ONE-SIDED POTENTIALS IN WEIGHTED CENTRAL MORREY SPACES

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The boundedness of one-sided potential operators defined, generally speaking, with respect to a Borel measure μ, in the classical and central Morrey spaces is established. Weighted estimates for these operators in the case of power-type weights are derived in central Morrey spaces and in complementary central Morrey spaces. Similar problems are studied for vanishing Morrey spaces.

1. Preliminaries

The well-known Riemann–Liouville and Weyl fractional integrals can be viewed as a one-sided variants of the Riesz potential playing an important role in harmonic analysis and partial differential equations (*PDE*s). The study of weighted theory for one-sided operators was first introduced by Sawyer [8], Andersen and Sawyer [3]. Many of their results show that for a class of smaller operators (one-sided operators) and a class of wider weights (one-sided weights), many of the famous findings of harmonic analysis still hold, however, it should be mentioned that, for example, one-sided Muckenhoupt classes are much wider than two-sided ones, which plays a crucial role in the one-weight theory.

One-sided weighted Morrey spaces were introduced by S. Shi and Z. Fu (see [9]). In that paper, the authors established the boundedness of some classical one-sided operators including the Riemann–Liouville fractional integrals on these spaces.

Let $0 < \alpha < n$. The fractional integral operator (Riesz potential operator)

$$J_\alpha(f)(x) = \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad x \in \mathbb{R}^n,$$

plays a fundamental role in harmonic analysis; it also finds applications in PDEs such as in the theory of Sobolev embeddings (see, e.g., Maz'ya [7]).

We are interested in the fractional integrals defined on $\mathbb{R}$ or $\mathbb{R}_+$. For $\mathbb{R}$ and $0 < \alpha < 1$, we define the fractional integral operators I_α, W_α and R_α given by

$$I_\alpha(f)(x) := \int_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}} dy, \quad W_\alpha(f)(x) := \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy,$$

$$R_\alpha(f)(x) := \int_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}} dy, \quad x \in \mathbb{R},$$

respectively, for a suitable f.

For $\mathbb{R}_+$ and $0 < \alpha < 1$, we consider the fractional integral operators $\mathcal{I}_\alpha$, $\mathcal{W}_\alpha$ and $\mathcal{R}_\alpha$ defined as follows:

$$\mathcal{I}_\alpha(f)(x) := \int_{\mathbb{R}_+} \frac{f(y)}{|x-y|^{1-\alpha}} dy, \quad \mathcal{W}_\alpha(f)(x) := \int_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy,$$

$$\mathcal{R}_\alpha(f)(x) := \int\limits_0^x \frac{f(y)}{(x-y)^{1-\alpha}}dy, \quad x \in \mathbb{R}_+,$$

respectively, for a suitable f.

We are also interested in fractional integral operators with measure. Let μ be a Borel measure on $\mathbb{R}_+$ and let

$$\mathcal{J}_{\alpha,\mu}(f)(x) := \int\limits_{\mathbb{R}_+} \frac{f(y)}{|x-y|^{1-\alpha}}d\mu(y), \quad \mathcal{W}_{\alpha,\mu}(f)(x) := \int\limits_{(x,\infty)} \frac{f(y)}{(y-x)^{1-\alpha}}d\mu(y),$$

$$\mathcal{R}_{\alpha,\mu}(f)(x) := \int\limits_{(0,x)} \frac{f(y)}{(x-y)^{1-\alpha}}d\mu(y), \quad x \in \mathbb{R}_+.$$

Classical Morrey spaces were introduced in 1938 by C. B. Morrey in relation to regularity problems of solutions of PDEs. They Suppose that μ is a Borel measure on $\mathbb{R}$, $0 \leq \lambda < 1$ and $1 \leq p < \infty$. Let $L^{p,\lambda}(\mathbb{R},\mu)$ be the Morrey space with measure μ, that is the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R},\mu)$ such that

$$\|f\|_{L^{p,\lambda}(\mathbb{R},\mu)} := \sup_{\mathbb{I}} \left(\frac{1}{|\mathbb{I}|^\lambda} \int\limits_{\mathbb{I}} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals $\mathbb{I}$ in $\mathbb{R}$.

If $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R},\mu) = L^p(\mathbb{R},\mu)$ is the Lebesgue space with measure μ and the norm is defined as follows:

$$\|f\|_{L^p(\mathbb{R},\mu)} := \left(\int\limits_{\mathbb{R}} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}.$$

If μ is the Lebesgue measure, then we use the symbol $L^{p,\lambda}(\mathbb{R})$ for $L^{p,\lambda}(\mathbb{R},\mu)$.

Central Morrey spaces were introduced by García–Cuerva and Herrero [5] (see also [2]). In this note, we are interested in one-sided central Morrey space $M_\beta^{p,\lambda}(\mathbb{R}_+,\mu)$, which is a collection of all μ-measurable functions f such that

$$\|f\|_{M_\beta^{p,\lambda}(\mathbb{R}_+,\mu)} := \sup_{r>0} \left(\frac{1}{r^\lambda} \int\limits_{(0,r]} |f(y)|^p y^\beta d\mu(y) \right)^{\frac{1}{p}} < \infty.$$

If $\beta = 0$, then we use the notation $M_\beta^{p,\lambda}(\mathbb{R}_+,\mu) := M^{p,\lambda}(\mathbb{R}_+,\mu)$.

Complementary classical Morrey space was introduced by Guliyev [6]. By $\mathbb{M}_\beta^{p,\lambda}(\mathbb{R}_+,\mu)$ we denote a complementary central Morrey space with measure μ, which is the set of all μ-measurable functions f such that

$$\|f\|_{\mathbb{M}_\beta^{p,\lambda}(\mathbb{R}_+,\mu)} := \sup_{r>0} \left(\frac{1}{r^\lambda} \int\limits_{(r,\infty)} |f(y)|^p y^\beta d\mu(y) \right)^{\frac{1}{p}} < \infty.$$

If $\beta = 0$, then we denote $\mathbb{M}_\beta^{p,\lambda}(\mathbb{R}_+,\mu)$ by the symbol $\mathbb{M}^{p,\lambda}(\mathbb{R}_+,\mu)$.

We need the definition of one-sided weighted vanishing Morrey space. Unlike classical Morrey spaces, in those spaces it is possible to have approximation by "nice" functions. We use the symbol $VM_\beta^{p,\lambda}(\mathbb{R}_+,\mu)$ for one-sided weighted vanishing Morrey space, being the class of all functions $f \in M_\beta^{p,\lambda}(\mathbb{R}_+,\mu)$ such that

$$\lim_{r \to 0} \left(\frac{1}{r^\lambda} \int\limits_{(0,r]} |f(y)|^p y^\beta d\mu(y) \right)^{\frac{1}{p}} = 0.$$

Classical vanishing Morrey spaces were introduced in the works of Vitanza (see [10,11]) to describe the regularity of elliptic PDEs more precisely than that in the Lebesgue spaces.

By $V\mathbb{M}_\beta^{p,\lambda}(\mathbb{R}_+, \mu)$ we denote one-sided weighted vanishing complementary Morrey space with measure μ, being the set of all functions $f \in \mathbb{M}_\beta^{p,\lambda}(\mathbb{R}_+, \mu)$ such that

$$\lim_{r \to \infty} \left(\frac{1}{r^\lambda} \int_{(r,\infty)} |f(y)|^p y^\beta d\mu(y) \right)^{\frac{1}{p}} = 0.$$

We need the definition of growth condition for μ.

Definition 1.1. We say that a measure μ on $\mathbb{R}$ (resp., on $\mathbb{R}_+$) satisfies the growth condition, if there exists $c > 0$ such that $\mu(I) \leq c|I|$ for all open intervals I.

The following statements are known for fractional integrals in $\mathbb{R}^n$, but we formulate them for $n = 1$ (i.e., in this case, J_α is I_α).

Theorem A (Spanne, unpublished). *Let $1 < p < \infty$, $0 < \alpha < 1$ and $q = \frac{p}{1-\alpha p}$. Then I_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R})$ if $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$.*

Theorem B ([1]). *Let $0 \leq \lambda < 1$, $1 < p < \infty$, $0 < \alpha < 1$ and $q = \frac{p}{1-\alpha p}$. Then I_α is bounded from $L^{p,\lambda}(\mathbb{R})$ to $L^{q,\lambda}(\mathbb{R})$.*

The following trace inequality characterization for I_α formulated in the case of the real line is well-known (see [4]).

Theorem C. *Let $1 < p < q < \infty$. Suppose that $0 < \alpha < \frac{1}{p}$, $0 < \lambda_1 < 1 - \alpha p$ and $\frac{\lambda_2}{q} = \frac{\lambda_1}{p}$. Then I_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R}, \nu)$ if and only if there is a positive constant c such that*

$$\nu(I) \leq c|I|^{q\left(\frac{1}{p}-\alpha\right)},$$

for all intervals I.

2. Main Results

In this section we formulate the main results of the note.

Theorem 2.1. *Let $1 < p < q < \infty$. Suppose that $0 < \alpha < \frac{1}{p}$, $0 < \lambda_1 < 1 - \alpha p$ and $\frac{\lambda_2}{q} = \frac{\lambda_1}{p}$. Let ν be a Borel measure on $\mathbb{R}$. Then the following four statements are equivalent:*

a) *I_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R}, \nu)$;*
b) *R_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R}, \nu)$;*
c) *W_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R}, \nu)$;*
d) *There is a positive constant c such that for all intervals I,*

$$\nu(I) \leq c|I|^{q\left(\frac{1}{p}-\alpha\right)}.$$

The next statement is a consequence of Theorem 2.1.

Theorem 2.2. *Let $1 < p < q < \infty$. Suppose that $0 < \alpha < \frac{1}{p}$, $0 < \lambda_1 < 1 - \alpha p$ and $\frac{\lambda_2}{q} = \frac{\lambda_1}{p}$. Then the following four statements are equivalent:*

a) *I_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R})$;*
b) *R_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R})$;*
c) *W_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R})$;*
d) *$q = \frac{p}{1-\alpha p}$.*

We have investigated the boundedness of the Riemann–Liouville integral operator defined on $\mathbb{R}_+$ acting between the weighted Morrey spaces.

Theorem 2.3. *Let the Borel measure μ on $\mathbb{R}_+$ satisfy the growth condition. Suppose that $1 < p \leq q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$. Suppose also that $\beta < p - 1$, $0 < \lambda_1 < 1$ and $\lambda_1 q = \lambda_2 p$. Then $\mathcal{R}_{\alpha,\mu}$ is bounded from $M_\beta^{p,\lambda_1}(\mathbb{R}_+, \mu)$ to $M_\gamma^{q,\lambda_2}(\mathbb{R}_+, \mu)$, where*

$$\gamma = \beta \frac{q}{p}. \tag{1}$$

For the Weyl integral operator $\mathcal{W}_{\alpha,\mu}$ we derived the boundedness in weighted complementary Morrey spaces.

Theorem 2.4. *Let the Borel measure μ on $\mathbb{R}_+$ satisfy the growth condition. Suppose that $1 < p \leq q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$. Suppose also that $p - 1 < \beta$, $0 < \lambda_1 < 1$ and $\lambda_1 q = \lambda_2 p$. Then $\mathcal{W}_{\alpha,\mu}$ is bounded from $\mathbb{M}_\beta^{p,\lambda_1}(\mathbb{R}_+, \mu)$ to $\mathbb{M}_\gamma^{q,\lambda_2}(\mathbb{R}_+, \mu)$, where*

$$\gamma = \beta \frac{q}{p} + \alpha q - q. \tag{2}$$

Further, the following statements hold.

Theorem 2.5. *Let the Borel measure μ on $\mathbb{R}_+$ satisfy the growth condition. Suppose that $1 < p \leq q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$. Suppose also that $\beta < p - 1$, $0 < \lambda_1 < 1$ and $\lambda_1 q = \lambda_2 p$. Then $\mathcal{R}_{\alpha,\mu}$ is bounded from $VM_\beta^{p,\lambda_1}(\mathbb{R}_+, \mu)$ to $VM_\gamma^{q,\lambda_2}(\mathbb{R}_+, \mu)$, where γ satisfies condition* (1).

Theorem 2.6. *Let the Borel measure μ on $\mathbb{R}_+$ satisfy the growth condition. Suppose that $1 < p \leq q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$. Suppose also that $p - 1 < \beta$, $0 < \lambda_1 < 1$ and $\lambda_1 q = \lambda_2 p$. Then $\mathcal{W}_{\alpha,\mu}$ is bounded from $V\mathbb{M}_\beta^{p,\lambda_1}(\mathbb{R}_+, \mu)$ to $V\mathbb{M}_\gamma^{q,\lambda_2}(\mathbb{R}_+, \mu)$, where γ satisfies condition* (2).

ACKNOWLEDGEMENT

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