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MARTINGALE HARDY SPACES BASED ON QUASI-BANACH FUNCTION LATTICES

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. This paper considers the atomic decomposition in martingale Hardy spaces generated by quasi-Banach function lattices. The results in this paper unify and extend the existing ones. Based on the Doob inequality, we make the paper self-contained except for weighted inequalities of martingale transforms. We obtain the vector-valued Doob maximal inequality for a large class of Banach function lattices over probability spaces as a by-product.

1. Introduction

The goal of this note is to obtain the atomic decomposition for martingale Hardy spaces generated by quasi-Banach function lattices. The results in this paper unify and extend the existing ones.

Here and below, we write $\mathbb{N}_0 \equiv \{0,1,2,\ldots\}$. We work in a probability space $(\Omega,\mathcal{F},P)$ equipped with a filtration $\mathbb{F}=\{\mathcal{F}_j\}_{j\in\mathbb{N}_0}$. We follow [5] to introduce briefly the theory of martingales. A tacit understanding of this note is that martingales and related notions such as stopping times are considered with respect to this filtration. Thus, a stopping time is a random variable $\tau:\Omega\to\mathbb{N}_0\cup\{\infty\}$ satisfying $\{\tau=k\}\in\mathcal{F}_k$ for all $k\in\mathbb{N}_0$. For a stopping time τ, we define the *stochastic interval* $[0,\tau]$ by $\{(\omega,j)\in\Omega\times(\mathbb{N}_0\cup\{\infty\}) : j\le\tau(\omega)\}$. We write $M=\{M_j\}_{j\in\mathbb{N}_0}$ for a stochastic process. For the sake of convenience, we set $M_{-1}\equiv 0$ and $\mathcal{F}_{-1}\equiv\mathcal{F}_0$. The space $L^0(\Omega)$ denotes the space of measurable functions on Ω modulo P-null functions, equipped with the topology of convergence in measure. For an integrable random variable X and $j\in\mathbb{N}_0$, we write $X_j\equiv E[X:\mathcal{F}_j]$, the *conditional expectation* with respect to $\mathcal{F}_j$. Thus, a sequence $M=\{M_j\}_{j\in\mathbb{N}_0}$ of integrable random variables is a martingale, if $(M_{j+1})_j=M_j$ for all $j\in\mathbb{N}_0$. For $j\in\mathbb{N}_0$ and a random variable X, we abbreviate $X\in\mathcal{F}_j$ if X is $\mathcal{F}_j$-measurable. A sequence $N=\{N_j\}_{j\in\mathbb{N}_0}$ of random variables is predictable if $N_j\in\mathcal{F}_{j-1}$ for all $j\in\mathbb{N}_0$. A (quasi)-Banach space $X\subset L^0(\Omega)$ is said to be a *(quasi)-Banach function lattice* over $(\Omega,\mathcal{F},P)$ (over Ω for short) if f is finite a.s., $g\in X$ and $\|g\|_X\le\|f\|_X$ whenever $f\in X$ and $g\in L^0(\Omega)$ satisfy $|g|\le|f|$ a.s..

Let $0<q\le\infty$. Given a sequence $\{f_k\}_{k\in\mathbb{Z}}\subset L^0(\Omega)$, we define

$$\|\{f_k\}_{k\in\mathbb{Z}}\|_{\ell^q}\equiv\left(\sum_{k=-\infty}^{\infty}|f_k|^q\right)^{\frac{1}{q}},$$

if $q<\infty$. A natural modification is made to define $\|\{f_k\}_{k\in\mathbb{Z}}\|_{\ell^\infty}$. The vector-valued space $X(\ell^q)$ is the set of all sequences $\{f_k\}_{k\in\mathbb{Z}}\subset L^0(\Omega)$ for which $\|\{f_k\}_{k\in\mathbb{Z}}\|_{X(\ell^q)}\equiv\|\|\{f_k\}_{k\in\mathbb{Z}}\|_{\ell^q}\|_X<\infty$, while the vector-valued space $\ell^q(X)$ is the set of all sequences $\{f_k\}_{k\in\mathbb{Z}}\subset L^0(\Omega)$ for which $\|\{f_k\}_{k\in\mathbb{Z}}\|_{\ell^q(X)}\equiv\|\{\|f_k\|_X\}_{k\in\mathbb{Z}}\|_{\ell^q}<\infty$. When we want to stress that the underlying space is Ω, we add Ω in the notation. For example, for $X=L^p(\Omega)$, we write $L^p(\Omega;\ell^q)$ and $\ell^q(L^p(\Omega))$ and for $X(\ell^q)$ and $\ell^q(X)$, respectively. If w is a weight, namely, a nonnegative measurable function over Ω and $X=L^p(w)$, whose norm is given by (3.1) below, then we write $L^p(w;\ell^q)$ and $\ell^q(L^p(w))$ for $X(\ell^q)$ and $\ell^q(X)$, respectively.

For a stochastic process $N=\{N_j\}_{j\in\mathbb{N}_0}$, define the *difference process* $dN=\{(dN)_j\}_{j\in\mathbb{N}_0}$ by $(dN)_j\equiv N_j-N_{j-1}$, $j\in\mathbb{N}_0$, where $N_{-1}\equiv 0$. For a martingale M, we define the sequences $\{S_j(M)\}_{j\in\mathbb{N}_0}$ and

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$\{s_j(M)\}_{j\in\mathbb{N}_0}$ of random variables by

$$S_j(M)\equiv\left(\sum_{k=0}^{j}|(dM)_k|^2\right)^{\frac12},\qquad s_j(M)\equiv\left(\sum_{k=0}^{j}E[|(dM)_k|^2\,:\,\mathcal{F}_{k-1}]\right)^{\frac12}.$$

Write $S(M)\equiv\lim\limits_{j\to\infty}S_j(M)$ and $s(M)\equiv\lim\limits_{j\to\infty}s_j(M)$. For a stochastic process $Z=\{Z_j\}_{j\in\mathbb{N}_0}$, we define

$$Z_j^*\equiv\sup_{k=0,1,\dots,j}|Z_k|,\qquad Z^*\equiv\lim_{j\to\infty}Z_j^*.$$

Then the correspondence $M\mapsto M^*$ is called the *Doob maximal operator*, where M is a martingale. The *X-based martingale S-Hardy space* $\mathcal{H}_X^S$, the *X-based martingale s-Hardy space* $\mathcal{H}_X^s$ and the *X-based martingale $*$-Hardy space* $\mathcal{H}_X^*$ are defined as the quasi-Banach spaces of all martingales M with $M_0=0$ for which $S(M)$, $s(M)$ and $M^*\in X$, respectively. We equip $\mathcal{H}_X^S$, $\mathcal{H}_X^s$ and $\mathcal{H}_X^*$ with the quasi-norms: for a martingale M with $M_0=0$, we write $\|M\|_{\mathcal{H}_X^S}\equiv\|S(M)\|_X$, $\|M\|_{\mathcal{H}_X^s}\equiv\|s(M)\|_X$ and $\|M\|_{\mathcal{H}_X^*}\equiv\|M^*\|_X$, respectively.

Here, we present the definition of atoms. Denote by $\mathbf{1}_E$ the indicator function of a set E. Due to the generality of X, we prefer to adopt the following normalization.

Definition 1.1.

(1) Let τ be a stopping time. A martingale $A=\{A_j\}_{j\in\mathbb{N}_0}$ is said to be an $(\infty;S,\tau)$-atom if it vanishes on $[0,\tau]$ and $S(A)\le\mathbf{1}_{\mathbb{R}}(\tau)$.

(2) Denote by $\mathcal{A}^S(\infty)$ the set of all sequences $\{(\mu^{(k)},A^{(k)},\tau^{(k)})\}_{k\in\mathbb{Z}}$ such that each $A^{(k)}$ is an $(\infty;S,\tau^{(k)})$-atom and that each $\mu^{(k)}$ is a nonnegative real number.

(3) The set $\mathcal{A}^S(\infty)_\uparrow$ is the subset of $\mathcal{A}^S(\infty)$ consisting of all elements $\{(\mu^{(k)},A^{(k)},\tau^{(k)})\}_{k\in\mathbb{Z}}$ satisfying $\tau^{(k)}\le\tau^{(k+1)}$ for all $k\in\mathbb{Z}$.

(4) Let τ be a stopping time. Define analogously the notions of $(\infty;s,\tau)$-*atoms* and $(\infty;*,\tau)$-*atoms* as well as the sets $\mathcal{A}^s(\infty)$, $\mathcal{A}^s(\infty)_\uparrow$, $\mathcal{A}^*(\infty)$ and $\mathcal{A}^*(\infty)_\uparrow$.

Here, we list some conventions for the notation about inequalities used in this paper. Let $A,B\ge0$. Then $A\lesssim B$ means that there exists a constant $C>0$ such that $A\le CB$, where C depends only on the parameters of importance. The symbol $A\sim B$ means that $A\lesssim B$ and $B\lesssim A$ happen simultaneously. If we need to stress that the implicit constant depends on some parameters, then we add them as subscripts. It matters that the implicit constants C in $\lesssim$ and $\sim$ never depend on $\mathbb{F}$.

This note aims to convince readers that Theorem 1.2 below is the root of many of the recent results on atomic decomposition.

Theorem 1.2. *Let X be a quasi-Banach function lattice over $(\Omega,\mathcal{F},P)$.*

(1) *Let $\{(\mu^{(k)},A^{(k)},\tau^{(k)})\}_{k\in\mathbb{Z}}\in\mathcal{A}^s(\infty)$ satisfy $\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\in X(\ell^1)$. Then the process M, given by*

$$M\equiv\sum_{k=-\infty}^{\infty}\mu^{(k)}A^{(k)},\tag{1.1}$$

belongs to $\mathcal{H}_X^s$ and satisfies

$$\|M\|_{\mathcal{H}_X^s}\le\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{X(\ell^1)}.\tag{1.2}$$

(2) *For all $M\in\mathcal{H}_X^s$, there exists $\{(\mu^{(k)},A^{(k)},\tau^{(k)})\}_{k\in\mathbb{Z}}\in\mathcal{A}^s(\infty)_\uparrow$ such that $\mu^{(k)}=2^k$, $k\in\mathbb{Z}$, that* (1.1) *is satisfied and that*

$$\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{X(\ell^u)}\lesssim_u\|M\|_{\mathcal{H}_X^s}\tag{1.3}$$

for all $0<u<\infty$.

Analogies to $\mathcal{H}_X^S$ and $\mathcal{H}_X^*$ are available.

Theorem 1.2, based on the works [3, 22], is the starting point of this paper. The remaining part of this note is organized as follows: Section 2 proves Theorem 1.2. Section 3 investigates the elementary properties of $\mathcal{H}_X^s$, $\mathcal{H}_X^S$ and $\mathcal{H}_X^*$. In Section 4, we present examples of X showing how some of the existing results can be reproduced by Theorem 1.2 and why other results are not its consequences. We can locate Section 4 as applications of Theorem 1.2.

2. Proof of Theorem 1.2

We use the following standard notation for martingales: For a stopping time τ and a random process $N = \{N^j\}_{j\in\mathbb{N}_0}$, we write $N^\tau \equiv \{N^{j\wedge\tau(\cdot)}\}_{j\in\mathbb{N}_0}$. The optimal sampling theorem shows that M^τ is a martingale for any martingale M and any stopping time τ. We solely consider $\mathcal{H}^s_X$, since $\mathcal{H}^S_X$ and $\mathcal{H}^*_X$ can be dealt with similarly. Theorem 1.2(1) is easy to prove. In fact, any atom A satisfies $A_0 = 0$, implying that $M_0 = 0$. Furthermore, from (1.1), we obtain

$$s(M) \le \sum_{k=-\infty}^{\infty} \mu^{(k)} s(A^{(k)}) \le \sum_{k=-\infty}^{\infty} \mu^{(k)} \mathbf{1}_{\mathbb{R}}(\tau^{(k)}).$$

Thus, (1.2) follows and $s(M) \in X$, implying that $M \in \mathcal{H}^s_X$. Thus, we concentrate on the proof of Theorem 1.2(2). Suppose $M \in \mathcal{H}^s_X \setminus \{0\}$; otherwise, we can take $\mu^{(k)} \equiv 0$ and $A^{(k)} \equiv 0$ for all $k \in \mathbb{Z}$. There exists $l \in \mathbb{N}$ such that $M_l \ne 0$. For each $k \in \mathbb{Z}$, we set

$$\tau^{(k)} \equiv \inf(\{j \in \mathbb{N}_0 \,:\, s_{j+1}(M) > 2^k\} \cup \{\infty\}). \tag{2.1}$$

We define an event $\mathcal{O}_k$ by $\mathcal{O}_k \equiv \{\tau^{(k)} < \infty\} = \{s(M) > 2^k\}$ and a process $A^{(k)} = \{(A^{(k)})_j\}_{j\in\mathbb{N}_0}$ by $(A^{(k)})_j \equiv 2^{-k-2}(M_{j\wedge\tau^{(k+1)}} - M_{j\wedge\tau^{(k)}})$, $j \in \mathbb{N}_0$. Then from the definition of $\mathcal{O}_k$'s, $\mathcal{O}_k \supset \mathcal{O}_{k+1}$. By the optimal sampling theorem, each $A^{(k)}$ is a martingale. Since $s(M^{\tau^{(k)}}) = s_{\tau^{(k)}}(M) \le 2^k$ and $s(M^{\tau^{(k+1)}}) \le 2^{k+1}$, $s(A^{(k)}) \le 1$. By the definition, $A^{(k)}$ vanishes on $[0, \tau^{(k)}]$ and hence $s(A^k) \le \chi_{\mathbb{R}}(\tau^{(k)})$. Therefore, each $A^{(k)}$ is an $(\infty; s, \tau^{(k)})$-atom.

Since $s_{j+1}(M) \in \mathcal{F}_j$ for all $j \in \mathbb{N}_0$, each $\tau^{(k)}$ is a stopping time. Recall that $M_{-1} = M_0 = 0$. Since $\tau^{(k)}(\omega) \to \infty$ a.s. as $k \to \infty$ and $\tau^{(k)}(\omega) \to \inf(\{l \in \mathbb{N} \,:\, M_l(\omega) \ne 0\} \cup \{\infty\}) - 1$ as $k \to -\infty$, $M_j = \displaystyle\sum_{k=-\infty}^{\infty} (M_{j\wedge\tau^{(k+1)}} - M_{j\wedge\tau^{(k)}})$ for each $j \in \mathbb{N}_0$, namely, $M = \displaystyle\sum_{k=-\infty}^{\infty} (M^{\tau^{(k+1)}} - M^{\tau^{(k)}})$. Thus we have (1.1).

Since $s(M)(\omega) < \infty$ for a.s. $\omega \in \Omega$, we can find $k \in \mathbb{Z}$ such that $\omega \in \mathcal{O}_k \setminus \mathcal{O}_{k+1}$ for such ω. In this case,

$$\sum_{l=-\infty}^{\infty} (2^l \mathbf{1}_{\mathcal{O}_l}(\omega))^u = \frac{2^{uk}}{1 - 2^{-u}}.$$

Since $2^k < s(M)(\omega) \le 2^{k+1}$ for such ω, we have

$$\sum_{l=-\infty}^{\infty} (2^l \mathbf{1}_{\mathcal{O}_l}(\omega))^u \le \frac{s(M)(\omega)^u}{1 - 2^{-u}} \tag{2.2}$$

for all $\omega \in \Omega$. Thus, (1.3) follows.

To conclude Section 2, we have a couple of remarks.

Remark 2.1. Note that the idea appearing in the above proof is used to obtain the decomposition results on Banach function lattices in $\mathbb{R}^n$. See [8, 25], for example.

If we reexamine the proof of Theorem 1.2, then we can polish what we obtained.

Remark 2.2. Let X be a quasi-Banach function lattice over $(\Omega, \mathcal{F}, P)$. For all martingales M such that $M_0 = 0$ and that $s(M) < \infty$ a.s., there exists $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ such that $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, that (1.1) is satisfied and that $\|\{\mu^{(k)} \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{\ell^u} \lesssim_u s(M)$ for all $0 < u < \infty$. See (2.2). Analogies to $\mathcal{H}^S_X$ and $\mathcal{H}^*_X$ are available.

3. Some Elementary Properties

We consider the relation between X and $\mathcal{H}_X^S$ in some special cases. Section 3.1 investigates the properties of martingale transforms. We refer to [18] for an exhaustive account of the boundedness of martingale transforms. Section 3.2 is a by-product of Section 3.1, where the vector-valued Doob maximal inequality is obtained. Section 3.3 investigates the relationship between X and $\mathcal{H}_X^S$ if X is a Banach function lattice subject to some mild conditions.

Here, we recall some terminologies in the theory of Banach function lattices and martingale theory before we go further.

Let $0 < p < \infty$. Recall that the *p-convexification* $Y^{(p)}$ of a quasi-Banach function lattice Y is the quasi-Banach space of all $f \in L^0(\Omega)$ for which $\|f\|_{Y^{(p)}} \equiv (\| \, |f|^p \, \|_Y)^{\frac{1}{p}} < \infty$. We define the *Köthe dual space* X' of a Banach function lattice X to be the space of all $f \in L^0(\Omega)$ for which $f \cdot g \in L^1(\Omega)$ for all $g \in X$. Notice that X' is a Banach function lattice over Ω equipped with the norm

$$\|f\|_{X'} \equiv \sup \Big\{ \|f \cdot g\|_{L^1(\Omega)} \, : \, \|g\|_X \leq 1 \Big\}.$$

Next, we recall some notions in the theory of martingales.

Definition 3.1. Let $M = \{M_j\}_{j \in \mathbb{N}_0}$ be a martingale.

(1) A martingale M is closable if there exists $M_\infty \in L^1(\Omega)$ such that $M_j = E[M_\infty \, : \, \mathcal{F}_j]$ for all $j \in \mathbb{N}_0$.
(2) A martingale M is uniformly integrable if

$$\lim_{\lambda \to \infty} \left(\sup_{j \in \mathbb{N}_0} E[|M_j| \mathbf{1}_{(\lambda, \infty]}(|M_j|)] \right) = 0.$$

A martingale is closable if and only if it is uniformly integrable and belongs to $\ell^\infty(L^1(\Omega))$. A tacit understanding in Section 3 is that we have Banach function lattices X, Y, Z satisfying $L^1(\Omega) \subset X, Y, Z \subset L^\infty(\Omega)$.

3.1. Martingale transform.

Let $N = \{N_j\}_{j \in \mathbb{N}_0}$ be a sequence of predictable random variables satisfying $\|N_k\|_{L^\infty(\Omega)} \leq 1$ for each $k \in \mathbb{N}_0$. We define

$$N * M \equiv \left\{ \sum_{k=0}^{j} N_k (dM)_k \right\}_{j \in \mathbb{N}_0}.$$

The mapping $N* : M \mapsto N * M$ is called a *martingale transform.* We investigate its boundedness property on a Banach lattice X. Recall that the correspondence $H \in L^1(\Omega) \mapsto H^* \in L^0(\Omega)$ is called the Doob maximal operator. We say that a nonnegative function $w \in L^0(\Omega) \setminus \{0\}$ is an A_1-weight, if $[w]_{A_1} \equiv \left\| \frac{w^*}{w} \right\|_{L^\infty(\Omega)} < \infty$. The set A_1 collects all A_1-weights. Let $0 < p < \infty$. For a weight w and $f \in L^0(\Omega)$, we define

$$\|f\|_{L^p(w)} \equiv \|f \cdot w^{\frac{1}{p}}\|_{L^p(\Omega)}. \tag{3.1}$$

The *weighted Lebesgue space* $L^p(w)$ is defined to be the quasi-Banach space of all $f \in L^0(\Omega)$ for which $\|f\|_{L^p(w)} < \infty$. The following weighted boundedness of martingale transforms can be found in [23, Theorem 5.3]:

$$\|(N * M)_\infty\|_{L^p(w)} \leq c_p [w]_{A_1} \|M_\infty\|_{L^p(w)} \tag{3.2}$$

if $M_\infty \in L^p(w)$ and $w \in A_1$, where $M = \{(M_\infty)_j\}_{j \in \mathbb{N}_0}$.

Using the extrapolation technique, Ho established the following result.

Theorem 3.2 ([9, Theorem 3.2])**.** *Let Y be a Banach function lattice and $1 < p < \infty$. Assume that the Doob maximal operator is bounded on Y'. Then the martingale operator $N*$ is bounded on $X \equiv Y^{(p)}$. Namely,* (3.2) *with $L^p(w)$ replaced by X holds if $M_\infty \in X$.*

If we reexamine the proof of Theorem 3.2, then we can generalize Theorem 3.2. We omit the proof of Theorem 3.3 below since it simply uses the idea of the proof of [9, Theorem 3.2]. See also [4].

Theorem 3.3. *Let $1 < p < \infty$ and let X be a Banach function lattice as in Theorem 3.2. Suppose that $U : [0, \infty) \to [0, \infty)$ is an increasing function. Define*

$$\mathcal{F}_p \equiv \left\{(f, g) \in L^0(\Omega)^2 \,:\, \|g\|_{L^p(w)} \le U([w]_{A_1})\|f\|_{L^p(w)}\right\}. \tag{3.3}$$

Then $g \in X$ and $\|g\|_X \lesssim \|f\|_X$ whenever $(f, g) \in \mathcal{F}_p$ satisfies $f \in X$.

3.2. **Maximal inequalities.** The results in Section 3.2 are by-products of the ones in Section 3.1. Although Lemma 3.4 is well known, we supply the whole proof to make the paper self-contained.

Lemma 3.4 (Doob's maximal inequality)**.** *Let g be a nonnegative random variable. Then for any closable martingale $M = \{M_j\}_{j\in\mathbb{N}_0}$ and $\lambda > 0$,*

$$\lambda E\left[\mathbf{1}_{(\lambda,\infty]}(M^*)g\right] \le E\left[|M_\infty|g^*\mathbf{1}_{(\lambda,\infty]}(M^*)\right].$$

Using the stopping time argument and the monotone convergence theorem, as a direct corollary of Lemma 3.4, we obtain

$$\lambda E\left[\mathbf{1}_{(\lambda,\infty]}(M^*)g\right] \le \sup_{j\in\mathbb{N}_0} E\left[|M_j|g^*\mathbf{1}_{(\lambda,\infty]}(M^*)\right],$$

for any martingale $M = \{M_j\}_{j\in\mathbb{N}_0}$, which is not always closable, and $\lambda > 0$.

Lemma 3.4 is well known, but for the sake of convenience for readers, we supply a short proof.

Proof. We abbreviate $g_k \equiv E[g \,:\, \mathcal{F}_k]$, $k \in \mathbb{N}_0$. Define a stopping time τ by

$$\tau \equiv \inf(\{k \in \mathbb{N}_0 \,:\, |M_k| > \lambda\} \cup \{\infty\}).$$

We decompose

$$E\left[\mathbf{1}_{(\lambda,\infty]}\left(M^*\right)g\right] = \sum_{k=0}^{\infty} E[\mathbf{1}_{\{\tau=k\}}g].$$

Since $\{\tau = k\} \in \mathcal{F}_k$ for each $k \in \mathbb{N}_0$, we have

$$E[\mathbf{1}_{\{\tau=k\}}g] = E[\mathbf{1}_{\{\tau=k\}}g_k] \le \frac{1}{\lambda}E[|M_k|\mathbf{1}_{\{\tau=k\}}g_k].$$

Since $\mathbf{1}_{\{\tau=k\}}g_k \in \mathcal{F}_k$ for each $k \in \mathbb{N}_0$, the triangle inequality for the conditional expectation yields

$$E[|M_k|\mathbf{1}_{\{\tau=k\}}g_k] \le E[E[|M_\infty|\mathbf{1}_{\{\tau=k\}}g_k \,:\, \mathcal{F}_k]] = E[|M_\infty|\mathbf{1}_{\{\tau=k\}}g_k] \le E\left[|M_\infty|\mathbf{1}_{\{\tau=k\}}g^*\right].$$

If we add this inequality over $k \in \mathbb{N}_0$, then we obtain the desired result. □

If we argue similarly to [24, Lemma 2.18], then we obtain the weighted L^p-boundedness for $1 < p < \infty$.

Corollary 3.5. *Let $1 < p < \infty$. Then for any closable martingale $M = \{M_j\}_{j\in\mathbb{N}_0}$ and any $w \in A_1$,*

$$\|M^*\|_{L^p(w)} \le p'[w]_{A_1}\|M_\infty\|_{L^p(w)}.$$

Using Corollary 3.5, we obtain the following vector-valued Doob maximal inequality.

Theorem 3.6. *Let X be a Banach function lattice as in Theorem 3.2. Then*

$$\left\|\{(M^{(k)})^*\}_{k\in\mathbb{Z}}\right\|_{X(\ell^r)} \lesssim_r \left\|\{M^{(k)}\}_{k\in\mathbb{Z}}\right\|_{X(\ell^r)}$$

for all $1 < r < \infty$ and $\{M^{(k)}\}_{k\in\mathbb{Z}} \in X(\ell^r)$.

Proof. For $X = L^p(w)$ with $w \in A_1$, in view of Theorem 3.3, it suffices to prove the following inequality:

$$\left\|\{(M^{(k)})^*\}_{k\in\mathbb{Z}}\right\|_{L^p(w;\ell^r)} \lesssim_{p,r} [w]_{A_1}\left\|\{M^{(k)}\}_{k\in\mathbb{Z}}\right\|_{L^p(w;\ell^r)}. \tag{3.4}$$

Let $\mathcal{F}_r$ be the set given by (3.3) with $p = r$ and $U(t) = t$, $t \ge 0$. Then

$$\left(\frac{1}{r'}\|\{M^{(k)}\}_{k\in\mathbb{Z}}\|_{\ell^r}, \|\{(M^{(k)})^*\}_{k\in\mathbb{Z}}\|_{\ell^r}\right) \in \mathcal{F}_r.$$

Then thanks to Corollary 3.5, we see that (3.4) holds with $X = Y^{(p)}$, and $Y = L^1(w)$ with $w \in A_1$. Therefore the proof is complete. □

In view of (3.2) and Theorems 3.2 and 3.6, we have the following conclusion.

Theorem 3.7. *Let X be a Banach function lattice as in Theorem 3.2. Then the martingale transform $N*$ and the Doob maximal operator are bounded on X.*

3.3. **Comparison of X and $\mathcal{H}^S_X$.** We aim to find a sufficient condition under which X, $\mathcal{H}^S_X$ and $\mathcal{H}^*_X$ are isomorphic. Let us say that a martingale $M = \{M_j\}_{j\in\mathbb{N}_0}$ is X-bounded if $M \in \ell^\infty(X)$. We check that any X-bounded martingale is closable if X is a Banach function lattice satisfying the condition in Theorem 3.2.

Corollary 3.8. *Let X be a Banach function lattice satisfying the condition in Theorem 3.2. Then any X-bounded martingale $M = \{M_j\}_{j\in\mathbb{N}_0}$ is uniformly integrable and hence closable.*

Proof. Recall that $p > 1$. Then

$$\sup_{j\in\mathbb{N}_0} \|M_j \mathbf{1}_{(\lambda,\infty]}(|M_j|)\|_{Y^{(\sqrt{p})}} \le \sup_{j\in\mathbb{N}_0} \lambda^{1-\sqrt{p}} \left(\|M_j\|_X\right)^{\sqrt{p}} = \lambda^{1-\sqrt{p}} \left(\|M\|_{\ell^\infty(X)}\right)^{\sqrt{p}} = \mathrm{o}(1),$$

as $\lambda \to \infty$. Since $Y^{(\sqrt{p})}$ is embedded into $L^1(\Omega)$ continuously, M is uniformly integrable and hence closable. □

Using Theorem 3.7, we characterize the spaces $\mathcal{H}^S_X$ and $\mathcal{H}^*_X$.

Theorem 3.9. *Let $1 < p, q < \infty$ and let X be a Banach lattice. Assume that there exist Banach function lattices Y, Z such that the Doob maximal operator is bounded on Y' and Z' and that $X = Y^{(p)}$ and $X' = Z^{(q)}$. Assume in addition that*

$$\mathcal{F} = \sigma\left(\bigcup_{j=0}^{\infty} \mathcal{F}_j\right). \tag{3.5}$$

(1) *The mapping $\Psi : X \in \mathcal{H}^*_X \mapsto X_\infty \in X$ is isomorphic.*
(2) *The mapping $\Phi : (M, Z) \in \mathcal{H}^S_X \times (X \cap \mathcal{F}_0) \mapsto M_\infty + Z \in X$ is isomorphic.*

Before we come to the proof, we indicate how to use the structure of X and X'. From Theorem 3.7, we see that the Doob maximal operator is bounded on X and X'.

Proof. Assertion (1) is a consequence of Theorem 3.7 and Corollary 3.8. We use (3.5) and Corollary 3.8 to show that martingales in $\mathcal{H}^*_X$ are closable. We concentrate on (2). By the closed graph theorem, it suffices to show that Φ is bijective and continuous. Let $r_j(\rho) \equiv \mathrm{sign}(\sin(2^j\pi\rho))$, $\rho \in [0, 1]$ be the j-th Rademacher sequence for each $j \in \mathbb{N}_0$. Then according to Theorem 3.7, we obtain

$$\left\| \sum_{j=0}^{\infty} r_j(\rho) dV_j \right\|_X \lesssim \|M_\infty\|_X \tag{3.6}$$

for all $M_\infty \in X$, where $V \equiv \{E[M_\infty : \mathcal{F}_j] - E[M_\infty : \mathcal{F}_0]\}_{j=0}^\infty$.

Estimate (3.6) together with Khintchine's inequality

$$\int_0^1 \left| \sum_{j=0}^{\infty} r_j(\rho) dV_j \right| \mathrm{d}\rho \sim S(V)$$

(see [24, Theorem 3.1]) show that

$$\|S(V)\|_X \lesssim \|M_\infty\|_X.$$

Thus, $V \in \mathcal{H}^S_X$. Since the Doob maximal operator is bounded on X, $E[M_\infty : \mathcal{F}_0] \in X \cap \mathcal{F}_0$. This implies $\Phi(V, E[M_\infty : \mathcal{F}_0]) = M_\infty$. Thus, the mapping Φ is surjective thanks to (3.5).

Let us show that the mapping Φ is continuous and injective. To this end, we have only to prove that

$$\|M\|_{\ell^\infty(X)} \lesssim \|M\|_{\mathcal{H}^S_X} \tag{3.7}$$

for all $M \in \mathcal{H}^S_X$. Indeed, once (3.7) is proved, we see that any martingale $M \in \mathcal{H}^S_X$ is X-bounded. Thus, (3.7) implies also that any martingale $M \in \mathcal{H}^S_X$ is closable. We can show the injectivity of Φ as follows: Suppose that $M \in \mathcal{H}^S_X$ and $Z \in X \cap \mathcal{F}_0$ satisfy $M_\infty + Z = 0$. Then $Z = E[M_\infty + Z : \mathcal{F}_0] = 0$, since $E[M_\infty : \mathcal{F}_0] = M_0 = 0$ for any $M \in \mathcal{H}^S_X$. Hence $M_\infty = M_\infty + Z = 0$. Thus, Φ is injective.

We can prove (3.7) as follows: Fix $j \in \mathbb{N}_0$. First, since $X'' = X$ with the coincidence of norms according to [1],

$$\|M_j\|_X = \sup_{N \in X', \|N\|_{X'}=1} |E[M_j N]|.$$

Fix $N \in X'$ with $\|N\|_{X'} = 1$ and $M_j N \geq 0$ a.s.. We abbreviate $N_k \equiv E[N : \mathcal{F}_k]$ for $k \in \mathbb{N}_0$. We set $N_{-1} \equiv 0$. Denote by dN the difference process of N: $(dN)_k = N_k - N_{k-1}$ for $k \in \mathbb{N}_0$. Then we have

$$E[M_j N] = \int_\Omega M_j(\omega) N_j(\omega) \mathrm{d}P(\omega) = E\left[\sum_{k=0}^{j} (dM)_k (dN)_k\right].$$

From the previous paragraph, we have

$$\|N\|_{\mathcal{H}^S_{X'}} \lesssim \|N\|_{X'} = 1.$$

Thus,

$$|E[M_j N]| \leq \|M\|_{\mathcal{H}^S_X} \|N\|_{\mathcal{H}^S_{X'}} \lesssim \|M\|_{\mathcal{H}^S_X}$$

by the Cauchy–Schwarz inequality and the definition of the norms of X and X'.

As a result, (3.7) is proved and we conclude that the mapping $(M, Z) \in \mathcal{H}^S_X \times (X \cap \mathcal{F}_0) \mapsto M_\infty + Z \in X$ is continuous and that Φ is injective and continuous. □

If we combine Theorems 1.2 and 3.9, then we obtain the following decomposition result on X.

Theorem 3.10. *Let X be a quasi-Banach function lattice over Ω equipped with a filtration $\mathbb{F}$ satisfying* (3.5) *and let $1 < p, q < \infty$. Assume that there exist Banach function lattices Y, Z such that the Doob maximal operator is bounded on Y' and Z' and that $X = Y^{(p)}$ and $X' = Z^{(q)}$.*

(1) *Let $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k \in \mathbb{Z}} \in \mathcal{A}^S(\infty)$ satisfy $\{\mu^{(k)} \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k \in \mathbb{Z}} \in X(\ell^1)$. Then*

$$M_\infty \equiv \sum_{k=-\infty}^{\infty} \mu^{(k)} (A^{(k)})_\infty \tag{3.8}$$

belongs to X and satisfies

$$\|M_\infty\|_X \lesssim \|\{\mu^{(k)} \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k \in \mathbb{Z}}\|_{X(\ell^1)}.$$

(2) *For all random variables $M_\infty \in X$, there exists $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k \in \mathbb{Z}} \in \mathcal{A}^S(\infty)_\uparrow$ such that $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, that* (3.8) *holds and that*

$$\|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k \in \mathbb{Z}}\|_{X(\ell^u)} \lesssim_u \|M_\infty\|_X \tag{3.9}$$

for all $0 < u < \infty$.

An analogy to $\mathcal{H}^*_X$ is available.

Among others, $X = L^p(\Omega)$ with $1 < p < \infty$ falls within the scope of Theorem 3.10.

4. Examples of X

We recall and compare some recent results on the atomic decomposition of martingale Hardy spaces, as well as the one in the classical book [27]. Section 4.1 reviews the most fundamental case of $X = L^p(\Omega)$ with $0 < p < \infty$. Section 4.2 considers martigale Orlicz–Hardy spaces. We compare the work of Miyamoto, Nakai and Sadasue [21]. Section 4.3 introduces the class $\mathcal{G}$ to define Musielak–Orlicz spaces and then considers martingale Musielak–Orlicz spaces. Section 4.4 deals with a variant of Section 4.3, where we define Musielak–Orlicz–Lorentz spaces and then consider martingale Musielak–Orlicz–Lorentz spaces. Although Section 4.4 seems parallel to Section 4.3, we need to transform Theorem 1.2 to compare our results with the ones obtained in [11]. Sections 4.5 and 4.6 can be viewed as special cases of Sections 4.3 and 4.4, respectively. Section 4.5 deals with martingale variable exponent Hardy spaces and Section 4.6 deals with martingale variable exponent Hardy–Lorentz spaces. Section 4.7 investigates martingale Hardy spaces generated by the weak variant of a given Banach function lattice. Note that there are many other Banach lattices which are used to consider martingale Hardy spaces, but will not be considered in Section 4: See [7,17,26], for example, for such quasi-Banach lattices.

4.1. $X = L^p(\Omega)$ **with** $0 < p < \infty$**–martingale Hardy spaces.** Let $0 < p < \infty$. Then as in [27], we abbreviate the *martingale Hardy spaces* $\mathcal{H}^S_{L^p(\Omega)}$, $\mathcal{H}^s_{L^p(\Omega)}$ and $\mathcal{H}^*_{L^p(\Omega)}$ to $\mathcal{H}^S_p$, $\mathcal{H}^s_p$ and $\mathcal{H}^*_p$, respectively. If $1 < p < \infty$, then Theorem 3.10 reveals the structure of $\mathcal{H}^S_p$ and $\mathcal{H}^*_p$. Based on [27], we will explain how to obtain the classical results from Theorem 1.2 assuming $0 < p \le 1$. Let $\{E_k\}_{k\in\mathbb{Z}}$ be a sequence of measurable sets and let $\{\mu^{(k)}\}_{k\in\mathbb{Z}}$ be a sequence of nonnegative real numbers. Letting $p = u$ in (3.9), we obtain

$$\|\{\mu^{(k)}\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{X(\ell^u)} = \left(\sum_{k=-\infty}^{\infty}(\mu^{(k)})^p P(E_k)\right)^{\frac{1}{p}}.$$

Meanwhile, assuming that $0 < p \le 1$, we have

$$\left(\sum_{k=-\infty}^{\infty}(\mu^{(k)})^p P(E_k)\right)^{\frac{1}{p}} = \|\{\mu^{(k)}\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{X(\ell^p)} \ge \|\{\mu^{(k)}\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{X(\ell^1)}.$$

Therefore Theorem 1.2 recaptures the well-known theorem in [27].

Theorem 4.1. *Let* $0 < p \le 1$.

(1) *Suppose that we have* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)$ *such that*

$$\sum_{k=-\infty}^{\infty}(\mu^{(k)})^p P(\tau^{(k)} < \infty) < \infty. \tag{4.1}$$

Then the process M*, given by* (1.1)*, belongs to* $\mathcal{H}^s_p$ *and satisfies*

$$\|M\|_{\mathcal{H}^s_p} \le \left(\sum_{k=-\infty}^{\infty}(\mu^{(k)})^p P(\tau^{(k)} < \infty)\right)^{\frac{1}{p}}.$$

(2) *For all* $M \in \mathcal{H}^s_p$*, there exists* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ *such that* $\mu^{(k)} = 2^k$*,* $k \in \mathbb{Z}$*, that* (1.1) *is satisfied and that*

$$\left(\sum_{k=-\infty}^{\infty}2^{kp} P(\tau^{(k)} < \infty)\right)^{\frac{1}{p}} \lesssim \|M\|_{\mathcal{H}^s_p}. \tag{4.2}$$

Analogies to $\mathcal{H}^S_p$ *and* $\mathcal{H}^*_p$ *are available.*

We end Section 4.1 with an application. We have been considering a bounded predictable process N. We will enlarge the class of N.

Theorem 4.2. *Let* X *be a Banach function lattice as in Theorem* 3.10*. Let* $N_\infty \in X'$*. Write* $N \equiv \{E[N_\infty : \mathcal{F}_{j-1}]\}_{j\in\mathbb{N}_0}$*. Then* $N* : \mathcal{H}^S_X \to \mathcal{H}^S_1$ *is bounded.*

Proof. Let $M \in \mathcal{H}^S_X$. Then $N * M_0 = 0$ and

$$S(N * M) \le S(M)N^*.$$

Thus, by the Cauchy–Schwarz inequality, the boundedness of the Doob maximal operator on X' and Theorem 3.10, we obtain the desired result. □

Theorem 4.2 is useful when we consider the quadratic variation for continuous martingales. We omit further details.

4.2. $X = L^\phi(\Omega)$–**martingale Orlicz–Hardy spaces.** In [21], Miyamoto, Nakai and Sadasue assumed that $\phi : [0,\infty) \to [0,\infty)$ is a measurable function satisfying

$$\phi(tr) \le c_\phi \max(t^l, t)\phi(r) \quad (t, r \in [0,\infty)) \tag{4.3}$$

for some $l \in (0,1]$ and $c_\phi > 0$. Here, we do not consider the dependency on $\omega \in \Omega$ of ϕ. The *Orlicz space* $L^\phi(\Omega)$ is defined to be the quasi-Banach space of all $f \in L^0(\Omega)$ such that $\phi(\lambda^{-1}|f|) \in L^1(\Omega)$ for some $0 < \lambda < \infty$. The norm of $f \in L^\phi(\Omega)$ is given by

$$\|f\|_{L^\phi(\Omega)} \equiv \inf\left\{\lambda > 0 \,:\, E[\Phi(\lambda^{-1}|f|)] \le 1\right\}.$$

We abbreviate the *martingale Orlicz Hardy spaces* $\mathcal{H}^S_{L^\phi}$, $\mathcal{H}^s_{L^\phi}$ and $\mathcal{H}^*_{L^\phi}$ to $\mathcal{H}^S_\phi$, $\mathcal{H}^s_\phi$ and $\mathcal{H}^*_\phi$, respectively. We transplant Theorem 1.2 to this setting.

Theorem 4.3. *Let $\phi : [0,\infty) \to [0,\infty)$ be a measurable function satisfying* (4.3)*.*

(1) *Let $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)$ satisfy $\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}} \in L^\phi(\Omega)$. Then the process M, given by* (1.1)*, belongs to $\mathcal{H}^s_\phi$ and satisfies $\|M\|_{\mathcal{H}^s_\phi} \le \|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^\phi(\Omega;\ell^1)}$.*

(2) *For all $M \in \mathcal{H}^s_\phi$, there exists $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ such that $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, that* (1.1) *is satisfied and that*

$$\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^\phi(\Omega;\ell^u)} \lesssim_u \|M\|_{\mathcal{H}^s_\phi} \tag{4.4}$$

for all $0 < u < \infty$.

Analogies to $\mathcal{H}^S_\phi$ and $\mathcal{H}^*_\phi$ are available.

We recall the result in [21, Theorem 2.3]. To do a comparison of Theorems 4.3 and 4.4, we start with (3) to enumerate.

Theorem 4.4. *Let $\phi : [0,\infty) \to [0,\infty)$ be a measurable function satisfying* (4.3)*.*

(3) *Let $1 < q < \infty$. Suppose that we have a sequence $\{\mu^{(k)}\}_{k\in\mathbb{Z}}$ of nonnegative real numbers, a sequence $\{A^{(k)}\}_{k\in\mathbb{Z}}$ of processes and a sequence $\{\tau^{(k)}\}_{k\in\mathbb{Z}}$ of stopping times such that each $A^{(k)}$ satisfies*

$$\|s(A^{(k)})\|_{L^q(\Omega)} \le P(\tau^{(k)} < \infty)^{\frac{1}{q}}, \quad A^{(k)} = 0 \text{ on } [0, \tau^{(k)}] \tag{4.5}$$

and that

$$\sum_{k=-\infty}^{\infty} \phi\left(\mu^{(k)}\right) P(\tau^{(k)} < \infty) \le 1. \tag{4.6}$$

Then the process M, given by (1.1)*, belongs to $\mathcal{H}^s_\phi$ and satisfies $\|M\|_{\mathcal{H}^s_\phi} \lesssim 1$.*

(4) *For all $M \in \mathcal{H}^s_\phi$ with $\|M\|_{\mathcal{H}^s_\phi} = 1$, there exists $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ such that $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, that* (1.1) *is satisfied and that*

$$\sum_{k=-\infty}^{\infty} \phi\left(2^k\right) P(\tau^{(k)} < \infty) \lesssim 1. \tag{4.7}$$

Analogies to $\mathcal{H}^S_\phi$ and $\mathcal{H}^*_\phi$ are available.

To prove Theorem 4.4(4) Miyamoto, Nakai and Sadasue used stopping times that differ from the one given by (2.1) (see [21, p. 677]). So, Theorem 4.3 does not completely recapture Theorem 4.4. Here we give more details on the difference between Theorems 4.3 and 4.4.

Remark 4.5.

(1) Let $1 < q < \infty$. Theorem 4.3(1) fails to cover Theorem 4.4(3). Miyamoto, Nakai and Sadasue used a different estimate that arises from the structure of ϕ. According to [21, Lemma 3.1],

$$E[\phi(|f|)] \le 2P(B)\phi\left(\frac{\|f\|_{L^q(\Omega)}}{P(B)^{\frac{1}{q}}}\right)$$

for all $f \in L^q(\Omega)$ and $B \in \mathcal{F}$ with $\{f \ne 0\} \subset B$.

(2) Let $q = \infty$. If we argue similarly to [21, Theorem 2.3(i)] by using different stopping times, we can modify the argument in Theorem 4.3(1) to recover Theorem 4.4(3). We omit further details.
(3) Likewise, if we reexamine and modify the proof of Theorem 4.3(2), then Theorem 4.3(2) recaptures Theorem 4.4(4).
(4) It seems interesting to compare the techniques in [10] with the ones of Theorems 4.3 and 4.4.

4.3. $X = L^{\phi}(\Omega)$**–martingale Musielak–Orlicz–Hardy spaces.** We follow the idea of [11, 20] to define Musielak–Orlicz spaces and martingale Musielak–Orlicz–Hardy spaces.

Definition 4.6.

(1) A function $\phi : \Omega \times [0,\infty] \to [0,\infty]$ is said to be a *Musielak–Orlicz function* if the function $\omega \in \Omega \mapsto \phi(\omega, |f(\omega)|)$ is measurable for any $f \in L^0(\Omega)$ and the function $t \in [0,\infty) \mapsto \phi(\omega, t) \in [0,\infty]$ is nondecreasing with $\phi(\omega, 0) = \lim_{t\downarrow 0} \phi(\omega, t) = 0$ and $\phi(\omega, \infty) = \lim_{t\to\infty} \phi(\omega, t) = \infty$ for a.s. $\omega \in \Omega$. The set of all Musielak–Orlicz functions will be denoted by $\mathcal{G}$.
(2) Let $\phi \in \mathcal{G}$. The *Musielak–Orlicz space* $L^{\phi}(\Omega)$ is defined to be the quasi-Banach space of all $f \in L^0(\Omega)$ such that

$$\int_{\Omega} \phi\left(\omega, \frac{|f(\omega)|}{\lambda}\right) \mathrm{d}P(\omega) < \infty$$

for some $0 < \lambda < \infty$. The norm of $f \in L^{\phi}(\Omega)$ is given by

$$\|f\|_{L^{\phi}(\Omega)} \equiv \inf\left\{\lambda > 0 \,:\, \int_{\Omega} \phi\left(\omega, \frac{|f(\omega)|}{\lambda}\right) \mathrm{d}P(\omega) \le 1\right\}.$$

(3) Let $p \in (0,\infty)$. A function $\phi \in \mathcal{G}$ is said to be *uniformly lower* (resp., *upper*) type p, if there exists $E \in \mathcal{F}$ with $P(E) = 1$ such that

$$\sup_{\substack{(\omega,t,s)\in E\times(0,\infty)\times(0,1),\\ \phi(\omega,t)>0}} \frac{\phi(\omega, st)}{s^p \phi(\omega, t)} < \infty \left(\text{resp.} \sup_{\substack{(\omega,t,s)\in E\times(0,\infty)\times[1,\infty),\\ \phi(\omega,t)>0}} \frac{\phi(\omega, st)}{s^p \phi(\omega, t)} < \infty\right).$$

As in [11, 20], we abbreviate the *martingale Musielak–Orlicz Hardy spaces* $\mathcal{H}^S_{L^{\phi}(\Omega)}$, $\mathcal{H}^s_{L^{\phi}(\Omega)}$ and $\mathcal{H}^*_{L^{\phi}(\Omega)}$ to $\mathcal{H}^S_{\phi}$, $\mathcal{H}^s_{\phi}$ and $\mathcal{H}^*_{\phi}$, respectively. We consider Theorem 1.2 with $X = L^{\phi}(\Omega)$.

Theorem 4.7. *Let* $\phi \in \mathcal{G}$.

(1) *Let* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)$ *satisfy* $\{\mu^{(k)}\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}} \in L^{\phi}(\Omega;\ell^1)$. *Then the process* M, *given by* (1.1), *belongs to* $\mathcal{H}^s_{\phi}$ *and satisfies* $\|M\|_{\mathcal{H}^s_{\phi}} \le \|\{\mu^{(k)}\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{L^{\phi}(\Omega;\ell^1)}$.
(2) *For all* $M \in \mathcal{H}^s_{\phi}$, *there exists* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_{\uparrow}$ *such that* $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, *that* (1.1) *is satisfied and that* $\|\{\mu^{(k)}\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{L^{\phi}(\Omega;\ell^u)} \lesssim_u \|M\|_{\mathcal{H}^s_{\phi}}$ *for all* $0 < u < \infty$.

Analogies to $\mathcal{H}^S_{\phi}$ *and* $\mathcal{H}^*_{\phi}$ *are available.*

Xie, Jiao and Yang obtained the following result [29].

Theorem 4.8. *Let* $1 < q \le \infty$ *and* $\phi : [0,\infty] \times \Omega \to [0,\infty]$ *be a Musielak–Orlicz function of uniformly lower type* $p \in (0,1]$.

(3) *Suppose that we have* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)$ *such that each* $A^{(k)}$ *satisfies* (4.5) *and that*

$$\sum_{k=-\infty}^{\infty} \int_{\{\tau^{(k)}<\infty\}} \varphi(\omega, \mu^{(k)}) \mathrm{d}P(\omega) \le 1. \tag{4.8}$$

Then the process M, *given by* (1.1), *belongs to* $\mathcal{H}^s_{\phi}$ *and satisfies* $\|M\|_{\mathcal{H}^s_{\phi}} \lesssim 1$.

(4) *For all* $M \in \mathcal{H}^s_\phi$ *with* $\|M\|_{\mathcal{H}^s_\phi} = 1$, *there exists* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ *such that* $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, *that* (1.1) *is satisfied and that*

$$\sum_{k=-\infty}^{\infty} \int_{\{\tau^{(k)}<\infty\}} \varphi(\omega, 2^k)\mathrm{d}P(\omega) \lesssim 1. \tag{4.9}$$

Analogies to $\mathcal{H}^S_\phi$ *and* $\mathcal{H}^*_\phi$ *are available.*

A remark similar to Section 4.2 applies to martingale Musielak–Orlicz spaces.

4.4. $X = L^{\phi,q}(\Omega)$**–martingale Musielak–Orlicz–Lorentz–Hardy spaces.** We consider similar function spaces to Musielak–Orlicz spaces. However, we encounter a big difference between Sections 4.3 and 4.4. We start with the definition of the quasi-Banach function lattice $L^{\phi,q}(\Omega)$.

Definition 4.9. Let $\phi \in \mathcal{G}$ and $0 < q < \infty$. The *Musielak–Orlicz–Lorentz space* $L^{\phi,q}(\Omega)$ is defined to be the quasi-Banach space of all $f \in L^0(\Omega)$ such that

$$\|f\|_{L^{\phi,q}(\Omega)} \equiv \left(\int_0^\infty \left(\rho \|\mathbf{1}_{(\rho,\infty]}(|f|)\|_{L^\phi(\Omega)}\right)^q \frac{\mathrm{d}\rho}{\rho} \right)^{\frac{1}{q}} < \infty.$$

Let $\phi \in \mathcal{G}$ and $0 < q < \infty$. It is noteworthy that $\|\chi_E\|_{L^{\phi,q}(\Omega)} \sim \|\chi_E\|_{L^\phi(\Omega)}$ for $E \in \mathcal{F}$ and that

$$\|\{2^k \mathbf{1}_{(2^k,\infty]}(|f|)\}_{k\in\mathbb{Z}}\|_{L^\phi(\Omega;\ell^q)} \sim \|f\|_{L^{\phi,q}(\Omega)} \tag{4.10}$$

for any $f \in L^0(\Omega)$. See [19, Lemma 2.4].

As in [11], we abbreviate the *martingale Musielak–Orlicz–Lorentz–Hardy spaces* $\mathcal{H}^S_{L^{\phi,q}(\Omega)}$, $\mathcal{H}^s_{L^{\phi,q}(\Omega)}$ and $\mathcal{H}^*_{L^{\phi,q}(\Omega)}$ to $\mathcal{H}^S_{\phi,q}$, $\mathcal{H}^s_{\phi,q}$ and $\mathcal{H}^*_{\phi,q}$, respectively.

We summarize what is known so far with the help of Theorem 1.2:

Theorem 4.10. *Let* $\phi \in \mathcal{G}$ *and* $0 < q < \infty$.

(1) *Let* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)$ *satisfy* $\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}} \in L^\phi(\Omega;\ell^1)$. *Then we have* $M \in \mathcal{H}^s_{\phi,q}$ *and* M *satisfies* $\|M\|_{\mathcal{H}^s_{\phi,q}} \le \left\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\right\|_{L^{\phi,q}(\Omega;\ell^1)}$, *where* M *is the process, given by* (1.1).

(2) *For all* $M \in \mathcal{H}^s_{\phi,q}$, *there exists* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ *such that* $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, *that* (1.1) *is satisfied and that* $\left\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\right\|_{L^{\phi,q}(\Omega;\ell^u)} \lesssim_u \|M\|_{\mathcal{H}^s_{\phi,q}}$ *for all* $0 < u < \infty$.

Analogies to $\mathcal{H}^S_{\phi,q}$ *and* $\mathcal{H}^*_{\phi,q}$ *are available.*

We now recall [11, Theorem 5.1].

Theorem 4.11. *Let* $\phi \in \mathcal{G}$ *be a function of uniformly lower type* p^-_ϕ *and of uniformly upper type* p^+_ϕ *with* $0 < p^-_\phi \le p^+_\phi < \infty$, *and let* $0 < q < \infty$.

(3) *If we assume* $\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}} \in \ell^q(L^\phi(\Omega))$ *instead of* $\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}} \in L^\phi(\Omega;\ell^1)$ *in Theorem* 4.10(1), *then* $\|M\|_{\mathcal{H}^s_{\phi,q}} \lesssim \left\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\right\|_{\ell^q(L^\phi(\Omega))}$.

(4) *For all* $M \in \mathcal{H}^s_{\phi,q}$, *there exists* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ *such that* $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, *that* (1.1) *is satisfied and that* $\left\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\right\|_{\ell^q(L^\phi(\Omega))} \lesssim \|M\|_{\mathcal{H}^s_{\phi,q}}$.

Analogies to $\mathcal{H}^S_{\phi,q}$ *and* $\mathcal{H}^*_{\phi,q}$ *are available.*

We deduce Theorem 4.11 from Theorem 1.2.

Proof. For an increasing sequence $\{\tau^{(k)}\}_{k\in\mathbb{Z}}$ of stopping times, we let

$$f \equiv \sum_{k=-\infty}^{\infty} 2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})$$

in (4.10). Then $\mathbf{1}_{(2^k,\infty]}(|f|) = \mathbf{1}_{\mathbb{R}}(\tau^{(k)})$ for all $k \in \mathbb{Z}$. Thus from (4.10),

$$\|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^{\phi,q}(\Omega;\ell^1)} \sim \|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{\ell^q(L^\phi(\Omega))}.$$

Hence Theorem 1.2(2) implies Theorem 4.11(4).

We will verify that Theorem 1.2(1) implies Theorem 4.11(3). To this end, it suffices to show that

$$\|\{2^k\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{L^{\phi,q}(\Omega;\ell^1)} \lesssim \|\{2^k\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{\ell^q(L^\phi(\Omega))} \tag{4.11}$$

for any sequence $\{E_k\}_{k\in\mathbb{Z}}$ in $\mathcal{F}$. Going through the argument similar to (2.2), we observe

$$\|\{2^k\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{L^{\phi,q}(\Omega;\ell^1)} \le 2\|\{2^k\mathbf{1}_{E_k\setminus(E_{k+1}\cup E_{k+2}\cup\cdots)}\}_{k\in\mathbb{Z}}\|_{L^{\phi,q}(\Omega;\ell^1)}.$$

From (4.10), we obtain

$$\|\{2^k\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{L^{\phi,q}(\Omega;\ell^1)} \lesssim \|\{2^k\mathbf{1}_{E_k\setminus(E_{k+1}\cup E_{k+2}\cup\cdots)}\}_{k\in\mathbb{Z}}\|_{\ell^q(L^\phi(\Omega))} \lesssim \|\{2^k\mathbf{1}_{E_k}\}_{k\in\mathbb{Z}}\|_{\ell^q(L^\phi(\Omega))}.$$

This implies (4.11). □

We end Section 4.4 with a historical remark on the space $L^{\phi,q}(\Omega)$, as well as the space $L^{\phi,\infty}(\Omega)$ that will be defined in Section 4.7 below. We remark that the vector-valued Doob maximal inequality for $L^{\phi,q}(\Omega)$ with $\phi\in\mathcal{G}$ and $0<q\le\infty$ is obtained in [11, Theorem 4.5]. Theorem 3.6 yields one only for $L^{\phi,q}(\Omega)$ with $\phi\in\mathcal{G}$ and $1\le q\le\infty$. If we use an interpolation inequality [20, Corollary 3.8], then we can recover the case of $0<q<1$.

4.5. $X = L^{p(\cdot)}(\Omega)$–**martingale variable Hardy spaces.** We recall the results in [28]. Here, we work in the right-open interval $[0,1)$, equipped with the Lebesgue measure $d\omega$. We also specify $\mathcal{F}_j$, $j\in\mathbb{N}$ to be the σ-algebra generated by $I_{jk}\equiv[2^{-j}k, 2^{-j}(k+1))$, $k=0,1,\ldots,2^j-1$, namely, we specify

$$\mathcal{F}_j \equiv \sigma(\{I_{jk}\}_{k=0}^{2^j-1}). \tag{4.12}$$

We consider the following class of variable exponents: We denote by $C^{\log}$ the set of all functions $p(\cdot)\in L^\infty([0,1))$ satisfying the "so-called" *globally log-Hölder continuous condition*, namely, there exists a positive constant $C_{\log(p)}$ such that, for any $\omega_1,\omega_2\in[0,1)$,

$$|p(\omega_1)-p(\omega_2)| \le \frac{C_{\log(p)}}{\log(e+1/|\omega_1-\omega_2|)}.$$

We assume that

$$p_- \equiv \inf_{\omega\in[0,1)} p(\omega) > 0.$$

We postulate $p(\cdot)$ on these conditions to guarantee that the Doob maximal operator is bounded in this setting. We can consider $\phi(x,t)=t^{p(x)}$ as a special case in Section 4.3. As in [28], we abbreviate the *martingale variable exponent Hardy spaces* $\mathcal{H}^S_{L^{p(\cdot)}(\Omega)}$, $\mathcal{H}^s_{L^{p(\cdot)}(\Omega)}$ and $\mathcal{H}^*_{L^{p(\cdot)}(\Omega)}$ to $\mathcal{H}^S_{p(\cdot)}$, $\mathcal{H}^s_{p(\cdot)}$ and $\mathcal{H}^*_{p(\cdot)}$, respectively.

Theorem 1.2 recaptures the results in [13, 14, 28].

Theorem 4.12. *Let $p(\cdot)\in C^{\log}$ and $\mathcal{F}_j$ be the σ-algebra defined by* (4.12) *for $j\in\mathbb{N}_0$.*

(1) *Let $\{(\mu^{(k)},A^{(k)},\tau^{(k)})\}_{k\in\mathbb{Z}}\in\mathcal{A}^s(\infty)$ satisfy $\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\in L^{p(\cdot)}(\Omega;\ell^1)$. Define the process M by* (1.1)*. Then $M\in\mathcal{H}^s_{p(\cdot)}$ and $\|M\|_{\mathcal{H}^s_{p(\cdot)}}\le\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^{p(\cdot)}(\Omega;\ell^1)}$.*

(2) *For all $M\in\mathcal{H}^s_{p(\cdot)}$, there exists $\{(\mu^{(k)},A^{(k)},\tau^{(k)})\}_{k\in\mathbb{Z}}\in\mathcal{A}^s(\infty)_\uparrow$ such that $\mu^{(k)}=2^k$, $k\in\mathbb{Z}$, that* (1.1) *is satisfied and that $\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^{p(\cdot)}(\Omega;\ell^u)}\lesssim_u\|M\|_{\mathcal{H}^s_{p(\cdot)}}$ for all $0<u<\infty$.*

Analogies to $\mathcal{H}^S_{p(\cdot)}$ and $\mathcal{H}^*_{p(\cdot)}$ are available.

A couple of helpful remarks are in order.

Remark 4.13.

(1) We can also replace $\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^{p(\cdot)}(\Omega;\ell^1)}$ by $\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^{p(\cdot)}(\Omega;\ell^u)}$ for some fixed $u\in(0,p_-]$ in Theorem 4.12 (1).

(2) As in [28, Theorem 3.1], it suffices to consider $\|\{\mu^{(k)}\mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^{p(\cdot)}(\Omega;\ell^u)}$ for some fixed $u\in(0,p_-]$ in Theorem 4.12 (2).

To conclude this section, we recall some related results:

- The vector-valued Doob inequality for $L^{p(\cdot)}([0,1))$ is essentially obtained in [2, Corollary 2.1].

- In [12], using the decreasing rearrangement, Jiao, Zeng and Zhou defined martingale variable exponent Hardy spaces over a probability space $(\Omega, \mathcal{F}, P)$ and obtained Theorem 1.2 for their spaces.
- In [6, 13, 14], Chen, Jiao, Weisz, Zhou and Zhao considered a more general framework. We learn that this framework also falls within the scope of Theorem 1.2 if we argue similarly to [22, Theorem 4.5].

4.6. $X = L^{p(\cdot),q}(\Omega)$–**martingale variable Lorentz–Hardy spaces.** Variable Lorentz spaces were introduced and investigated by Kempka and Vybíral [16]. Let $0 < q < \infty$ and $p(\cdot) \in C^{\log}$. The *variable Lorentz space* $L^{p(\cdot),q}([0,1))$ is defined to be the space of all $f \in L^0([0,1))$ such that

$$\|f\|_{L^{p(\cdot),q}([0,1))} \equiv \left(\int_0^\infty \left(\rho \|\mathbf{1}_{(\rho,\infty]}(|f|)\|_{p(\cdot)}\right)^q \frac{\mathrm{d}\rho}{\rho} \right)^{\frac{1}{q}} < \infty.$$

See [16, §2.3]. As in [15], we abbreviate the *martingale variable Lorentz–Hardy spaces* $\mathcal{H}^S_{L^{p(\cdot),q}(\Omega)}$, $\mathcal{H}^s_{L^{p(\cdot),q}(\Omega)}$ and $\mathcal{H}^*_{L^{p(\cdot),q}(\Omega)}$ to $\mathcal{H}^S_{p(\cdot),q}$, $\mathcal{H}^s_{p(\cdot),q}$ and $\mathcal{H}^*_{p(\cdot),q}$, respectively. As we did in [6, 13, 14], we can change the underlying space $[0,1)$ to more general probability spaces. Here, for the sake of simplicity, we maintain the setting of Section 4.5.

We transplant Theorem 1.2 to $L^{p(\cdot),q}(\Omega)$.

Theorem 4.14. *Let $0 < q < \infty$, $p(\cdot) \in C^{\log}$ and $\mathcal{F}_j$ be the σ-algebra defined by (4.12) for $j \in \mathbb{N}_0$.*

(1) *Let $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)$ satisfy $\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}} \in L^{p(\cdot),q}(\Omega;\ell^1)$. Then the process M, given by (1.1) with $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, belongs to $\mathcal{H}^s_{p(\cdot),q}$ and satisfies*

$$\|M\|_{\mathcal{H}^s_{p(\cdot),q}} \le \|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^{p(\cdot),q}(\Omega;\ell^1)}.$$

(2) *For all $M \in \mathcal{H}^s_{p(\cdot),q}$, there exists $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ such that (1.1) is satisfied for $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, and that $\|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^{p(\cdot),q}(\Omega;\ell^u)} \lesssim_u \|M\|_{\mathcal{H}^s_{p(\cdot),q}}$ for all $0 < u < \infty$.*

Analogies to $\mathcal{H}^S_{p(\cdot),q}$ and $\mathcal{H}^*_{p(\cdot),q}$ are available.

We can deduce Theorem 4.15 below from Theorem 4.14 in a similar manner to deduce Theorem 4.11 from Theorem 1.2.

Theorem 4.15. [15] *Let $0 < q < \infty$, $p(\cdot) \in C^{\log}$ and $\mathcal{F}_j$ be the σ-algebra defined by (4.12) for $j \in \mathbb{N}_0$.*

(3) *Assume $\{\mu^{(k)} \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}} \in \ell^q(L^{p(\cdot)}(\Omega))$ instead of $\{\mu^{(k)} \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}} \in L^{p(\cdot),q}(\Omega;\ell^1)$ in Theorem 4.14(1). Then*

$$\|M\|_{\mathcal{H}^s_{p(\cdot),q}} \lesssim \|\{\mu^{(k)} \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{\ell^q(L^{p(\cdot)}(\Omega))}. \tag{4.13}$$

(4) *For all $M \in \mathcal{H}^s_{p(\cdot),q}$, there exists $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ such that (1.1) is satisfied for $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, and that $\|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{L^{p(\cdot),q}(\Omega;\ell^u)} \lesssim_u \|M\|_{\mathcal{H}^s_{p(\cdot),q}}$ for all $0 < u < \infty$.*

Analogies to $\mathcal{H}^S_{p(\cdot),q}$ and $\mathcal{H}^*_{p(\cdot),q}$ are available.

4.7. **Weak martingale Hardy spaces generated by a quasi-Banach function lattice.** We now revisit the work [19]. For a quasi-Banach function lattice X over Ω, we define the corresponding *weak space* $\mathrm{w}X$ as the quasi-Banach space of all $f \in L^0(\Omega)$ for which the quasi-norm $\|f\|_{\mathrm{w}X} \equiv \sup_{\lambda>0} \lambda \|\mathbf{1}_{(\lambda,\infty]}(|f|)\|_X$ is finite. By mimicking what we did in Section 4.4, we have the following characterization, which recaptures [19].

Theorem 4.16. *Let X be a Banach function lattice.*

(1) *Suppose that we have $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)$ such that $\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathrm{w}X(\ell^1)$. Then the process M, given by (1.1), belongs to $\mathcal{H}^s_{\mathrm{w}X}$ and satisfies*

$$\|M\|_{\mathcal{H}^s_{\mathrm{w}X}} \le \|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{\mathrm{w}X(\ell^1)}.$$

(2) [19] *Let* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)$ *satisfy* $\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}} \in \ell^\infty(X)$. *Then the process* M, *given by* (1.1) *with* $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, *belongs to* $\mathcal{H}^s_{\mathrm{w}X}$ *and satisfies*

$$\|M\|_{\mathcal{H}^s_{\mathrm{w}X}} \lesssim \|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{\ell^\infty(X)}.$$

(3) *For all* $M \in \mathcal{H}^s_{\mathrm{w}X}$, *there exists* $\{(\mu^{(k)}, A^{(k)}, \tau^{(k)})\}_{k\in\mathbb{Z}} \in \mathcal{A}^s(\infty)_\uparrow$ *such that* $\mu^{(k)} = 2^k$, $k \in \mathbb{Z}$, *that* (1.1) *is satisfied and that*

$$\|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{\ell^u(X)} \sim \|\{2^k \mathbf{1}_{\mathbb{R}}(\tau^{(k)})\}_{k\in\mathbb{Z}}\|_{\mathrm{w}X(\ell^u)} \lesssim_u \|M\|_{\mathcal{H}^s_{\mathrm{w}X}} \tag{4.14}$$

for all $0 < u < \infty$.

Analogies to $\mathcal{H}^S_{\mathrm{w}X}$ *and* $\mathcal{H}^*_{\mathrm{w}X}$ *are available.*

To conclude Section 4.7, we give examples of X.

Example 4.17.

(1) Let $\phi \in \mathcal{G}$ as in Sections 4.3 and 4.4. The *Musielak–Orlicz–Lorentz space* or the *weak Musielak–Orlicz space* $L^{\phi,\infty}(\Omega)$ is defined to be the quasi-Banach space of all $f \in L^0(\Omega)$ such that

$$\|f\|_{L^{\phi,\infty}(\Omega)} \equiv \sup_{\rho>0} \rho \|\mathbf{1}_{(\rho,\infty]}(|f|)\|_{L^\phi(\Omega)} < \infty.$$

In [11], Jiao, Weisz, Xie and Yang investigated the *martingale Musielak–Orlicz–Lorentz–Hardy spaces* (the *weak martingale Musielak–Orlicz–Hardy spaces*) $\mathcal{H}^S_{\phi,\infty}(\Omega)$, $\mathcal{H}^s_{\phi,\infty}(\Omega)$ and $\mathcal{H}^*_{\phi,\infty}(\Omega)$.

(2) Work in the setting in Sections 4.5 and 4.6. The *variable Lorentz space*, or the *weak variable Lebesgue space* $L^{p(\cdot),\infty}([0,1))$ is defined to be the space of all $f \in L^0([0,1))$ such that

$$\|f\|_{L^{p(\cdot),\infty}([0,1))} \equiv \sup_{\rho>0} \rho \|\mathbf{1}_{(\rho,\infty]}(|f|)\|_{p(\cdot)} < \infty.$$

Jiao, Zhou, Weisz and Wu investigated the *variable Lorentz–Hardy space* (the *weak variable Hardy space*) $\mathcal{H}^S_{p(\cdot),\infty}(\Omega)$, $\mathcal{H}^s_{p(\cdot),\infty}(\Omega)$ and $\mathcal{H}^*_{p(\cdot),\infty}(\Omega)$ in [15].

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