Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 3, 505–507

TWO-WEIGHT CRITERIA FOR MULTIPLE FRACTIONAL INTEGRALS IN MIXED-NORMED LEBESGUE SPACES

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. Two-weight norm estimates for multiple fractional integral operators are established in the mixed-normed Lebesgue spaces. As a consequence, we have a complete characterization of the trace inequality for a strong fractional maximal operator $M_{\overrightarrow{\alpha}}$. Fefferman–Stein type inequality for $M_{\overrightarrow{\alpha}}$ is also derived. Finally, we present one-weight characterization for multiple Riesz potential operator in mixed-normed Lebesgue spaces.

1. Preliminaries

Our aim is to study the two-weight norm inequality

$$\Big\|V(M_{\overrightarrow{\alpha}} f)\Big\|_{L^{\overrightarrow{q}}} \leq C\Big\|Wf\Big\|_{L^{\overrightarrow{p}}}, \quad \overrightarrow{q} := (q_1, \dots, q_n), \quad \overrightarrow{p} := (p_1, \dots, p_n), \tag{1}$$

where $L^{\overrightarrow{p}}$ and $L^{\overrightarrow{q}}$ are the mixed-normed Lebesgue spaces defined on $\mathbb{R}^{d\times n}$, $M_{\overrightarrow{\alpha}}$ is the strong fractional maximal operator and V and W are weight functions on $\mathbb{R}^{d\times n}$. Here, under the symbol $\mathbb{R}^{d\times n}$ we mean an n-fold Cartesian product of $\mathbb{R}^d$, i.e., $\mathbb{R}^{d\times n} := \mathbb{R}^d \times \cdots \times \mathbb{R}^d$.

If $\overrightarrow{p} = (p, \dots, p)$, then $L^{\overrightarrow{p}}$ coincides with the classical Lebesgue space $L^p(\mathbb{R}^{d\times n})$.

As a consequence, we have a complete characterization of the D. Adams-type [1] trace inequality

$$\Big\|V(M_{\overrightarrow{\alpha}} f)\Big\|_{L^{\overrightarrow{q}}} \leq C\Big\|f\Big\|_{L^{\overrightarrow{p}}} \tag{2}$$

for strong fractional maximal operator $M_{\overrightarrow{\alpha}}$.

Throughout this note, by the symbols $\overrightarrow{p}$ and $\overrightarrow{q}$ we mean $n-$ tuples $(n \geq 1)$, $\overrightarrow{p} = (p_1, \dots, p_n)$ and $\overrightarrow{q} = (q_1 \dots, q_n)$, respectively. The relation $\overrightarrow{p} < \overrightarrow{q}$ means that $p_j < q_j$ for every $j = 1, \dots, n$. The identity $\varphi(\overrightarrow{p}, \overrightarrow{q}, \dots, \overrightarrow{\alpha}) = 0$ means that $\varphi(p_j, q_j, \dots, \alpha_j) = 0$, $j = 1, \dots, n$. We also assume that $\overrightarrow{p}' = (p_1', \dots, p_n')$, where $p_j' = \frac{p_j}{p_j - 1}$, $j = 1, \dots, n$.

The space $L^{\overrightarrow{p}}$, $1 < \overrightarrow{p} < \infty$, is defined with respect to the norm

$$\begin{aligned}\|\varphi\|_{L^{\overrightarrow{p}}} &:= \|\varphi\|_{L^{\overrightarrow{p}}(\mathbb{R}^{d\times n})} := \Big\|\Big\|\cdots\|\varphi\|_{L^{p_1}(\mathbb{R}^d)}\cdots\Big\|_{L^{p_{n-1}}(\mathbb{R}^d)}\Big\|_{L^{p_n}(\mathbb{R}^d)} \\ &:= \Bigg\{\int\limits_{\mathbb{R}^d}\Bigg[\int\limits_{\mathbb{R}^d}\cdots\Bigg[\int\limits_{\mathbb{R}^d}|\varphi(x_1, \dots, x_n)|^{p_1}dx_1\Bigg]^{\frac{p_2}{p_1}}\cdots dx_{n-1}\Bigg]^{\frac{p_n}{p_{n-1}}}dx_n\Bigg\}^{\frac{1}{p_n}}\end{aligned}$$

which is reflexive, and its dual space is $L^{\overrightarrow{p}'}$. For this and some other essential properties of the space $L^{\overrightarrow{p}}$ we refer, e.g., to [2].

It is easy to see that in a particular case, where $\varphi(x_1, \dots, x_n) = \varphi_1(x_1)\dots\varphi_n(x_n)$, we have

$$\|\varphi\|_{L^{\overrightarrow{p}}(\mathbb{R}^{d\times n})} = \prod_{j=1}^{n}\|\varphi_j\|_{L^{p_j}(\mathbb{R}^d)}.$$

For $f \in L_{\text{loc}}(\mathbb{R}^{d\times n})$, a strong fractional maximal operator $M_{\overrightarrow{\alpha}}$ is defined as follows:

2020 *Mathematics Subject Classification.* 26A33, 42B35, 47B38.

Key words and phrases. Strong fractional maximal operator; Mixed normed Lebesgue spaces; Fefferman-Stein inequality; One-weight inequality; Two-weight inequality.

$$(M_{\overrightarrow{\alpha}} f)(x_1,\dots,x_n) = \sup \frac{1}{\prod_{j=1}^n |Q_j|^{1-\frac{\alpha_j}{d}}} \int\limits_{Q_1\times\dots\times Q_n} |f(y_1,\dots,y_n)|dy_1\dots dy_n,$$

where the supremum is taken over all cubes $Q_j \subset \mathbb{R}^d$ with sides parallel to the coordinate axes such that $x_j \in Q_j$, $j = 1,\dots,n$, $\overrightarrow{\alpha} = (\alpha_1,\dots,\alpha_n)$.

It can be checked immediately that there is a positive constant $C_{\overrightarrow{\alpha},n,d}$ such that for $f \geq 0$,

$$(M_{\overrightarrow{\alpha}} f)(x_1,\dots,x_n) \leq C_{\overrightarrow{\alpha},n,d}(I_{\overrightarrow{\alpha}} f)(x_1,\dots,x_n),$$

where $I_{\overrightarrow{\alpha}} f$ is the multiple fractional integral operator given by the formula

$$(I_{\overrightarrow{\alpha}} f)(x_1,\dots,x_n) = \int\limits_{\mathbb{R}^n}\cdots\int\limits_{\mathbb{R}^n} \frac{f(y_1,\dots,y_n)}{\prod_{j=1}^n |x_j-y_j|^{d-\alpha_j}} dy_1\cdots dx_n.$$

The one–weight problem for the classical fractional maximal operator

$$(M_\alpha f)(x) = \sup_{Q\ni x} \frac{1}{|Q|^{1-\frac{\alpha}{d}}} \int\limits_Q |f(y)|dy, \quad 0<\alpha<d, \quad x\in\mathbb{R}^d,$$

was solved by B. Muckenhoupt and R. L. Wheeden [6] under the so-called Muckenhoupt-Wheeden condition. For the solution of the two–weight problem for M_α under various type of conditions we refer to [7,8] (see also monograph [3, Ch. 4]).

The two-weight problem for the operator $M_{\overrightarrow{\alpha}}$ in the case $p_1 = \dots = p_n$ was studied in [4]. The complete characterization of the one-weight problem for the strong Hardy–Littlewood maximal operator for multiple weight $w(x_1,\dots,x_n) = w_1(x_1)\dots w_n(x_n)$ was obtained by D. Kurtz [5].

2. Main Results

To formulate the general statement of this note, we need the following

Definition 2.1. Let $\mathcal{D}$ be a dyadic grid in $\mathbb{R}^d$. We say that a weight function ρ on $\mathbb{R}^d$ satisfies the dyadic reverse doubling condition ($\rho \in RD^d(\mathbb{R}^d)$) if there is a constant $C>1$ such that for all $Q, Q' \in \mathcal{D}$ with $Q' \subset Q$, $|Q| = 2^d|Q'|$, the inequality

$$C\int\limits_{Q'} \rho(x)dx \leq \int\limits_Q \rho(x)dx$$

holds.

Definition 2.2. We say that $\overrightarrow{p} \prec \overrightarrow{q}$ if $1 < \max\{p_j\}_{j=1}^n < \min\{q_j\}_{j=1}^n < \infty$. Further, we say that $1 < \overrightarrow{r} < \infty$ if $1 < \min\{r_1,\dots,r_n\} \leq \max\{r_1,\dots,r_n\} < \infty$.

Theorem A. *Let $1 < \overrightarrow{p} \prec \overrightarrow{q} < \infty$ and let V and W be the weights of the functions on $\mathbb{R}^{d\times n}$, provided that $W(x_1,\dots,x_n) = \prod_{j=1}^n W_j(x_j)$ with $W_j^{-p_j'} \in RD(\mathbb{R}^d)$, $j=1,\dots,n$. Then two-weight inequality* (1) *holds if and only if*

$$\sup_{Q_1,\dots,Q_n} \Big\|\chi_{Q_1}(x_1)\dots\chi_{Q_n}(x_n)V(x_1,\dots,x_n)\Big\|_{L^{\overrightarrow{q}}}$$
$$\times \Big\|\chi_{Q_1}(x_1)\dots\chi_{Q_n}(x_n)W^{-1}(x_1,\dots,x_n)\Big\|_{L^{\overrightarrow{p}'}} \prod_{j=1}^n |Q_j|^{\frac{\alpha_j}{d}-1} < \infty.$$

Theorem A implies the following statements.

Theorem B (Characterization of the trace inequality)**.** *Let $1 < \overrightarrow{p} \prec \overrightarrow{q} < \infty$ and let V be a weight function on $\mathbb{R}^{d\times n}$. Then the trace inequality* (2) *holds if and only if*

$$\sup_{Q_1,\dots,Q_n} \Big\|\chi_{Q_1}(x_1)\dots\chi_{Q_n}(x_n)V(x_1,\dots,x_n)\Big\|_{L^{\overrightarrow{q}}} \prod_{j=1}^n |Q_j|^{\frac{\alpha_j}{d}-\frac{1}{p_j}} < \infty.$$

Theorem C (Fefferman–Stein inequality)**.** *Let* $1 < \overrightarrow{p} \preceq \overrightarrow{q} < \infty$, $\overrightarrow{\alpha} < \frac{d}{\overrightarrow{p}}$, *and let* V *be weight function on* $\mathbb{R}^{d\times n}$. *Then the following inequality*

$$\Big\|V\big(M_{\overrightarrow{\alpha}} f\big)\Big\|_{L^{\overrightarrow{q}}} \le C\Big\|f\big(\widetilde{M}_{\overrightarrow{p},\overrightarrow{q},\overrightarrow{\alpha}} V\big)\Big\|_{L^{\overrightarrow{p}}}$$

holds, where $\widetilde{M}_{\overrightarrow{p},\overrightarrow{q},\overrightarrow{\alpha}}$ *is the fractional maximal operator having the form*

$$(\widetilde{M}_{\overrightarrow{p},\overrightarrow{q},\overrightarrow{\alpha}} V)(x_1,\dots,x_n) := \sup \frac{1}{\prod\limits_{j=1}^{n} |Q_j|^{\frac{1}{p_j}-\frac{\alpha_j}{d}}} \Big\|\chi_{Q_1}(x_1)\dots\chi_{Q_n}(x_n)V(x_1,\dots,x_n)\Big\|_{L^{\overrightarrow{q}}}$$

and the supremum is taken over all cubes $Q_j \subset \mathbb{R}^d$ *with sides parallel to the coordinate axis such that* $x_j \in Q_j$, $j = 1,\dots,n$.

Finally we formulate the one-weight characterization for operators $M_{\overrightarrow{\alpha}}$ and $I_{\overrightarrow{\alpha}}$:

Theorem D (One-weight characterization)**.** *Let* $1 < \overrightarrow{p} < \infty$, $\frac{1}{\overrightarrow{p}} - \frac{1}{\overrightarrow{q}} = \frac{\overrightarrow{\alpha}}{d}$. *Let* $W(x_1,x_2,\dots,x_n) = W_1(x_1)\dots W_n(x_n)$, *where* W_j *are weight functions on* $\mathbb{R}^d$, $j = 1,\dots,n$. *Then the following statements are equivalent:*

(i) the one-weight inequality

$$\Big\|W(I_{\overrightarrow{\alpha}} f)\Big\|_{L^{\overrightarrow{q}}} \le C\Big\|Wf\Big\|_{L^{\overrightarrow{p}}}$$

holds;

(ii) the one-weight inequality

$$\Big\|W(M_{\overrightarrow{\alpha}} f)\Big\|_{L^{\overrightarrow{q}}} \le C\Big\|Wf\Big\|_{L^{\overrightarrow{p}}}$$

is fulfilled;

(iii)

$$\sup_{Q_1,\dots,Q_n} \Big\|\chi_{Q_1}(x_1)\dots\chi_{Q_n}(x_n)W(x_1,\dots,x_n)\Big\|_{L^{\overrightarrow{q}}} \times \Big\|\chi_{Q_1}(x_1)\dots\chi_{Q_n}(x_n)W^{-1}(x_1,\dots,x_n)\Big\|_{L^{\overrightarrow{p}'}} \prod_{j=1}^{n} |Q_j|^{\frac{\alpha_j}{d}-1} < \infty.$$

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(Received 15.07.2023)

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