ON SOME PROPERTIES OF UNIFORM DISTRIBUTION SEQUENCES  

ALEKS KIRTADZE  

Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. Some properties of uniform distribution sequences for invariant extensions of linear Lebesgue measures are considered.

For a real number \( x \), let \( x = x - [x] \) be a fractional part of \( x \), where \( [x] \) denotes the integer part of \( x \), that is, the greatest integer which is less or equal to \( x \). Let \( \{x_n : n \in \mathbb{N}\} \) be a given sequence of real numbers. Notice that the fractional part of any real number is contained in the unit interval \( I = [0, 1) \).

A sequence of real numbers \( \{x_n : n \in \mathbb{N}\} \) is said to be uniformly distributed sequence modulo 1 (abbreviated u.d.s. mod 1) if for each \( a, b \), with \( 0 \leq a < b \leq 1 \), we have

\[
\lim_{n \to \infty} \frac{\text{card}(\{x_k : 1 \leq k \leq n\} \cap [a, b])}{n} = b - a.
\]

The above-mentioned equation can be written in the following form:

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} 1_{[a,b)}(x_k) = \int_{\mathbb{R}} 1_{[a,b)}(x) \, dx,
\]

where \( 1_{[a,b)} \) denotes the characteristic function on the interval \([a,b) \subset I\).

The following theorem is valid.

Theorem 1. The sequence \( \{x_n : n \in \mathbb{N}\} \) of real numbers is u.d.s. mod 1 if and only if for every real-valued continuous function \( f \) defined on the closed interval \( I = [0,1] \); we have

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} f(x_k) = \int_{0}^{1} f(x) \, dx.
\]

(For the above definitions and theorem, see [1–3, 6–8]).

In the present paper, an approach to some questions of the theory of uniform distribution sequences is discussed. Such an approach is suitable for certain situations, where the given \([0,1]\) interval is equipped with the class of invariant extensions of the linear Lebesgue measure on \([0,1]\), and in this case we consider the theorems, analogous to those due to E. Hlawka and H. Weyl (see, for example, [6]).

For our purpose, we will need some auxiliary notions and facts from the Measure Theory.

Throughout this article, we use the following standard notation:

- \( \mathbb{R} \) is the set of all real numbers;
- \( \mathbb{N} \) is the set of all natural numbers;
- \( c \) is the cardinality of the continuum (i.e., \( c = 2^\omega \));
- \( \lambda \) is the linear Lebesgue measure on \( \mathbb{R} \);
- \( \text{dom}(\mu) \) is the domain of a given measure \( \mu \);
- \( \mu_1 \supset \mu \) - a measure \( \mu_1 \) is an extension of the given measure \( \mu \).

Let \( E \) be a nonempty set, \( G \) be a group of transformations of \( E \), and let \( X \) be a subset of \( E \).

2020 Mathematics Subject Classification. 28A05, 28D05.

Key words and phrases. Uniform distribution sequence; Invariant extension of the Lebesgue measure.
Lemma 1. There exists a family \( \{X_i : i \in [0, 1]\} \) of subsets of the real line \( \mathbb{R} \) such that:

1) \( X_i \cap X_j = \emptyset \).
2) If \( F \) is an arbitrary closed subset of the real line \( \mathbb{R} \) such that \( \lambda(F) > 0 \), then \( \text{card}(X_i \cap F) = c \).
3) \( \bigcup_{i \in I} X_i \) is an almost \( \mathbb{R} \)-invariant set in \( \mathbb{R} \), where \( I' \) is an arbitrary subset of \( [0, 1] \).

Lemma 2. There exists a family \( \{Y_i : i \in [0, 1]\} \) of subsets of the real line \( \mathbb{R} \) such that:

(a) for any sequence \( \{i_k : k \in \mathbb{N}\} \subset [0, 1] \), the intersection
\[
\bigcap_{k \in \mathbb{N}} Y_{i_k},
\]
where
\[
\overline{Y}_{i_k} = Y_{i_k} \cup \overline{Y}_{i_k} = \mathbb{R} \setminus Y_{i_k}
\]
is an almost invariant set.

(b) for any sequence \( \{i_k : k \in \mathbb{N}\} \subset [0, 1] \) and for any closed subset \( F \) of the real line \( \mathbb{R} \) with \( \lambda(F) > 0 \), we have
\[
\text{card} \left( \left( \bigcap_{k \in \mathbb{N}} \overline{Y}_{i_k} \right) \cap F \right) = c.
\]
(For the proofs of Lemma 1 and Lemma 2, see [4]).

According to the above-mentioned lemmas, we come to the following statement.

Lemma 3. There exists a family \( \{\mu_t : t \in [0, 1]\} \) of measures defined on some shift-invariant \( \sigma \)-algebra \( S(\mathbb{R}) \) of subsets of the real axis \( \mathbb{R} \) such that:

1) each measure \( \mu_t \) is a shift-invariant extension of the linear Lebesgue measure \( \lambda \);
2) measures \( \mu_t \) and \( \mu_t' \) are mutually singular, \( (t \neq t') \).

Moreover, \( \mu_t(\mathbb{R} \setminus X_t) = 0 \) for each \( t \in [0, 1] \), where \( \{X_t : t \in [0, 1]\} \) follows from Lemma 2.

Proof. For an arbitrary \( t \in [0, 1] \), we denote by \( K_t \) a shift-invariant \( \sigma \)-ideal generated by the set \( \mathbb{R} \setminus X_t \).

Applying Marczewski’s method, we can extend the Lebesgue measure \( \lambda \) to the measure \( \mu_t \). We obtain the family \( \{\overline{\mu_t} : t \in [0, 1]\} \) of shift-invariant extensions of the Lebesgue measure \( \lambda \).

Denote by \( S(\mathbb{R}) \) the shift-invariant \( \sigma \)-algebra of subsets of the real line \( \mathbb{R} \), generated by the union
\[
L(\mathbb{R}) \cup F(\mathbb{R}) \cup \{X_t : t \in [0, 1]\},
\]
where \( L(\mathbb{R}) \) denotes a class of all Lebesgue measurable subsets of the real line \( \mathbb{R} \) and
\[
F(\mathbb{R}) = \{X : X \subset \mathbb{R}, \text{card}(X) < c\}.
\]

For each \( t \in [0, 1] \), we assume that
\[
\mu_t = \overline{\mu_t}|_{S(\mathbb{R})}.
\]
The family of measures \( \{\mu_t : t \in [0, 1]\} \) satisfies the conditions of Lemma 3.

Remark 1. Let us consider the family \( \{\mu_t : t \in [0, 1]\} \) of shift-invariant extensions of the measure \( \lambda \) obtained from Lemma 3. Let \( \lambda_t \) denote the restriction of the measure \( \mu_t \) to the class
\[
S[0, 1] = \{Y \cap [0, 1] : Y \in S(\mathbb{R})\},
\]
where \( S(\mathbb{R}) \) follows from Lemma 3. It is obvious that for each \( t \in [0, 1] \), the measure \( \lambda_t \) is concentrated on the set \( Z_t = X_t \cap [0, 1] \), provided that
\[
\lambda_t([0, 1] \setminus Z_t) = 0.
\]
Consider the family of probability measures \( \{ \lambda_t : t \in [0, 1] \} \) and the family \( \{ Z_t : t \in [0, 1] \} \) of subsets of \([0,1]\) which come from Remark 1 and let \( \lambda_t^\infty \) denote an infinite power of \( \lambda_t \).

The next lemma is valid.

**Lemma 4.** For \( t \in [0, 1] \), we denote by \( \mathbf{L}([0,1], \lambda_t) \) the class of \( \lambda_t \)-integrable functions. Then for \( f \in \mathbf{L}([0,1], \lambda_t) \), we have

\[
\lambda_t^\infty \left( \left\{ x_k : k \in \mathbb{N} \right\} \in [0,1]^\infty : \lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(x_k) = \int_0^1 f(x) d\lambda_t(x) \right) = 1.
\]

(For the proof of Lemma 4, see [5]).

A sequence of real numbers \( \{ x_k : k \in \mathbb{N} \} \in [0,1]^\infty \) is said to be \( \lambda \)-uniformly distributed sequence \((\lambda\text{-u.d.s.})\) if for each \( c, d \), with \( 0 \leq c < d \leq 1 \), we have

\[
\lim_{n \to \infty} \frac{\text{card}(\{ x_k : 1 \leq k \leq n \} \cap [c,d])}{n} = d - c.
\]

A sequence of real numbers \( \{ x_k : k \in \mathbb{N} \} \in \mathbb{R}^\infty \) is said to be uniformly distributed module 1 if the sequence of its fractional parts \( \{ x_k : k \in \mathbb{N} \} \) is \( \lambda \text{-u.d.s.} \).

**Remark 2.** It is obvious that \( \{ x_k : k \in \mathbb{N} \} \in [0,1]^\infty \) is uniformly distributed module 1 if and only if \( \{ x_k : k \in \mathbb{N} \} \) is \( \lambda \text{-u.d.s.} \).

A sequence of real numbers \( \{ x_k : k \in \mathbb{N} \} \in [0,1]^\infty \) is said to be \( \lambda_t \)-uniformly distributed sequence \((\lambda_t\text{-u.d.s.})\) if for each \( c, d \), with \( 0 \leq c < d \leq 1 \), we have

\[
\lim_{n \to \infty} \frac{\text{card}(\{ x_k : 1 \leq k \leq n \} \cap [c,d] \cap \lambda_t(1) \} \cap \{ c, d \})}{n} = d - c.
\]

We say that a family \( \{ f_k : k \in \mathbb{N} \} \) of elements of \( \mathbf{L}([0,1], \lambda_t) \) defines a \( \lambda_t \)-u.d.s. on \([0,1]\), if for each \( \{ x_n : n \in \mathbb{N} \} \subset [0,1]^\infty \), the validity of the condition

\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} f_k(x_k) = \int_0^1 f_k(x) d\lambda_t(x)
\]

for \( k \in \mathbb{N} \) implies that \( \{ x_n : n \in \mathbb{N} \} \) is \( \lambda_t \text{-u.d.s.} \).

Notice that the indicator functions of the sets \( [a,b] \cap \mathbb{Z}_t \) with rational \( a, b \) is an example of such a family.

**Theorem 2.** Let \( T_t \) be the set of all real-valued sequences from \([0,1]^\infty \) which are \( \lambda_t \text{-u.d.s.} \). Then \( \lambda_t^\infty (T_t) = 1 \).

**Proof.** Let \( \{ f_k : k \in \mathbb{N} \} \) be a countable subclass of \( \mathbf{L}([0,1], \lambda_t) \) that defines a \( \lambda_t \text{-u.d.s.} \) on \([0,1]\). For \( k \in \mathbb{N} \), we set

\[
B_k = \left\{ \{ x_k : k \in \mathbb{N} \} : \{ x_k : k \in \mathbb{N} \} \in [0,1]^\infty, \lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} f_k(x_k) = \int_0^1 f_k(x) d\lambda_t(x) \right\}.
\]

By Lemma 4, we know that

\[
\lambda_t^\infty (B_k) = 1
\]

for \( k \in \mathbb{N} \), which implies

\[
\lambda_t^\infty \left( \bigcap_{k \in \mathbb{N}} B_k \right) = 1.
\]

Hence we have

\[
\lambda_t^\infty \left( \{ x_k : k \in \mathbb{N} \} \in [0,1]^\infty : (\forall k)(k \in \mathbb{N}) \Rightarrow \lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} f_k \left( x_k = \int_0^1 f_k(x) d\lambda_t(x) \right) = 1.\right.
\]
The latter relation means that $\lambda^\infty$-almost every elements of $[0, 1]^\infty$ are $\lambda_t$-u.d.s, or equivalently, $\lambda^\infty_t(T_i) = 1$.

**Theorem 3.** For $t \in [0, 1]$, we put

$$Z_t[0, 1] = \{ \tilde{f} = f(x) \times \chi_{Z_t(x)}; f \in C[0, 1] \}. $$

Then the sequence $\{x_n : n \in \mathbb{N}\}$ is $\lambda_t$-u.d.s if and only if the condition

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k \leq n} \tilde{f}(x_k) = \int_0^1 \tilde{f}(x)d\lambda_t(x) $$

holds for each $\tilde{f} \in Z_t[0, 1]$.

The proof of Theorem 2 is similar to that of H. Weyl’s Theorem (see [6]).

**Remark 3.** Some results presented in the paper were accepted jointly by Professor Gogi Pantsulaia. Here is a modified version of our previous unpublished survey.

**REFERENCES**


(Received 15.06.2023)

A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

Georgian Technical University, 77 Kostava Str., Tbilisi, 0160, Georgia

Email address: kirtadze2@yahoo.com