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PARTIAL EQUIPMENT OF THE MANIFOLD TANGENT SPACE $TT(T(Vn))$

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. In this paper we consider the manifold tangent space $T(T(Vn))$ of the tangent space $T(Vn)$. Invariant I-forms of the space $T(T(Vn))$ are defined and their structural equations are obtained. Linear connections of the space $T(T(Vn))$ are established in the case of partial equipment of the tangent space $T(T(Vn))$.

It is well known that various geometrical structures on a given space are defined by concrete fields of differential-geometric objects. If new fibers join the given fiber spaces, then for these new fibers there sometimes appear new geometric structures generated by the original structures. In that case the new geometrical structures are regarded as peculiar analogs of the facts of internal geometry of the equipped surface.

Let us consider the manifold tangent space $T(T(Vn))$ with local coordinates $(x^i, y^{\bar{i}}, y^i, z^{\bar{i}})$, $i,j,k = \overline{1,n}$, $\bar{i},\bar{j},\bar{k} = \overline{1,n}$, where $x^i, y^{\bar{i}}$ are the coordinates of the basis $T(Vn)$, and y^i, $z^{\bar{i}}$ are the coordinates of the fiber T_z, $z \in T(Vn)$. In other words, the vector fields $\mathfrak{X}$

$$\mathfrak{X} = y^i \frac{\partial}{\partial x^i} + z^{\bar{i}} \frac{\partial}{\partial y^{\bar{i}}}$$

generate the fiber space $T(T(Vn))$. Then it is obvious that the local coordinates $(x^i, y^{\bar{i}}, y^i, z^{\bar{i}})$ of a point of the fiber space $T(T(Vn))$ transform as follows:

$$\overline{x}^i = \overline{x}^i(x^k), \quad \overline{y}^i = x^i_k y^k, \quad \overline{y}^{\bar{i}} = x^{\bar{i}}_{\bar{k}} y^{\bar{k}}, \quad \overline{z}^{\bar{i}} = x^{\bar{i}}_{\bar{k}} z^{\bar{k}} + x^{\bar{i}}_{\bar{k}j} y^{\bar{k}} y^j.$$

On the space $T(T(Vn))$ we can define the following forms:

$$\theta^i = dy^i + \omega^i_k y^k, \quad \theta^{\bar{i}} = dy^{\bar{i}} + \omega^{\bar{i}}_{\bar{k}} y^{\bar{k}}, \quad \vartheta^{\bar{i}} = dz^{\bar{i}} + \omega^{\bar{i}}_{\bar{k}} z^{\bar{k}} + \omega^{\bar{i}}_{\bar{k}j} y^{\bar{k}} y^j.$$

Performing an external differentiation of these equalities and using the structural equations for the I-form ω^i_k, $\omega^{\bar{i}}_{\bar{k}}$, $\omega^{\bar{i}}_{\bar{k}j}$ [2] we obtain

$$\begin{aligned} D\theta^i &= \theta^k \wedge \omega^i_k + \omega^k \wedge \theta^i_k, \\ D\theta^{\bar{i}} &= \theta^{\bar{k}} \wedge \omega^{\bar{i}}_{\bar{k}} + \omega^k \wedge \theta^{\bar{i}}_k, \\ D\vartheta^{\bar{i}} &= \vartheta^{\bar{k}} \wedge \omega^{\bar{i}}_{\bar{k}} + \theta^{\bar{k}} \wedge \theta^{\bar{i}}_{\bar{k}} + \theta^k \wedge \theta^{\bar{i}}_k + \omega^k \wedge \vartheta^{\bar{i}}_k, \end{aligned}$$

where

$$\theta^{\bar{i}}_k = \omega^{\bar{i}}_{\bar{j}k} y^{\bar{j}}, \ \theta^i_k = \omega^i_{kj} y^j, \quad \theta^{\bar{i}}_{\bar{k}} = \omega^{\bar{i}}_{\bar{k}j} y^j, \quad \vartheta^{\bar{i}}_j = \omega^{\bar{i}}_{\bar{k}j} z^{\bar{k}} + \omega^{\bar{i}}_{\bar{k}ij} y^{\bar{k}} y^i.$$

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From the transformation law of local coordinates of the tangent space $T(T(Vn))$ it follows that

$$d\overline{x}^i = x^i_k dx^k,$$
$$d\overline{y}^i = x^i_{kj} y^k dx^j + x^i_k dy^k,$$
$$d\overline{y}^{\overline{i}} = x^{\overline{i}}_{\overline{k}j} y^{\overline{k}} dx^j + x^{\overline{i}}_{\overline{k}} dy^{\overline{k}},$$
$$d\overline{z}^{\overline{i}} = (x^{\overline{i}}_{\overline{k}j} z^{\overline{k}} + x^{\overline{i}}_{\overline{k}pj} y^{\overline{k}} y^p) dx^j + x^{\overline{i}}_{\overline{k}j} y^j dy^{\overline{k}} + x^{\overline{i}}_{\overline{k}j} y^{\overline{k}} dy^j + x^{\overline{i}}_{\overline{k}} dz^{\overline{k}}.$$

The quantities $\{dx^k, dy^{\overline{k}}, dy^k, dz^{\overline{k}}\}$ define the co-basis of the co-tangent space $\overset{*}{T}T(T(Vn))$.

It is obvious that the space $\overset{*}{T}T(T(Vn))$ always has an invariant subspace spanned over the co-basis $\{dx^k\}$. Let us consider the case when the space $\overset{*}{T}T(T(Vn))$ is partially equipped. Note that the partial equipment of the space $\overset{*}{T}T(T(Vn))$ is understood in the sense that with the change of coordinates the co-basis transformation matrix takes the form:

$$\begin{Vmatrix} x^i_k & 0 & 0 & 0 \\ x^{\overline{i}}_{\overline{k}j} y^{\overline{k}} & x^{\overline{i}}_{\overline{k}} & 0 & 0 \\ 0 & 0 & x^i_k & 0 \\ 0 & 0 & x^{\overline{i}}_{\overline{k}j} y^{\overline{k}} & x^{\overline{i}}_{\overline{k}} \end{Vmatrix}. \tag{1}$$

From formulas (1) we see that the space $\overset{*}{T}T(T(Vn))$ has one more invariant subspace which is defined by means of the co-basis Dy^i:

$$Dy^i = dy^i + \Gamma^i_j dx^j.$$

From the condition of invariance it follows that the quantities Γ^i_j generate the differential-geometric object according to the following transformation law:

$$\overline{\Gamma}^i_k = x^i_p \overset{*}{x}{}^q_k \Gamma^p_q - \overset{*}{x}{}^q_k x^i_{qj} y^j. \tag{2}$$

Moreover, in order that the space $\overset{*}{T}T(T(Vn))$ be partially equipped, we have to define, as seen from formulas (1), a partially invariant subspace. The latter is defined by means of the co-basis $Dz^{\overline{i}}$:

$$Dz^{\overline{i}} = dz^{\overline{i}} + G^{\overline{i}}_k Dy^k + M^{\overline{i}}_{\overline{k}} dy^{\overline{k}}.$$

By the partial invariance condition we have

$$D\overline{z}^{\overline{i}} = x^{\overline{i}}_{\overline{j}} Dz^{\overline{j}} + x^{\overline{i}}_{\overline{j}k} y^{\overline{j}} Dy^k.$$

It follows that the quantities $G^{\overline{i}}_k$, $M^{\overline{i}}_{\overline{k}}$ generate the differential-geometric object according to the following transformation law of its components:

$$x^{\overline{i}}_{\overline{j}} M^{\overline{j}}_{\overline{k}} = x^{\overline{j}}_{\overline{k}} \overline{M}^{\overline{i}}_{\overline{j}} + x^{\overline{i}}_{\overline{k}j} y^j, \tag{3}$$
$$x^{\overline{i}}_{\overline{j}} G^{\overline{j}}_k = x^p_k \overline{G}^{\overline{i}}_p + \overline{M}^{\overline{i}}_{\overline{j}} x^{\overline{j}}_{\overline{p}k} y^{\overline{p}} - x^{\overline{i}}_{\overline{j}p} y^{\overline{j}} \Gamma^p_k + x^{\overline{i}}_{\overline{j}k} z^{\overline{j}} + x^{\overline{i}}_{\overline{j}pk} y^{\overline{j}} y^p. \tag{4}$$

The quantities Γ^i_j, $G^{\overline{i}}_k$, $M^{\overline{i}}_{\overline{k}}$ define the partial equipment of the space $T(T(Vn))$. Also, the quantities $\Gamma^i_j, G^{\overline{i}}_k$, $M^{\overline{i}}_{\overline{k}}$ define the differental-geometric object of linear connection of the space $T(T(Vn))$, which we call the triangular co-connection of the space $T(T(Vn))$.

In that case, after the change of the local coordinates the reference frame

$$\left\{ \frac{\partial}{\partial x^i}, \frac{\partial}{\partial y^{\overline{i}}}, \frac{\partial}{\partial y^i}, \frac{\partial}{\partial z^{\overline{i}}} \right\}$$

of the tangent fiber space $TT(T(Vn))$ transforms as follows:

$$\frac{\partial}{\partial z^{\bar{i}}} = x^{\bar{k}}_{\bar{i}} \frac{\partial}{\partial \overline{z}^{\bar{k}}}\,,$$

$$\frac{\partial}{\partial y^{k}} = x^{\bar{i}}_{\bar{j}k} y^{\bar{j}} \frac{\partial}{\partial \overline{z}^{\bar{i}}} + x^{i}_{k} \frac{\partial}{\partial \overline{y}^{i}}\,,$$

$$\frac{\partial}{\partial y^{\bar{j}}} = x^{\bar{i}}_{\bar{j}k} y^{k} \frac{\partial}{\partial \overline{z}^{\bar{i}}} + x^{\bar{i}}_{\bar{j}} \frac{\partial}{\partial \overline{y}^{\bar{i}}}\,,$$

$$\frac{\partial}{\partial x^{k}} = (x^{\bar{i}}_{\bar{j}k} z^{\bar{j}} + x^{\bar{i}}_{\bar{j}pk} y^{\bar{j}} y^{p}) \frac{\partial}{\partial \overline{z}^{\bar{i}}} + x^{i}_{kp} y^{p} \frac{\partial}{\partial \overline{y}^{i}} + x^{\bar{i}}_{\bar{j}k} y^{\bar{j}} \frac{\partial}{\partial \overline{y}^{\bar{i}}} + x^{i}_{k} \frac{\partial}{\partial \overline{x}^{i}}\,.$$

The triangular co-connection always generates a triangular connection by means of which the partial equipment of the space $TT(T(Vn))$ is defined (and vice versa).

The partial equipment of the tangent space $TT(T(Vn))$ can be defined by using the vectors $D_{\bar{i}}$ and D_i:

$$D_{\bar{i}} = \frac{\partial}{\partial y^{\bar{i}}} - G^{\bar{j}}_{\bar{i}} \frac{\partial}{\partial z^{\bar{j}}},$$

the invariance condition

$$D_{\bar{i}} = x^{\bar{j}}_{\bar{i}} \overline{D}_{\bar{j}}$$

and

$$D_i = \frac{\partial}{\partial x^i} - E^{\bar{j}}_{i} D_{\bar{j}} - L^{k}_{i} \frac{\partial}{\partial y^k}.$$

The partial invariance condition is

$$D_i = x^{k}_{i} \overline{D}_k + x^{\bar{j}}_{\bar{k}i} y^{\bar{k}} \overline{D}_{\bar{j}}.$$

It is obvious that the quantities $L^k_i, G^{\bar{j}}_{\bar{i}}, E^{\bar{j}}_i$ generate differential-geometric objects and a higher-order definite connection. We call this connection the triangular connection of the tangent space $TT(T(Vn))$.

Let us consider the fiber space $T(Vn)$ with triplet connection Γ^i_j, $\Gamma^{i}_{\bar{k}j}, \Gamma^i_{jk}$. There arises a question whether it is possible to construct the triangular connection by using the triplet connection and its differential continuation, i.e. to define the internal object of the triplet connection by means of the triplet connection. It turns out that the answer to this question is positive and, what is more, the triangular connection can be constructed without using the linear connection Γ^i_j, but using only the object of the generated vertical affine connection $\Gamma^{i}_{\bar{k}j}$, affine connection Γ^i_{jk} and the partial continuation of the object of connection $\Gamma^{i}_{\bar{k}j}$.

We introduce the notations:

$$C^i_j \equiv \Gamma^i_{jk} y^k, \quad N^{\bar{i}}_{\bar{j}} \equiv \Gamma^{i}_{\bar{j}k} y^k,$$

$$P^{\bar{k}}_{i} \equiv \Gamma^{k}_{\bar{j}i} z^{\bar{j}} + \Gamma^{k}_{\bar{j}ip} y^p y^{\bar{j}} + \Gamma^{k}_{\bar{j}i} \Gamma^{j}_{\bar{p}q} y^{\bar{p}} y^q + \Gamma^{p}_{\bar{j}i} \Gamma^{k}_{\bar{p}q} y^{\bar{j}} y^q - \Gamma^k_{pj} \Gamma^p_{iq} y^j y^q.$$

The transformation law of these quantities has the form:

$$x^i_k C^k_p = x^k_p \overline{C}^i_k + x^i_{kp} y^k, \tag{5}$$

$$x^{\bar{i}}_{\bar{j}} N^{\bar{j}}_{\bar{k}} = x^{\bar{j}}_{\bar{k}} \overline{N}^{\bar{i}}_{\bar{j}} + x^{\bar{i}}_{\bar{k}p} y^p, \tag{6}$$

$$x^{\bar{i}}_{\bar{k}} P^{\bar{k}}_{j} = x^k_j \overline{P}^{\bar{i}}_k + \overline{N}^{\bar{i}}_{\bar{k}} x^{\bar{k}}_{\bar{p}j} y^{\bar{p}} - x^{\bar{i}}_{\bar{p}k} y^{\bar{p}} C^k_j + x^{\bar{i}}_{\bar{k}j} z^{\bar{k}} + x^{\bar{i}}_{\bar{k}pj} y^{\bar{k}} y^p. \tag{7}$$

Formulas (2), (3), (4) and (5), (6), (7) show that the quantities $\Gamma^i_j, G^{\bar{i}}_{k}$, $M^{\bar{i}}_{\bar{k}}$ and C^i_j, $N^{\bar{i}}_{\bar{j}}$, $P^{\bar{k}}_i$ obey one and the same transformation law of their components. Hence we make the following conclusion.

Theorem. *The object of affine connection Γ^i_{jk}, object of connection $\Gamma^{i}_{\bar{k}j}$ and its partial differential continuation $\Gamma^{k}_{\bar{j}ip}$ always generate the partial equipment of the fibered space $TT(T(Vn))$* [1–7].

References

1. F. I. Kagan, Affine connections on a tangent bundle. (Russian) *Izv. Vysš. Učebn. Zaved. Matematika* **1975**, no. 2(153), 31–42.
2. G. Todua, On internal tensor structures of the tangent bundle of space $L_m(V_n)$ with a triplet connection. *Bull. Georgian Natl. Acad. Sci.* **173** (2006), no. 1, 22–25.
3. G. Todua, Lifts. On tensor structures. *Trans. A. Razmadze Math. Inst.* **170** (2016), no. 2, 272–279.
4. G. Todua, On the internal tensor structures of the fibration $T(\mathrm{Lm}(Vn))$. *Trans. A. Razmadze Math. Inst.* **172** (2018), no. 1, 126–132.
5. G. Todua, On the vector fiber surface of the space $Lm(Vn)$ of triplet connectedness. *Trans. A. Razmadze Math. Inst.* **174** (2020), no. 3, 377–379.
6. G. Todua, On the F structures of the space $T(Lm(Vn))$. *Trans. A. Razmadze Math. Inst.* **175** (2021), no. 1, 143–147.
7. K. Yano, Sh. Kobayashi, Prolongations of tensor fields and connections to tangent bundles. I. General theory. *J. Math. Soc. Japan* **18** (1966), 194–210.

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