Transactions of A. Razmadze
Mathematical Institute
Vol. **177** (2023), issue 3, 495–499

ONE-SIDED POTENTIALS IN WEIGHTED CENTRAL MORREY SPACES

GIORGI IMERLISHVILI[1,2], ALEXANDER MESKHI[1,3], MARIA ALESSANDRA RAGUSA[4,5]

Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. The boundedness of one-sided potential operators defined, generally speaking, with respect to a Borel measure μ, in the classical and central Morrey spaces is established. Weighted estimates for these operators in the case of power-type weights are derived in central Morrey spaces and in complementary central Morrey spaces. Similar problems are studied for vanishing Morrey spaces.

1. Preliminaries

The well-known Riemann–Liouville and Weyl fractional integrals can be viewed as a one-sided variants of the Riesz potential playing an important role in harmonic analysis and partial differential equations (PDEs). The study of weighted theory for one-sided operators was first introduced by Sawyer [8], Andersen and Sawyer [3]. Many of their results show that for a class of smaller operators (one-sided operators) and a class of wider weights (one-sided weights), many of the famous findings of harmonic analysis still hold, however, it should be mentioned that, for example, one-sided Muckenhoupt classes are much wider than two-sided ones, which plays a crucial role in the one-weight theory.

One-sided weighted Morrey spaces were introduced by S. Shi and Z. Fu (see [9]). In that paper, the authors established the boundedness of some classical one-sided operators including the Riemann–Liouville fractional integrals on these spaces.

Let $0 < \alpha < n$. The fractional integral operator (Riesz potential operator)

$$J_\alpha(f)(x) = \int\limits_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy, \quad x \in \mathbb{R}^n,$$

plays a fundamental role in harmonic analysis; it also finds applications in PDEs such as in the theory of Sobolev embeddings (see, e.g., Maz'ya [7]).

We are interested in the fractional integrals defined on $\mathbb{R}$ or $\mathbb{R}_+$. For $\mathbb{R}$ and $0 < \alpha < 1$, we define the fractional integral operators I_α, W_α and R_α given by

$$I_\alpha(f)(x) := \int\limits_{\mathbb{R}} \frac{f(y)}{|x-y|^{1-\alpha}} dy, \quad W_\alpha(f)(x) := \int\limits_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy,$$

$$R_\alpha(f)(x) := \int\limits_{-\infty}^x \frac{f(y)}{(x-y)^{1-\alpha}} dy, \quad x \in \mathbb{R},$$

respectively, for a suitable f.

For $\mathbb{R}_+$ and $0 < \alpha < 1$, we consider the fractional integral operators $\mathcal{I}_\alpha$, $\mathcal{W}_\alpha$ and $\mathcal{R}_\alpha$ defined as follows:

$$\mathcal{I}_\alpha(f)(x) := \int\limits_{\mathbb{R}_+} \frac{f(y)}{|x-y|^{1-\alpha}} dy, \quad \mathcal{W}_\alpha(f)(x) := \int\limits_x^\infty \frac{f(y)}{(y-x)^{1-\alpha}} dy,$$

2020 *Mathematics Subject Classification.* 26A33, 42B35 47B38.

Key words and phrases. One-sided potentials; Fractional integrals; Power-type weights; Central Morrey spaces; Complementary Morrey spaces; Vanishing Morrey spaces; Boundedness.

$$\mathcal{R}_\alpha(f)(x) := \int\limits_0^x \frac{f(y)}{(x-y)^{1-\alpha}}dy, \quad x \in \mathbb{R}_+,$$

respectively, for a suitable f.

We are also interested in fractional integral operators with measure. Let μ be a Borel measure on $\mathbb{R}_+$ and let

$$\mathcal{J}_{\alpha,\mu}(f)(x) := \int\limits_{\mathbb{R}_+} \frac{f(y)}{|x-y|^{1-\alpha}}d\mu(y), \quad \mathcal{W}_{\alpha,\mu}(f)(x) := \int\limits_{(x,\infty)} \frac{f(y)}{(y-x)^{1-\alpha}}d\mu(y),$$

$$\mathcal{R}_{\alpha,\mu}(f)(x) := \int\limits_{(0,x)} \frac{f(y)}{(x-y)^{1-\alpha}}d\mu(y), \quad x \in \mathbb{R}_+.$$

Classical Morrey spaces were introduced in 1938 by C. B. Morrey in relation to regularity problems of solutions of PDEs. They Suppose that μ is a Borel measure on $\mathbb{R}$, $0 \le \lambda < 1$ and $1 \le p < \infty$. Let $L^{p,\lambda}(\mathbb{R},\mu)$ be the Morrey space with measure μ, that is the space of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R},\mu)$ such that

$$\|f\|_{L^{p,\lambda}(\mathbb{R},\mu)} := \sup_{\mathbb{I}} \bigg(\frac{1}{|\mathbb{I}|^\lambda} \int\limits_{\mathbb{I}} |f(y)|^p d\mu(y) \bigg)^{\frac{1}{p}} < \infty,$$

where the supremum is taken over all intervals $\mathbb{I}$ in $\mathbb{R}$.

If $\lambda = 0$, then $L^{p,\lambda}(\mathbb{R},\mu) = L^p(\mathbb{R},\mu)$ is the Lebesgue space with measure μ and the norm is defined as follows:

$$\|f\|_{L^p(\mathbb{R},\mu)} := \bigg(\int\limits_{\mathbb{R}} |f(y)|^p d\mu(y) \bigg)^{\frac{1}{p}}.$$

If μ is the Lebesgue measure, then we use the symbol $L^{p,\lambda}(\mathbb{R})$ for $L^{p,\lambda}(\mathbb{R},\mu)$.

Central Morrey spaces were introduced by García–Cuerva and Herrero [5] (see also [2]). In this note, we are interested in one-sided central Morrey space $M^{p,\lambda}_\beta(\mathbb{R}_+,\mu)$, which is a collection of all μ-measurable functions f such that

$$\|f\|_{M^{p,\lambda}_\beta(\mathbb{R}_+,\mu)} := \sup_{r>0} \bigg(\frac{1}{r^\lambda} \int\limits_{(0,r]} |f(y)|^p y^\beta d\mu(y) \bigg)^{\frac{1}{p}} < \infty.$$

If $\beta = 0$, then we use the notation $M^{p,\lambda}_\beta(\mathbb{R}_+,\mu) := M^{p,\lambda}(\mathbb{R}_+,\mu)$.

Complementary classical Morrey space was introduced by Guliyev [6]. By $\mathbb{M}^{p,\lambda}_\beta(\mathbb{R}_+,\mu)$ we denote a complementary central Morrey space with measure μ, which is the set of all μ-measurable functions f such that

$$\|f\|_{\mathbb{M}^{p,\lambda}_\beta(\mathbb{R}_+,\mu)} := \sup_{r>0} \bigg(\frac{1}{r^\lambda} \int\limits_{(r,\infty)} |f(y)|^p y^\beta d\mu(y) \bigg)^{\frac{1}{p}} < \infty.$$

If $\beta = 0$, then we denote $\mathbb{M}^{p,\lambda}_\beta(\mathbb{R}_+,\mu)$ by the symbol $\mathbb{M}^{p,\lambda}(\mathbb{R}_+,\mu)$.

We need the definition of one-sided weighted vanishing Morrey space. Unlike classical Morrey spaces, in those spaces it is possible to have approximation by "nice" functions. We use the symbol $VM^{p,\lambda}_\beta(\mathbb{R}_+,\mu)$ for one-sided weighted vanishing Morrey space, being the class of all functions $f \in M^{p,\lambda}_\beta(\mathbb{R}_+,\mu)$ such that

$$\lim_{r\to 0} \bigg(\frac{1}{r^\lambda} \int\limits_{(0,r]} |f(y)|^p y^\beta d\mu(y) \bigg)^{\frac{1}{p}} = 0.$$

Classical vanishing Morrey spaces were introduced in the works of Vitanza (see [10,11]) to describe the regularity of elliptic PDEs more precisely than that in the Lebesgue spaces.

By $V\mathbb{M}_{\beta}^{p,\lambda}(\mathbb{R}_+,\mu)$ we denote one-sided weighted vanishing complementary Morrey space with measure μ, being the set of all functions $f \in \mathbb{M}_{\beta}^{p,\lambda}(\mathbb{R}_+,\mu)$ such that

$$\lim_{r\to\infty}\left(\frac{1}{r^{\lambda}}\int\limits_{(r,\infty)}|f(y)|^{p}y^{\beta}d\mu(y)\right)^{\frac{1}{p}}=0.$$

We need the definition of growth condition for μ.

Definition 1.1. We say that a measure μ on $\mathbb{R}$ (resp., on $\mathbb{R}_+$) satisfies the growth condition, if there exists $c>0$ such that $\mu(I)\le c|I|$ for all open intervals I.

The following statements are known for fractional integrals in $\mathbb{R}^n$, but we formulate them for $n=1$ (i.e., in this case, J_α is I_α).

Theorem A (Spanne, unpublished)**.** *Let $1<p<\infty$, $0<\alpha<1$ and $q=\frac{p}{1-\alpha p}$. Then I_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R})$ if $\frac{\lambda_1}{p}=\frac{\lambda_2}{q}$.*

Theorem B ([1])**.** *Let $0\le\lambda<1$, $1<p<\infty$, $0<\alpha<1$ and $q=\frac{p}{1-\alpha p}$. Then I_α is bounded from $L^{p,\lambda}(\mathbb{R})$ to $L^{q,\lambda}(\mathbb{R})$.*

The following trace inequality characterization for I_α formulated in the case of the real line is well-known (see [4]).

Theorem C. *Let $1<p<q<\infty$. Suppose that $0<\alpha<\frac{1}{p}$, $0<\lambda_1<1-\alpha p$ and $\frac{\lambda_2}{q}=\frac{\lambda_1}{p}$. Then I_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R},\nu)$ if and only if there is a positive constant c such that*

$$\nu(I)\le c|I|^{q\left(\frac{1}{p}-\alpha\right)},$$

for all intervals I.

2. Main Results

In this section we formulate the main results of the note.

Theorem 2.1. *Let $1<p<q<\infty$. Suppose that $0<\alpha<\frac{1}{p}$, $0<\lambda_1<1-\alpha p$ and $\frac{\lambda_2}{q}=\frac{\lambda_1}{p}$. Let ν be a Borel measure on $\mathbb{R}$. Then the following four statements are equivalent:*

a) *I_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R},\nu)$;*

b) *R_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R},\nu)$;*

c) *W_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R},\nu)$;*

d) *There is a positive constant c such that for all intervals I,*

$$\nu(I)\le c|I|^{q\left(\frac{1}{p}-\alpha\right)}.$$

The next statement is a consequence of Theorem 2.1.

Theorem 2.2. *Let $1<p<q<\infty$. Suppose that $0<\alpha<\frac{1}{p}$, $0<\lambda_1<1-\alpha p$ and $\frac{\lambda_2}{q}=\frac{\lambda_1}{p}$. Then the following four statements are equivalent:*

a) *I_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R})$;*

b) *R_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R})$;*

c) *W_α is bounded from $L^{p,\lambda_1}(\mathbb{R})$ to $L^{q,\lambda_2}(\mathbb{R})$;*

d) *$q=\frac{p}{1-\alpha p}$.*

We have investigated the boundedness of the Riemann–Liouville integral operator defined on $\mathbb{R}_+$ acting between the weighted Morrey spaces.

Theorem 2.3. *Let the Borel measure μ on $\mathbb{R}_+$ satisfy the growth condition. Suppose that $1 < p \le q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$. Suppose also that $\beta < p-1$, $0 < \lambda_1 < 1$ and $\lambda_1 q = \lambda_2 p$. Then $\mathcal{R}_{\alpha,\mu}$ is bounded from $M_\beta^{p,\lambda_1}(\mathbb{R}_+,\mu)$ to $M_\gamma^{q,\lambda_2}(\mathbb{R}_+,\mu)$, where*

$$\gamma = \beta\frac{q}{p}. \tag{1}$$

For the Weyl integral operator $\mathcal{W}_{\alpha,\mu}$ we derived the boundedness in weighted complementary Morrey spaces.

Theorem 2.4. *Let the Borel measure μ on $\mathbb{R}_+$ satisfy the growth condition. Suppose that $1 < p \le q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$. Suppose also that $p-1 < \beta$, $0 < \lambda_1 < 1$ and $\lambda_1 q = \lambda_2 p$. Then $\mathcal{W}_{\alpha,\mu}$ is bounded from $\mathbb{M}_\beta^{p,\lambda_1}(\mathbb{R}_+,\mu)$ to $\mathbb{M}_\gamma^{q,\lambda_2}(\mathbb{R}_+,\mu)$, where*

$$\gamma = \beta\frac{q}{p} + \alpha q - q. \tag{2}$$

Further, the following statements hold.

Theorem 2.5. *Let the Borel measure μ on $\mathbb{R}_+$ satisfy the growth condition. Suppose that $1 < p \le q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$. Suppose also that $\beta < p-1$, $0 < \lambda_1 < 1$ and $\lambda_1 q = \lambda_2 p$. Then $\mathcal{R}_{\alpha,\mu}$ is bounded from $VM_\beta^{p,\lambda_1}(\mathbb{R}_+,\mu)$ to $VM_\gamma^{q,\lambda_2}(\mathbb{R}_+,\mu)$, where γ satisfies condition (1).*

Theorem 2.6. *Let the Borel measure μ on $\mathbb{R}_+$ satisfy the growth condition. Suppose that $1 < p \le q < \infty$ and $\alpha = \frac{1}{p} - \frac{1}{q}$. Suppose also that $p-1 < \beta$, $0 < \lambda_1 < 1$ and $\lambda_1 q = \lambda_2 p$. Then $\mathcal{W}_{\alpha,\mu}$ is bounded from $V\mathbb{M}_\beta^{p,\lambda_1}(\mathbb{R}_+,\mu)$ to $V\mathbb{M}_\gamma^{q,\lambda_2}(\mathbb{R}_+,\mu)$, where γ satisfies condition (2).*

Acknowledgement

This research [PHDF-22-6359] has been supported by the Shota Rustaveli National Science Foundation of Georgia (SRNSFG).

References

1. D. R. Adams, A note on Riesz potentials. *Duke Math. J.* **42** (1975), no. 4, 765–778.
2. J. Alvarez, M. Guzmán-Partida, J. D. Lakey, Spaces of bounded λ-central mean oscillation, Morrey spaces, and λ-central Carleson measures. *Collect. Math.* **51** (2000), no. 1, 1–47.
3. K. F. Andersen, E. T. Sawyer, Weighted norm inequalities for the Riemann-Liouville and Weyl fractional integral operators. *Trans. Amer. Math. Soc.* **308** (1988), no. 2, 547–558.
4. A. Eridani, V. Kokilashvili, A. Meskhi, Morrey spaces and fractional integral operators. *Expo. Math.* **27** (2009), no. 3, 227–239.
5. J. García–Cuerva, M. J. L. Herrero, A theory of Hardy spaces associated to the Herz spaces. *Proc. London Math. Soc.* **69** (1994), no. 3, 605–628.
6. V. S. Guliyev, *Integral Operators on Function Spaces on the Homogeneous Groups and on Domains in $\mathbb{R}^n$.* (Russian) Doctor's Degree Dissertation. Moscow, Mat. Inst. Steklov, 1994.
7. V. Maz'ya, *Sobolev Spaces.* Translated from the Russian by T. O. Shaposhnikova. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin, 1985.
8. E. Sawyer, Weighted inequalities for the one-sided Hardy-Littlewood maximal functions. *Trans. Amer. Math. Soc.* **297** (1986), no. 1, 53–61.
9. S. Shi, Z. Fu, Estimates of some operators on one-sided Weighted Morrey Spaces. *Abstr. Appl. Anal.* 2013, Art. ID 829218, 9 pp.
10. C. Vitanza, Functions with vanishing Morrey norm and elliptic partial differential equations. *Proceedings of methods of real analysis and partial differential equations, Capri*, 147–150, 1990.
11. C. Vitanza, Regularity results for a class of elliptic equations with coefficients in Morrey spaces. *Ricerche Mat.* **42** (1993), no. 2, 265–281.

(Received 15.03.2023)

[1]A. Razmadze Mathematical Institute of I. Javakhishvili Tbilisi State University, 2 Merab Aleksidze II Lane, Tbilisi 0193, Georgia

[2]Georgian Technical University, VI Building, Faculty of Informatics and Control Systems, Kostava 77, 0175 Tbilisi, Georgia

[3]Kutaisi International University, Youth Avenue, Turn 5/7, 4600 Kutaisi, Georgia

[4]Dipartimento di Matematica e Informatica, Università di Catania, Viale Andrea Doria, 6-95125 Catania, Italy

[5]Faculty of Fundamental Science, Industrial University of Ho Chi Minh City, Vietnam
Email address: alexander.meskhi@tsu.ge; alexander.meskhi@kiu.edu.ge
Email address: imerlishvili18@gmail.com
Email address: maragusa@dmi.unict.it