_{\text{ext}}\left(L + \sum_{i=1}^{n} s_i, n; \mathbf{k}_n + \mathbf{s}_n; \mathbf{0}_n\right).$$

We agree to write below

$$b(L, n; \mathbb{X}) \quad \text{instead of} \quad b(L, n; \mathbf{0}_n; \mathbf{0}_n; \mathbb{X})$$

and

$$\mathbb{X}_{\text{ext}}(L, n) \quad \text{instead of} \quad \mathbb{X}_{\text{ext}}(L, n; \mathbf{0}_n; \mathbf{0}_n).$$

2.2. **Solution to problem for $\mathbb{X} \in \{\mathbb{R}_+, \mathbb{Q}_+\}$: a particular case.** In this subsection, we present two proofs of the following reformulation of the AM-GM inequality, which provides a complete solution to Problem 2.1 in case when $\mathbb{X} \in \{\mathbb{R}_+, \mathbb{Q}_+\}$ and $\mathbf{k}_n = \mathbf{s}_n = \mathbf{0}_n$.

Theorem 2.3. *Let $\mathbb{X} \in \{\mathbb{R}_+, \mathbb{Q}_+\}$. For a fixed natural number n and the number $L \in \mathbb{X}$ with $L > 0$, we have*

$$b(L, n; \mathbb{X}) = \left(\frac{L}{n}\right)^n \tag{2.3}$$

and

$$\mathbb{X}_{\text{ext}}(L, n) = \left\{\left(\frac{L}{n}, \dots, \frac{L}{n}\right)\right\}. \tag{2.4}$$

To give our proofs of Theorem 2.3, we introduce first a notation and prove a lemma of independent interest.

For a natural number $n > 1$ and the numbers q, x_i, $i = 1, \dots, n$, we write:

$$I_n = \{1, \dots, n\} \ \text{ and } \ I_{n,q}(x_1, \dots, x_n) = \{i \in I_n : x_i \neq q\}.$$

Lemma 2.4. *Let $\mathbb{X} \in \{\mathbb{R}_+, \mathbb{Q}_+, \mathbb{Z}_+\}$, $n > 1$ be a natural number, $q \in \mathbb{X}$, $q > 0$ be a number and*

$$(x_1, \dots, x_n) \in \mathbb{X}(nq, n),$$

i.e.,

$$x_i \in \mathbb{X}, \quad i = 1, \dots, n \ \text{ and } \ \sum_{i=1}^{n} x_i = nq.$$

Suppose that

$$I_{n,q}(x_1, \dots, x_n) \neq \emptyset. \tag{2.5}$$

Then

$$\operatorname{card}\left(I_{n,q}(x_1, \dots, x_n)\right) \geq 2 \tag{2.6}$$

and there exists

$$(x'_1, \dots, x'_n) \in \mathbb{X}(nq, n)$$

with the following properties:

$$\prod_{i=1}^{n} x'_i > \prod_{i=1}^{n} x_i \tag{2.7}$$

and

$$\operatorname{card}\left(I_{n,q}(x'_1, \dots, x'_n)\right) \leq \operatorname{card}\left(I_{n,q}(x_1, \dots, x_n)\right) - 1 < \operatorname{card}\left(I_{n,q}(x_1, \dots, x_n)\right). \tag{2.8}$$

Proof. Suppose that (2.6) is not true. Then from this assumption and (2.5), we can conclude that $I_{n,q}(x_1,\dots,x_n)=\{j\}$ for some $j\in I_n$; this implies:

$$nq=\sum_{i=1}^{n}x_i=x_j+\sum_{i\in I_n\setminus\{j\}}x_i=x_j+(n-1)q.$$

Hence $x_j=q$ contradicts $j\in I_{n,q}(x_1,\dots,x_n)$. Therefore (2.6) is proved.

Now, it is easy to see that for some $j\in I_{n,q}(x_1,\dots,x_n)$ and $k\in I_{n,q}(x_1,\dots,x_n)$, we have

$$x_j>q \;\text{ and }\; x_k<q,$$

and so,

$$(x_j-q)(q-x_k)>0. \tag{2.9}$$

Define now a sequence x'_i, $i=1,\dots,n$ as follows:

$$x'_j=q,\quad x'_k=x_j+x_k-q,\quad x'_i=x_i \;\text{ for }\; i\in I_n\setminus\{j,k\}.$$

Clearly, $(x'_1,\dots,x'_n)\in\mathbb{X}(nq,n)$, and we also have

$$x'_jx'_k>x_jx_k. \tag{2.10}$$

In fact, taking into account (2.9), we can write:

$$x'_jx'_k-x_jx_k=q\cdot(x_j+x_k-q)-x_jx_k=(x_j-q)\cdot(q-x_k)>0$$

and so, (2.10) is proved.

Now, using (2.10), we get

$$\prod_{i=1}^{n}x'_i=x'_jx'_k\cdot\prod_{i\in I_n\setminus\{j,k\}}x'_i=x'_jx'_k\cdot\prod_{i\in I_n\setminus\{j,k\}}x_i>x_jx_k\cdot\prod_{i\in I_n\setminus\{j,k\}}x_i=\prod_{i=1}^{n}x_i$$

and so, (2.7) is proved.

It is clear that

$$I_{n,q}(x'_1,\dots,x'_n)\subset I_{n,q}(x_1,\dots,x_n)\setminus\{j\}$$

and (2.8) holds, as well. □

The first proof of Theorem 2.3.

First, we consider the case for $\mathbb{X}=\mathbb{R}_+$. Clearly, it suffices to show that

$$(\mathbb{R}_+)_{\text{ext}}(L,n)=\left\{\left(\frac{L}{n},\dots,\frac{L}{n}\right)\right\}. \tag{2.11}$$

As we have noted in Subsection 2.1, *we can suppose that* $(\mathbb{R}_+)_{\text{ext}}(L,n)\neq\emptyset$. So, fix some $(x_1,\dots,x_n)\in(\mathbb{R}_+)_{\text{ext}}(L,n)$ and define $q=\frac{L}{n}$. Then

$$\prod_{i=1}^{n}x_i=b(L,n;\mathbb{R}_+). \tag{2.12}$$

Suppose for a moment that $I_{n,q}(x_1,\dots,x_n)\neq\emptyset$. Then by Lemma 2.4, we can find

$$(x'_1,\dots,x'_n)\in\mathbb{R}_+(L,n)$$

such that

$$\prod_{i=1}^{n}x'_i>\prod_{i=1}^{n}x_i.$$

From this inequality and (2.12), we get

$$\prod_{i=1}^{n}x'_i>b(L,n;\mathbb{R}_+),$$

but this contradicts the definition of $b(L,n;\mathbb{R}_+)$.

Therefore $I_{n,q}(x_1,\dots,x_n)=\emptyset$; hence $x_i=q$, $i=1,\dots,n$ and so, (2.11) is true.

Consequently, (2.3) and (2.4) are proved when $\mathbb{X}=\mathbb{R}_+$.

Consider the case for $\mathbb{X} = \mathbb{Q}_+$.

Since in this case we also have $L \in Q_+$, equalities (2.3) and (2.4) remain true.

Hence, the first proof of Theorem 2.3 is complete.

The second proof of Theorem 2.3 does not assume that $(\mathbb{R}_+)_{\text{ext}}(L,n) \neq \emptyset$.

Define $q = \frac{L}{n}$. It suffices to show that

$$(x_1,\dots,x_n) \in \mathbb{X}(L,n) \text{ and } I_{n,q}(x_1,\dots,x_n) \neq \emptyset \Longrightarrow \prod_{i=1}^{n} x_i < q^n.$$

So, fix an *arbitrary* $(x_1,\dots,x_n) \in \mathbb{X}(L,n)$ with $I_{n,q}(x_1,\dots,x_n) \neq \emptyset$ and let us verify that

$$\prod_{i=1}^{n} x_i < q^n. \tag{2.13}$$

In this case, by Lemma 2.4, we can find and fix some

$$(x_1^{(1)},\dots,x_n^{(1)}) \in \mathbb{X}(L,n)$$

such that

$$\prod_{i=1}^{n} x_i^{(1)} > \prod_{i=1}^{n} x_i \tag{2.14}$$

and

$$\operatorname{card}\big(I_{n,q}(x_1^{(1)},\dots,x_n^{(1)})\big) < \operatorname{card}\left(I_{n,q}(x_1,\dots,x_n)\right).$$

Case 1.1. For $(x_1^{(1)},\dots,x_n^{(1)})$*, we have* $I_{n,q}(x_1^{(1)},\dots,x_n^{(1)}) = \emptyset$.

In this case, we have $x_i^{(1)} = q$, $i = 1,\dots,n$ and so,

$$\prod_{i=1}^{n} x_i^{(1)} = q^n.$$

This equality together with (2.14) gives

$$\prod_{i=1}^{n} x_i < \prod_{i=1}^{n} x_i^{(1)} = q^n$$

and so, (2.13) is satisfied.

Case 1.2. For $(x_1^{(1)},\dots,x_n^{(1)})$*, we have* $I_{n,q}(x_1^{(1)},\dots,x_n^{(1)}) \neq \emptyset$.

In this case, necessarily $n > 2$. We can apply Lemma 2.4 to $(x_1^{(1)},\dots,x_n^{(1)}) \in \mathbb{X}(L,n)$ instead of $(x_1,\dots,x_n) \in \mathbb{X}(L,n)$ to find and fix some

$$(x_1^{(2)},\dots,x_n^{(2)}) \in \mathbb{X}(L,n)$$

such that

$$\prod_{i=1}^{n} x_i^{(2)} > \prod_{i=1}^{n} x_i^{(1)} \tag{2.15}$$

and

$$\operatorname{card}\left(I_{n,q}(x_1^{(2)},\dots,x_n^{(2)})\right) < \operatorname{card}\left(I_{n,q}(x_1^{(1)},\dots,x_n^{(1)})\right) < \operatorname{card}\left(I_{n,q}(x_1,\dots,x_n)\right).$$

Case 1.3. For $(x_1^{(2)},\dots,x_n^{(2)})$*, we have* $I_{n,q}(x_1^{(2)},\dots,x_n^{(2)}) = \emptyset$.

In this case, we have $x_i^{(2)} = q$, $i = 1,\dots,n$ and so,

$$\prod_{i=1}^{n} x_i^{(2)} = q^n.$$

This equality together with (2.14) and (2.15) gives

$$\prod_{i=1}^{n} x_i < \prod_{i=1}^{n} x_i^{(1)} < \prod_{i=1}^{n} x_i^{(2)} = q^n$$

and so, (2.13) is satisfied again.

Continuing in this way, we can find a natural number $m < n$ and a sequence

$$(x_1^{(i)}, \dots, x_n^{(i)}) \in \mathbb{X}(L, n), \quad i = 1, \dots, m$$

such that

$$I_{n,q}(x_1^{(i)}, \dots, x_n^{(i)}) \neq \emptyset, \quad i = 1, \dots, m-1 \text{ and } I_{n,q}(x_1^{(m)}, \dots, x_n^{(m)}) = \emptyset.$$

Then

$$\prod_{i=1}^n x_i < \prod_{i=1}^n x_i^{(1)} < \prod_{i=1}^n x_i^{(2)} < \cdots < \prod_{i=1}^n x_i^{(m)} = q^n.$$

Consequently, $(x_1, \dots, x_n)$ satisfies (2.13). □

2.3. **Solution to the problem: the case $\mathbb{X} \in \{\mathbb{R}_+, \mathbb{Q}_+\}$.** The following theorem provides a solution to Problem 2.1 in general when $\mathbb{X} \in \{\mathbb{R}_+, \mathbb{Q}_+\}$.

Theorem 2.5. *Let $n \geq 1$ be a natural number, $L \in \mathbb{X}$, $L > 0$,*

$$\mathbf{k}_n := (k_1, \dots, k_n) \in \mathbb{X}^n, \quad \mathbf{s}_n := (s_1, \dots, s_n) \in \mathbb{X}^n,$$

and

$$q := \frac{L + \sum_{i=1}^n s_i}{n}.$$

Assuming further that

$$d := L - \sum_{i=1}^n k_i > 0, \tag{2.16}$$

we put $b(L, n; \mathbf{k}_n; \mathbf{s}_n)$ instead of $b(L, n; \mathbf{k}_n; \mathbf{s}_n; \mathbb{X})$ and denote

$$\alpha = \min_{1 \leq i \leq n} (k_i + s_i), \quad \beta = \max_{1 \leq i \leq n} (k_i + s_i),$$

$$\mathcal{I}_1 = \{i \in I_n : s_i + k_i \leq q\}, \quad \mathcal{I}_2 = \{i \in I_n : s_i + k_i > q\},$$

$$m := \operatorname{card}(\mathcal{I}_1), \quad \mathcal{I}_1 := \{i_1, \dots, i_m\}.$$

Then the following statements are valid:

(a)

$$q \geq \frac{d}{n} + \alpha > \alpha, \tag{2.17}$$

in particular, $m \geq 1$ and

$$\prod_{i=1}^n (k_i + s_i + \frac{d}{n}) \leq b(L, n; \mathbf{k}_n; \mathbf{s}_n) \leq q^n. \tag{2.18}$$

(b) *If $m = n$, i.e., if $q \geq \beta$, then we have the equalities*

$$b(L, n; \mathbf{k}_n; \mathbf{s}_n) = q^n \text{ and } \mathbb{X}_{\text{ext}}(L, n; \mathbf{k}_n; \mathbf{s}_n) = \{(x_1, \dots, x_n)\},$$

where $x_i = q - s_i$, $i = 1, \dots, n$.

(c) *If $m < n$, i.e., if $q < \beta$, then*

$$b(L, n; \mathbf{k}_n; \mathbf{s}_n) < q^n \tag{2.19}$$

and

$$b(L, n; \mathbf{k}_n, \mathbf{s}_n) = b(L_m, m; , \mathbf{k}_m; \mathbf{s}_m) \prod_{i \in \mathcal{I}_2} (k_i + s_i), \tag{2.20}$$

where $L_m := L - \sum_{i \in \mathcal{I}_2} k_i$, $\mathbf{k}_m := (k_{i_1}, \dots, k_{i_m})$ and $\mathbf{s}_m := (s_{i_1}, \dots, s_{i_m})$.

(d) *If $m = 1$, then*

$$b(L, n; \mathbf{k}_n; \mathbf{s}_n) = (L_1 + s_{i_1}) \prod_{i \in I_n \setminus \{i_1\}} (k_i + s_i)$$

and

$$(x_1, \dots, x_n) \in \mathbb{X}_{\text{ext}}(L, n; \mathbf{k}_n; \mathbf{s}_n),$$

where $L_1 := L - \sum_{i \in I_n \setminus \{i_1\}} k_i$, $x_{i_1} := L_1$ and $x_i := k_i$ for $i \in I_n \setminus \{i_1\}$.

Proof. (a) (2.17) follows from (2.16). In fact,

$$q = \frac{d + \sum_{i=1}^{n}(k_i + s_i)}{n} \geq \frac{d + n\alpha}{n} = \frac{d}{n} + \alpha > \alpha.$$

We have $m \geq 1$ by (2.17).

The left inequality of (2.18) is true because

$$\Big(k_1 + \frac{1}{n}d, \ldots, k_n + \frac{1}{n}d\Big) \in \mathbb{X}(L, n; \mathbf{k}_n).$$

The right inequality of (2.18) follows directly from Theorem 2.3, as for $(x_1, \ldots, x_n) \in \mathbb{X}(L, n; \mathbf{k}_n)$, we have

$$(x_1 + s_1, \ldots, x_n + s_n) \in \mathbb{X}\Big(L + \sum_{i=1}^{n} s_i, n; \mathbf{k}_n\Big),$$

and hence

$$\prod_{i=1}^{n}(x_i + s_i) \leq b\Big(L + \sum_{i=1}^{n} s_i, n; \mathbf{k}_n, \mathbf{0}_n; \mathbb{X}\Big) \leq b\Big(L + \sum_{i=1}^{n} s_i, n; \mathbf{0}_n, \mathbf{0}_n; \mathbb{X}\Big) = q^n.$$

(b) From $q \geq \beta \geq s_i + k_i$, $i = 1, \ldots, n$, we conclude that for

$$x_i := q - s_i, \quad i = 1, \ldots, n,$$

we have

$$(x_1, \ldots, x_n) \in \mathbb{X}(L, n; \mathbf{k}_n).$$

Hence

$$q^n = \prod_{i=1}^{n}(x_i + s_i) \leq b(L, n; \mathbf{k}_n; \mathbf{s}_n).$$

From this inequality and the right-hand inequality of (2.18), we get that (b) is true.

(c) Suppose that (2.19) fails, i. e., we have the equality

$$b(L, n; \mathbf{k}_n; \mathbf{s}_n) = q^n.$$

Then for some

$$(x_1, \ldots, x_n) \in \mathbb{X}(L, n; \mathbf{k}_n),$$

we shall have

$$\prod_{i=1}^{n}(x_i + s_i) = b(L, n; \mathbf{k}_n; \mathbf{s}_n) = q^n.$$

That is, in the AM-GM inequality, for $x_i + s_i$, $i = 1, \ldots, n$, we have the equality. From this we can conclude that

$$q = x_i + s_i, \quad i = 1, \ldots, n.$$

Hence $q - s_i = x_i \geq k_i$, $i = 1, \ldots, n$ and so, $q \geq s_i + k_i$, $i = 1, \ldots, n$; in particular, $q \geq \beta$, but this contradicts our assumption that $q < \beta$.

We will now prove (2.20) in two steps.

Step 1. (AS) *If* $(x_1, \ldots, x_n) \in \mathbb{X}_{\text{ext}}(L, n; \mathbf{k}_n, \mathbf{s}_n)$, *i.e., if* $(x_1, \ldots, x_n) \in \mathbb{X}(L, n; \mathbf{k}_n)$ *and*

$$\prod_{j=1}^{n}(x_j + s_j) = b(L, n; \mathbf{k}_n, \mathbf{s}_n), \tag{2.21}$$

then

$$x_i = k_i, \quad \forall i \in \mathcal{I}_2. \tag{2.22}$$

To prove (AS), we fix some $(x_1, \ldots, x_n) \in \mathbb{X}(L, n; \mathbf{k}_n)$ satisfying (2.21) and let us show that then (2.22) is satisfied, too.

Suppose that this is not so, i.e., $x_l > k_l$ for some $l \in \mathcal{I}_2$. Fix $t > 0$ such that $x_l - t > k_l$. Fix then $i \in \{1, \ldots, n\}$ with $i \neq l$ and put

$$y_i := x_i + t, \quad y_l := x_l - t,$$

in case $n > 2$,

$$y_j := x_j \text{ for } j \in \{1, \dots, n\} \setminus \{i, l\}.$$

Then we have

$$(y_1, \dots, y_n) \in \mathbb{X}(L, n; \mathbf{k}_n).$$

From this and (2.21), we can conclude that

$$\prod_{j=1}^{n} (x_j + s_j) \geq \prod_{j=1}^{n} (y_j + s_j).$$

This implies

$$(x_i + s_i)(x_l + s_l) \geq (x_i + t + s_i)(x_l - t + s_l).$$

Using this inequality along with the implications

$$(x_i + s_i)(x_l + s_l) \geq (x_i + t + s_i)(x_l - t + s_l) \implies \frac{x_l + s_l}{x_l - t + s_l} \geq \frac{x_i + t + s_i}{x_i + s_i}$$
$$\implies \frac{x_l + s_l}{x_l - t + s_l} - 1 \geq \frac{x_i + t + s_i}{x_i + s_i} - 1 \implies \frac{t}{x_l - t + s_l} \geq \frac{t}{x_i + s_i},$$

we get

$$x_i + s_i \geq x_l - t + s_l.$$

This implies

$$\sum_{i \in \{1, \dots, n\} \setminus \{l\}} (x_i + s_i) \geq \sum_{i \in \{1, \dots, n\} \setminus \{l\}} (x_l - t + s_l) = (n-1)x_l + (n-1)(s_l - t).$$

Therefore

$$L + \sum_{j=1}^{n} s_j = x_l + s_l + \sum_{i \in \{1, \dots, n\} \setminus \{l\}} (x_i + s_i)$$
$$\geq x_l + s_l + (n-1)x_l + (n-1)(s_l - t) = nx_l + ns_l - (n-1)t.$$

This yields

$$q = \frac{L + \sum_{j=1}^{n} s_j}{n} \geq x_l + s_l - \frac{n-1}{n} t \geq x_l + s_l - t \geq k_l + s_l,$$

which contradicts our choice of l: $q < k_l + s_l$. Therefore (2.21) implies (2.22) and (AS) is proved.

Step 2. From (AS) proved in *Step 1*, we derive now equality (2.20).

Fix $(x_1, \dots, x_n) \in \mathbb{X}(L, n; \mathbf{k}_n)$ satisfying (2.21). From the statement proved in *Step 1*, we can write

$$\prod_{j=1}^{m} (x_{i_j} + s_{i_j}) \prod_{j \in \mathcal{I}_2} (k_j + s_j) = b(L, n; \mathbf{k}_n; \mathbf{s}_n) \tag{2.23}$$

and

$$\sum_{j=1}^{m} x_{i_j} = L_m. \tag{2.24}$$

From (2.24) and the definition of $b(L_m, m; \mathbf{k}_m, \mathbf{s}_m)$, we can write

$$\prod_{j=1}^{m} (x_{i_j} + s_{i_j}) \leq b(L_m, m; \mathbf{k}_m; \mathbf{s}_m). \tag{2.25}$$

Fix now some $(y_1, \dots, y_m) \in \mathbb{X}(L_m, m; \mathbf{k}_m)$ such that

$$\prod_{j=1}^{m} (y_j + s_{i_j}) = b(L_m, m; \mathbf{k}_m; \mathbf{s}_m). \tag{2.26}$$

Consider the indices $i_{m+1}, \dots, i_n$ such that

$$\mathcal{I}_2 = \{i_{m+1}, \dots, i_n\}.$$

Since $(y_1,\dots,y_m,k_{i_{m+1}},\dots,k_{i_n})\in\mathbb{X}(L,n;\mathbf{k}_n)$, from (2.21), we get

$$\prod_{j=1}^{m}(y_j+s_{i_j})\prod_{j=m+1}^{n}(k_{i_j}+s_{i_j})\le b(L,n;\mathbf{k}_n,\mathbf{s}_n)=\prod_{j=1}^{m}(x_{i_j}+s_{i_j})\prod_{j\in\mathcal{I}_2}(k_j+s_j).$$

Hence

$$\prod_{j=1}^{m}(y_j+s_{i_j})\le\prod_{j=1}^{m}(x_{i_j}+s_{i_j}).$$

From this inequality and from (2.26), we obtain

$$b(L_m,m;\mathbf{k}_m,\mathbf{s}_m)\le\prod_{j=1}^{m}(x_{i_j}+s_{i_j}).$$

This inequality along with (2.25) gives

$$b(L_m,m;\mathbf{k}_m,\mathbf{s}_m)=\prod_{j=1}^{m}(x_{i_j}+s_{i_j}).$$

Combining this equality with (2.23), we come to (2.20).

(d) follows from (2.20) of (c) because $b(L_1,1;s_{i_1})=L_1+s_{i_1}$. □

Remark 2.6. Note that Theorem 2.5(b) gives the complete solution to Problem 2.1, while Theorem 2.5(c) fails to provide a complete solution, because (for $n>2$ and $m>1$) it does not give a **formula** for $b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{X})$; it provides just a method of its calculation at a **finite** (in the worst case of $(n-2)$) number of steps. This method shows that when the "initial data" are rational, then $b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{Q}_+)$ is a rational number, as well (i. e., when $\mathbb{X}=\mathbb{Q}_+$, then $b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{Q}_+)\in\mathbb{Q}_+$ and so, $(\mathbb{Q}_+)_{\text{ext}}(L,n;\mathbf{k}_n;\mathbf{s}_n)\neq\emptyset$).

3. Problem for the Case $\mathbb{X}=\mathbb{Z}_+$

3.1. **Solution to problem for $\mathbb{X}=\mathbb{Z}_+$: a particular case.** In this subsection, we deal with Problem 2.1 in case $\mathbb{X}=\mathbb{Z}_+$ and $\mathbf{k}_n=\mathbf{s}_n=\mathbf{0}_n$.

We present a proof of the following assertion, which in the case under consideration provides a complete solution to Problem 2.1 and a slight modification of a similar statement obtained earlier in [5] (see also [15]).

Proposition 3.1 ([5]). *Let $L\ge n>1$ be natural numbers, $[q]$ be the integer part of $q:=\frac{L}{n}$, and $r:=L-n[q]$.*

Then the following statements are valid.

(a) *If $(x_1,\dots,x_n)\in(\mathbb{Z}_+)_{\text{ext}}(L,n)$, then $x_i\ge1$, $i=1,\dots,n$ and*

$$|x_i-x_j|\in\{0,1\},\quad i,j=1,\dots,n. \tag{3.1}$$

(b) *The equality*

$$b(L,n;\mathbb{Z}_+)=(1+[q])^r[q]^{n-r} \tag{3.2}$$

holds.

Moreover, for $(x_1,\dots,x_n)\in\mathbb{Z}_+(L,n)$, we have

$$(x_1,\dots,x_n)\in(\mathbb{Z}_+)_{\text{ext}}(L,n),$$

if and only if

$$\operatorname{card}(\{i\in I_n:x_i=1+[q]\})=r\ \text{ and }\ \operatorname{card}(\{i\in I_n:x_i=[q]\})=n-r.$$

(c) *If $r=0$, then*

$$b(L,n;\mathbb{Z}_+)=b(L,n;\mathbb{R}_+)=q^n \tag{3.3}$$

and if $r>0$, then

$$b(L,n;\mathbb{Z}_+)<b(L,n;\mathbb{R}_+)=q^n, \tag{3.4}$$

or, equivalently,

$$(1+[q])^r[q]^{n-r} < \left([q]+\frac{r}{n}\right)^n. \tag{3.5}$$

(d) *If $r > 0$, then*

$$b(L,n;\mathbb{Z}_+) \le [b(L,n;\mathbb{R}_+)] = [q^n], \tag{3.6}$$

or, equivalently,

$$(1+[q])^r[q]^{n-r} \le \left[\left([q]+\frac{r}{n}\right)^n\right];$$

The equality in (3.6) **takes place only in the following cases:**
Case 1. $n = 2$.
Case 2. $n = 3$ *and* $L \in \{4, 5, 7, 8\}$.
Case 3. $n > 3$ *and* $L = n + 1$.

Proof. (a) Observe first of all that evidently

$$(x_1,\dots,x_n) \in (\mathbb{Z}_+)_{\text{ext}}(L,n) \implies x_i \ge 1, \quad i = 1,\dots,n. \tag{3.7}$$

Fix now $(x_1,\dots,x_n) \in (\mathbb{Z}_+)_{\text{ext}}(L,n)$ and show that (3.1) holds.

Suppose for a moment that (3.1) is not true; i.e., for some $i, j \in \{1,\dots,n\}$, we have $|x_i - x_j| > 1$. We can suppose without loss of generality that $x_1 - x_2 > 1$. Put now $y_1 = x_1 - 1$, $y_2 = x_2 + 1$ and, in case $n > 2$, $y_i = x_i$, $i = 3,\dots,n$. Clearly, $(y_1,\dots,y_n) \in \mathbb{Z}_+(L,n)$. This and the definition of $b(L,n;Z_+)$ and (3.7) imply

$$b(L,n;Z_+) \ge \prod_{i=1}^n y_i = y_1y_2\prod_{i=3}^n x_i > x_1x_2\prod_{i=3}^n x_i = \prod_{i=1}^n x_i;$$

but the obtained inequality

$$b(L,n;Z_+) > \prod_{i=1}^n x_i$$

contradicts the relation $(x_1,\dots,x_n) \in (\mathbb{Z}_+)_{\text{ext}}(L,n)$.

(b) Let us show now the validity of (3.2). Fix some $(x_1,\dots,x_n) \in (\mathbb{Z}_+)_{\text{ext}}(L,n)$. Then

$$b(L,n;\mathbb{Z}_+) = \prod_{i=1}^n x_i. \tag{3.8}$$

We can suppose without loss of generality that $x_1 \ge \cdots \ge x_n$. From (3.1), we have $|x_i - x_j| \in \{0,1\}$, $i, j = 1,\dots,n$. In particular, we have $x_1 - x_n \in \{0,1\}$.

Suppose first that $x_1 = x_n$. Then $nx_1 = L = n[q] + r$ and (3.8) implies that (3.2) is true with $r = 0$.

Suppose now that $x_1 > x_n$ and $m < n$ are the first natural number such that $x_1 = x_m$. Then we have $x_i = x_m + 1$, $i = 1,\dots,m$ and $x_i = x_m$, $i = m+1,\dots,n$. This equalities and (3.8) imply that

$$b(L,n;\mathbb{Z}_+) = (1+x_m)^m x_m^{n-m}. \tag{3.9}$$

The equalities $x_i = x_m + 1$, $i = 1,\dots,m$ and $x_i = x_m$, $i = m+1,\dots,n$ as $\sum_{i=1}^n x_i = L$ imply also that

$$m(x_m+1) + (n-m)x_m = L = n[q] + r$$

and so,

$$nx_m + m = n[q] + r.$$

Consequently, $x_m = [q]$ and $r = m > 0$ and (3.9) implies that (3.2) is true.

(c) Clearly, when $r = 0$, then (3.3) is true.

Let now $r > 0$. As

$$q = \frac{L}{n} = [q] + \frac{r}{n},$$

inequality (3.4) and inequality (3.5) are equivalent. By AM-GM, we have the inequality

$$(1+[q])^r[q]^{n-r} = (1+[q])\cdot\cdots\cdot(1+[q])\cdot[q])\cdot\cdots\cdot[q]$$
$$< \Big(\frac{(1+[q])+\cdots+(1+[q])+[q])+\cdots+[q]}{n}\Big)^n$$
$$= \Big(\frac{r(1+[q])+(n-r)[q])}{n}\Big)^n = \Big([q]+\frac{r}{n}\Big)^n,$$

and (3.5) is proved.

(d) Evidently, (3.6) follows from (3.4). The assertion contained in the rest part of (d) is of independent interest and its proof is already published in [4]; for the reader's convenience we reproduce this proof here (see Remark 3.2 below). □

Remark 3.2. Let $n > 1$ and $1 \le r < n$ be natural numbers. Then we have

$$(1+x)^r x^{n-r} \le \Big[\Big(x+\frac{r}{n}\Big)^n\Big], \quad x = 1, 2, \dots \tag{3.10}$$

and the following statements give a complete description of the set

$$V_{n,r} := \Big\{x \in \mathbb{N} : (1+x)^r x^{n-r} = \Big[\Big(x+\frac{r}{n}\Big)^n\Big]\Big\}.$$

(a) $V_{2,1} = \mathbb{N}$, $V_{3,1} = V_{3,2} = \{1, 2\}$ and $V_{n,1} = \{1\}$, $n = 4, 5, \dots$.
(b) $n > 3$, $r > 1 \implies V_{n,r} = \emptyset$.

Proof of Remark 3.2. First, we formulate several useful observations.

$$V_{n,r} = \{x \in \mathbb{N} : \Big(x+\frac{r}{n}\Big)^n - (1+x)^r x^{n-r} < 1\}; \tag{3.11}$$

$$\Big(x+\frac{r}{n}\Big)^n - (1+x)^r x^{n-r} = \sum_{k=0}^{n}\Big(\frac{r^k}{n^k}C_{n,k} - C_{r,k}\Big)x^{n-k}, \tag{3.12}$$

where

$$C_{n,k} := \frac{n!}{k!(n-k)!}$$

and, as usual, we assume that $C_{r,k} = 0$ if $k > r$;

$$\frac{r^k}{n^k}C_{n,k} - C_{r,k} = 0, \quad k = 0, 1; \tag{3.13}$$

$$\frac{r^k}{n^k}C_{n,k} - C_{r,k} > 0, \quad k = 2, \dots, n. \tag{3.14}$$

Proof of (3.10). See Proposition 3.1(c)
Proof of (3.11). It is true because $(x+\frac{r}{n})^n$, $x = 1, 2, \dots$ are not integers.
Proof of (3.12). We have by the binomial theorem.
Proof of (3.13). Direct verification.
Proof of (3.14). The inequality is equivalent to the inequality

$$\Big(1-\frac{1}{n}\Big)\Big(1-\frac{2}{n}\Big)\cdots(1-\frac{k-1}{n}) > \Big(1-\frac{1}{r}\Big)\Big(1-\frac{2}{r}\Big)\cdots\Big(1-\frac{k-1}{r}\Big),$$

which is, of course, true when $r < n$.

Proof of Remark 3.2(a). The equalities $V_{2,1} = \mathbb{N}$ and $V_{3,1} = V_{3,2} = \{1, 2\}$ are easy to verify by using (3.11).

We have $1 \in V_{n,1}$ from (3.11) and from the known inequality $\big(1+\frac{1}{n}\big)^n - 2 < 1$. The remaining relation

$$x \ge 2,\ n > 3 \implies x \notin V_{n,1}$$

follows from the next more general statement:

$$x \ge 2,\ n > 3 \implies x \notin V_{n,r}. \tag{3.15}$$

To prove (3.15), note that owing to (3.12), (3.13) and (3.14), we can write

$$\Big(x+\frac{r}{n}\Big)^n-(1+x)^r x^{n-r}>\Big(\frac{r^2}{n^2}C_{n,2}-C_{r,2}\Big)x^{n-2}=\frac{x^{n-2}r}{2}\Big(1-\frac{r}{n}\Big).$$

From this relation we get (3.15). Indeed, if $n\ge 4$ and $x\ge 2$, then

$$\frac{x^{n-2}r}{2}(1-\frac{r}{n})\ge 2^{n-3}\frac{r(n-r)}{n},$$

and since $r(n-r)$ achieves its minimum value at $r=1$ or at $r=n-1$, we have the estimate

$$\frac{x^{n-2}r}{2}\Big(1-\frac{r}{n}\Big)\ge 2^{n-3}\frac{r(n-r)}{n}\ge 2^{n-3}\Big(1-\frac{1}{n}\Big)>1,$$

and from (3.11), we can conclude that $x\notin V_{n,r}$.

Proof of Remark 3.2(b). From (3.15) we have that if $x\ge 2$, then $x\notin V_{n,r}$.

Now let us prove that if $n\ge 4$ and $r>1$, then $1\notin V_{n,r}$, as well. For this purpose, according to (3.11), it suffices to show that the following (*slightly unexpected*) inequality

$$n\ge 4,\ 1<r<n \Longrightarrow \Big(1+\frac{r}{n}\Big)^n-2^r>1$$

holds. Therefore it remains now to prove the following inequality:

$$n\ge 4,\ 1<r<n \Longrightarrow \Big((1+\frac{r}{n})^{\frac{n}{r}}\Big)^r-2^r>1. \tag{3.16}$$

It is clear that for the given r the left-hand side of (3.16) increases as n increases. Let us consider separately two cases $r=2$ and $r\ge 3$.

If $r=2$, it is enough to check that (3.16) is true when $n=4$. Plugging in (3.16) $r=2$ and $n=4$ we are getting true inequality $\frac{17}{16}>1$ and hence we are done.

For $r\ge 3$, it is enough to show that (3.16) holds when $n=r+1$. In this case, the left-hand side of (3.16) will be

$$\Big(2-\frac{1}{r+1}\Big)^{r+1}-2^r=\frac{C_{r+1,2}2^{r-1}}{(r+1)^2}-\frac{C_{r+1,3}2^{r-2}}{(r+1)^3}+\sum_{k=4}^{r+1}C_{r+1,k}2^{r+1-k}\frac{(-1)^k}{(r+1)^k}.$$

Since the sum

$$\sum_{k=4}^{r+1}C_{r+1,k}2^{r+1-k}\frac{(-1)^k}{(r+1)^k}$$

is nonnegative, it is enough to show that

$$\frac{C_{r+1,2}2^{r-1}}{(r+1)^2}-\frac{C_{r+1,3}2^{r-2}}{(r+1)^3}>1.$$

We have

$$\frac{C_{r+1,2}2^{r-1}}{(r+1)^2}-\frac{C_{r+1,3}2^{r-2}}{(r+1)^3}=2^{r-2}\Big(1-\frac{1}{r+1}\Big)\Big(\frac{5}{6}+\frac{1}{3(r+1)}\Big)>2^{r-2}\Big(1-\frac{1}{r+1}\Big)\frac{5}{6}\ge 2\times\frac{3}{4}\times\frac{5}{6}=\frac{15}{12}>1$$

and so, (3.16) holds when $n=r+1$. □

3.2. **Solution to problem for $\mathbb{X}=\mathbb{Z}_+$, in general.** The following analogue of Theorem 2.5 provides a solution to Problem 2.1, in general, when $\mathbb{X}=\mathbb{Z}_+$.

Theorem 3.3. *Let $n\ge 1$ be a natural number, $L\in\mathbb{Z}_+$, $L>0$,*

$$\mathbf{k}_n:=(k_1,\dots,k_n)\in(\mathbb{Z}_+)^n,\quad \mathbf{s}_n:=(s_1,\dots,s_n)\in(\mathbb{Z}_+)^n,$$

$$q:=\frac{L+\sum_{i=1}^n s_i}{n}\quad \textit{and}\quad r:=L+\sum_{i=1}^n s_i-n[q].$$

We assume further that

$$d:=L-\Big(\sum_{i=1}^n k_i+n\Big)\ge 0 \tag{3.17}$$

and denote $\alpha = \min_{1\le i\le n}(k_i+s_i)$, $\beta = \max_{1\le i\le n}\ (k_i+s_i)$,

$$\mathcal{I}_1 = \{i \in I_n : s_i + k_i \le q\}, \quad \mathcal{I}_2 = \{i \in I_n : s_i + k_i > q\},$$
$$m := \operatorname{card}(\mathcal{I}_1), \quad \mathcal{I}_1 := \{i_1, \dots, i_m\}.$$

Then the following statements are valid:

(a)

$$q \ge 1 + \frac{d}{n} + \alpha > \alpha, \tag{3.18}$$

in particular, $m \ge 1$ *and*

$$\left(\prod_{i=1}^{n-1}(1+k_i+s_i)\right)\cdot(1+k_n+d+s_n) \le b(L,n;\mathbf{k}_n,\mathbf{s}_n;\mathbb{Z}_+) \le (1+[q])^r[q]^{n-r}. \tag{3.19}$$

(b) *If* $m = n$, *i.e., if* $q \ge \beta$, *then we have the following* **equality**:

$$b(L,n;\mathbf{k}_n,\mathbf{s}_n;\mathbb{Z}_+) = (1+[q])^r[q]^{n-r}.$$

Moreover, for $(x_1,\dots,x_n) \in \mathbb{Z}_+(L,n;\mathbf{k}_n)$, *we have* $(x_1,\dots,x_n) \in (\mathbb{Z}_+)_{\rm ext}(L,n;\mathbf{k}_n;\mathbf{s}_n)$ *if and only if*

$$\operatorname{card}\left(\{i \in I_n : x_i = 1+[q]-s_i\}\right) = r \ \text{ and } \ \operatorname{card}\left(\{i \in I_n : x_i = [q]-s_i\}\right) = n-r.$$

(c) *If* $m < n$, *i.e., if* $q < \beta$, *then*

$$q = [q] \implies b(L,n;\mathbf{k}_n,\mathbf{s}_n;\mathbb{Z}_+) < q^n; \tag{3.20}$$

and

$$b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{Z}_+) = b(L_m,m;,\mathbf{k}_m;\mathbf{s}_m;\mathbb{Z}_+)\prod_{i\in\mathcal{I}_2}(k_i+s_i), \tag{3.21}$$

where $L_m := L - \sum_{i\in\mathcal{I}_2} k_i$, $\mathbf{k}_m := (k_{i_1},\dots,k_{i_m})$ *and* $\mathbf{s}_m := (s_{i_1},\dots,s_{i_m})$.

(d) *If* $m = 1$, *then*

$$b(L,n;\mathbf{k}_n;\mathbf{s}_n;\mathbb{Z}_+) = (L_1 + s_{i_1})\prod_{i\in I_n\setminus\{i_1\}}(k_i+s_i)$$

and

$$(x_1,\dots,x_n) \in \mathbb{X}_{\rm ext}(L,n;\mathbf{k}_n;\mathbf{s}_n),$$

where $L_1 := L - \sum_{i\in I_n\setminus\{i_1\}} k_i$, $x_{i_1} = L_1$ *and* $x_i = k_i$ *for* $i \in I_n \setminus \{i_1\}$.

Proof. (a) (3.18) follows from (3.17). In fact,

$$q = \frac{d + \sum_{i=1}^n (k_i+s_i) + n}{n} \ge \frac{d + n\alpha + n}{n} = \frac{d}{n} + \alpha + 1 > \alpha.$$

We have $m \ge 1$ by (3.18).

The first inequality of (3.19) is true because

$$(1+k_1,\dots,1+k_{n-1},1+k_n+d) \in \mathbb{Z}_+(L,n;\mathbf{k}_n).$$

The second inequality of (3.19) follows directly from Proposition 3.1, as for $(x_1,\dots,x_n) \in \mathbb{Z}_+(L,n;\mathbf{k}_n)$, we have

$$(x_1+s_1,\dots,x_n+s_n) \in \mathbb{Z}_+\left(L + \sum_{i=1}^n s_i, n; \mathbf{k}_n\right),$$

and hence

$$\prod_{i=1}^n (x_i+s_i) \le b\left(L + \sum_{i=1}^n s_i, n; \mathbf{k}_n, \mathbf{0}_n; \mathbb{Z}_+\right) \le b\left(L + \sum_{i=1}^n s_i, n; \mathbf{0}_n, \mathbf{0}_n; \mathbb{Z}_+\right) = (1+[q])^r[q]^{n-r}.$$

(b) Clearly, since β is an integer, from $q \ge \beta$, we have $[q] \ge \beta$ and so, $[q] \ge k_i + s_i$, $i = 1,\dots,n$. This implies that

$$(x_1^{(r)},\dots,x_n^{(r)}) \in \mathbb{Z}_+(L,n;\mathbf{k}_n)$$

and

$$(1+[q])^r[q]^{n-r} = \prod_{i=1}^{n}(x_i^{(r)} + s_i) \leq b(L, n; \mathbf{k}_n, \mathbf{s}_n; \mathbb{Z}_+).$$

This equality together with the right inequality of (3.19) show that (b) is true.

(c) Suppose that (3.20) fails, i.e., we have the equality

$$b(L, n; \mathbf{k}_n, \mathbf{s}_n; \mathbb{Z}_+) = q^n$$

Then for some

$$(x_1, \ldots, x_n) \in \mathbb{Z}_+(L, n; \mathbf{k}_n),$$

we will have

$$\prod_{i=1}^{n}(x_i + s_i) = b(L, n; \mathbf{k}_n, \mathbf{s}_n; \mathbb{Z}_+) = q^n,$$

i.e., in the AM-GM inequality for $x_i + s_i$, $i = 1, \ldots, n$, we have the equality, whence we can conclude that

$$q = x_i + s_i, \quad i = 1, \ldots, n.$$

Hence $q - s_i = x_i \geq k_i$, $i = 1, \ldots, n$ and so, $q \geq s_i + k_i$, $i = 1, \ldots, n$; in particular, $q \geq \beta$, but this contradicts our assumption that $q < \beta$.

(c) The proof is similar to that of Theorem 2.5(c). For reader's convenience, we repeat it with the corresponding slight modifications.

Step 1. (AS2) If $(x_1, \ldots, x_n) \in (\mathbb{Z}_+)_{\text{ext}}(L, n; \mathbf{k}_n; \mathbf{s}_n)$; i.e., $(x_1, \ldots, x_n) \in \mathbb{Z}_+(L, n; \mathbf{k}_n)$ and

$$\prod_{j=1}^{n}(x_j + s_j) = b(L, n; \mathbf{k}_n; \mathbf{s}_n; \mathbb{Z}_+), \tag{3.22}$$

then

$$x_i = k_i, \quad \forall i \in \mathcal{I}_2. \tag{3.23}$$

To prove (AS2), fix some $(x_1, \ldots, x_n) \in \mathbb{Z}_+(L, n; \mathbf{k}_n)$ satisfying (3.22)) and let us show that then (3.23) is satisfied, too.

Suppose that this is not the case, i. e., $x_l > k_l$ for some $l \in \mathcal{I}_2$. Fix $t > 0$ such that $x_l - t > k_l$. Fix then $i \in \{1, \ldots, n\}$ with $i \neq l$ and put

$$y_i := x_i + t, \quad y_l := x_l - t,$$

and in case $n > 2$,

$$y_j := x_j \ \text{ for } \ j \in \{1, \ldots, n\} \setminus \{i, l\}.$$

Then we have

$$(y_1, \ldots, y_n) \in \mathbb{Z}_+(L, n; \mathbf{k}_n).$$

From this and in view of (3.22), we can conclude that

$$\prod_{j=1}^{n}(x_j + s_j) \geq \prod_{j=1}^{n}(y_j + s_j).$$

This implies that

$$(x_i + s_i)(x_l + s_l) \geq (x_i + t + s_i)(x_l - t + s_l).$$

This inequality and the following implications:

$$(x_i + s_i)(x_l + s_l) \geq (x_i + t + s_i)(x_l - t + s_l) \implies \frac{x_l + s_l}{x_l - t + s_l} \geq \frac{x_i + t + s_i}{x_i + s_i}$$
$$\implies \frac{x_l + s_l}{x_l - t + s_l} - 1 \geq \frac{x_i + t + s_i}{x_i + s_i} - 1 \implies \frac{t}{x_l - t + s_l} \geq \frac{t}{x_i + s_i},$$

give

$$x_i + s_i \geq x_l - t + s_l.$$

This implies that

$$\sum_{i\in\{1,\dots,n\}\setminus\{l\}}(x_i+s_i)\ge\sum_{i\in\{1,\dots,n\}\setminus\{l\}}(x_l-t+s_l)=(n-1)x_l+(n-1)(s_l-t).$$

Therefore

$$L+\sum_{j=1}^{n}s_j=x_l+s_l+\sum_{i\in\{1,\dots,n\}\setminus\{l\}}(x_i+s_i)$$
$$\ge x_l+s_l+(n-1)x_l+(n-1)(s_l-t)=nx_l+ns_l-(n-1)t,$$

whence

$$q=\frac{L+\sum_{j=1}^{n}s_j}{n}\ge x_l+s_l-\frac{n-1}{n}t\ge x_l+s_l-t>k_l+s_l.$$

Consequently, $[q]\ge k_l+s_l$, which contradicts our choice of l: $[q]<k_l+s_l$. Therefore, (2.21) implies (2.22) and hence (AS2) is proved.

Step 2. It remains to prove (3.21).

Fix $(x_1,\dots,x_n)\in\mathbb{Z}_+(L,n;\mathbf{k}_n)$ satisfying (3.22). From (AS2) proved in *Step 1*, we can write

$$\prod_{j=1}^{m}(x_{i_j}+s_{i_j})\prod_{j\in\mathcal{I}_2}(k_j+s_j)=b(L,n;\mathbf{k}_n,\mathbf{s}_n;\mathbb{Z}_+) \tag{3.24}$$

and

$$\sum_{j=1}^{m}x_{i_j}=L_m. \tag{3.25}$$

From (3.25) and the definition of $b(L_m,m;\mathbf{k}_m,\mathbf{s}_m;\mathbb{Z}_+)$, we can write

$$\prod_{j=1}^{m}(x_{i_j}+s_{i_j})\le b(L_m,m;\mathbf{k}_m,\mathbf{s}_m;\mathbb{Z}_+). \tag{3.26}$$

Fix now some $(y_1,\dots,y_m)\in\mathbb{Z}_+(L_m,m;\mathbf{k}_m)$ such that

$$\prod_{j=1}^{m}(y_j+s_{i_j})=b(L_m,m;\mathbf{k}_m,\mathbf{s}_m;\mathbb{Z}_+). \tag{3.27}$$

Consider now the indices $i_{m+1},\dots,i_n$ such that

$$\mathcal{I}_2=\{i_{m+1},\dots,i_n\}.$$

Since $(y_1,\dots,y_m,k_{i_{m+1}},\dots,k_{i_n})\in\mathbb{Z}_+(L,n;\mathbf{k}_n)$, from (3.22), we can deduce

$$\prod_{j=1}^{m}(y_j+s_{i_j})\prod_{j=m+1}^{n}(k_{i_j}+s_{i_j})\le\prod_{j=1}^{m}(x_{i_j}+s_{i_j})\prod_{j\in\mathcal{I}_2}(k_j+s_j).$$

Hence

$$\prod_{j=1}^{m}(y_j+s_{i_j})\le\prod_{j=1}^{m}(x_{i_j}+s_{i_j}).$$

This inequality along with equality (3.27) give

$$b(L_m,m;\mathbf{k}_m,\mathbf{s}_m;\mathbb{Z}_+)\le\prod_{j=1}^{m}(x_{i_j}+s_{i_j}). \tag{3.28}$$

From (3.28) and (3.26), we obtain

$$b(L_m,m;\mathbf{k}_m,\mathbf{s}_m;\mathbb{Z}_+)=\prod_{j=1}^{m}(x_{i_j}+s_{i_j}).$$

From this equality and (3.24), we conclude that (3.21) is true.

(d) follows from (3.21) of (c) because $b(L_1,1;s_{i_1};\mathbb{Z}_+)=L_1+s_{i_1}$. □

Remark 3.4. Theorem 3.3(c) differs slightly from Theorem 2.5(c); it does not contain the following statement:

$$r \neq 0 \ \text{ and } \ q < \beta \Longrightarrow b(L, n; \mathbf{k}_n, \mathbf{s}_n; \mathbb{Z}_+) < (1 + [q])^r [q]^{n-r}.$$

This implication may not be true, in general, as the following example shows.

Example. Take $n = 2$, $k_1 = k_2 = 0$, $s_1 = 0$ and $s_2 = L + 1$. We shall have $[q] = L < L + 1 = s_2 + k_2$, $r = 1$; however,

$$b\,(L, 2; (0,0); (0, L+1); \mathbb{Z}_+) = L(L+1) = (1 + [q])[q].$$

Remark 3.5. As in the case of Theorem 2.5, we note that Theorem 3.3(b) gives the complete solution of Problem 2.1, while Theorem 3.3(c) fails to give a complete solution of Problem 2.1, because it, (again for $n > 2$ and $m > 1$), does not provide a **formula** for $b(L, n; \mathbf{k}_n; \mathbf{s}_n; \mathbb{Z}_+)$; it provides only a reduction of the problem to the case $m < n$.

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References

1. E. F. Beckenbach, R. Bellman, *Inequalities.* Ergebnisse der Mathematik und ihrer Grenzgebiete, (N.F.), Band 30. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1961.
2. P. S. Bullen, D. S. Mitrinović, P. M. Vasić, *Means and their Inequalities.* Translated and revised from the Serbo-Croatian. Mathematics and its Applications (East European Series), 31. D. Reidel Publishing Co., Dordrecht, 1988.
3. P. S. Bullen, *Handbook of Means and their Inequalities.* Mathematics and its Applications, 560. Kluwer Academic Publishers Group, Dordrecht, 2003.
4. G. Chelidze, M. Nikoleishvili, V. Tarieladze, On a problem of integer valued optimization. *Semin. I. Vekua Inst. Appl. Math. Rep.* **47** (2021), 21–25.
5. G. Chelidze, V. Kvaratskhelia, M. Nikoleishvili, *A discrete extremum problem*, (preprint) 2012.
6. K. M. Chong, An inductive proof of the A.M.-G.M. inequality. *Amer. Math. Monthly* **83** (1976), no. 5, 369.
7. H. Dörrie, Über einige Anwendungen des Satzes vom arithmetischen und geometrischen Mittel. *Z. Math. Naturw. Unterr* **52** (1921), 103–108.
8. H. Dörrie, *100 Great Problems of Elementary Mathematics: Their History and Solution.* translated by David Antin, New York, Dover Publications, Inc., 1965.
9. D. J. H. Garling, *Inequalities: a Journey into Linear Analysis.* Cambridge University Press, Cambridge, 2007.
10. P. P. Korovkin, *Inequalities.* (Russian) Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow-Leningrad, 1952.
11. H. Kreis, Arithmetisches und geometrisches Mittel. (German) *Elem. Math.* **1** (1946), 37–38.
12. A. Lohwater, *Introduction to Inequalities.* Marjorie Lohwater, 1982.
13. L. Maligranda, The AM-GM inequality is equivalent to the Bernoulli inequality. *Math. Intelligencer* **34** (2012), no. 1, 1–2.
14. P. R. Mercer, *More Calculus of a Single Variable.* Undergraduate Texts in Mathematics. Springer, New York, 2014.
15. M. Nikoleishvili, V. Tarieladze, About a problem of extremum. In: *Proceedings of the Fifth International Scientific-Practical Conference "Scientific Issues of the Modernity" at Sukhishvili Teaching University*, 401–403, Gori, Georgia, December, 13, 2014.
16. J. A. Stedall, *The Greate Invention of Algebra.* Thomas Harriot's treatise on equations. Oxford University Press, Oxford, 2003.
17. J. F. Steffensen, The geometrical mean. *J. Institute Actuaries* **62**(1931), no. 1, 117–118.
18. V. M. Tikhomirov, *Stories About Maxima and Minima.* Translated from the 1986 Russian original by Abe Shenitzer. Mathematical World, 1. American Mathematical Society, Providence, RI; Mathematical Association of America, Washington, DC, 1990.
19. `https://en.wikipedia.org/wiki/Inequalityofarithmeticandgeometricmeans`.

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[1]Muskhelishvili Institute of Computational Mathematics of the Georgian Technical University, 0159 Tbilisi, Georgia

[2]Kutaisi International University, Youth Avenue, Turn 5/7, 4600 Kutaisi, Georgia

[3]Georgian Institute of Public Affairs (GIPA), 0105, Tbilisi, Georgia
Email address: g.chelidze@gtu.ge; giorgi.chelidze@kiu.edu.ge
Email address: mikheil.nikoleishvili@gmail.com
Email address: vajatarieladze@yahoo.com; v.tarieladze@gtu.ge