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WEIGHTED ESTIMATES OF THE UNILATERAL POTENTIALS

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Dedicated to the memory of Academician Vakhtang Kokilashvili

Abstract. We consider weighted unilateral ball potentials with radial quasi-monotone weights, as well as unilateral potentials related to the half-space $\mathbb{R}^n_+$, with quasi-monotone weights depending on $x_n > 0$. We give the sufficient and, in some cases, necessary conditions for the $L^p \to L^q$-boundedness of these potentials.

For some subclasses of quasi-monotone weights, we prove a pointwise estimate of the weighted potentials via the non-weighted potential and the weighted Hardy operator, which may be used for an arbitrary Banach function space with the lattice property.

1. Introduction

We study weighted estimates for the so-called unilateral potentials, called also one-sided potentials. There are known two versions of the unilateral potentials. One is known as "ball potentials" and defined as

$$B^\alpha_+ f(x) = \int\limits_{|y|<|x|} \frac{(|x|^2-|y|^2)^\alpha}{|x-y|^n} f(y)\,dy, \qquad \alpha>0, \tag{1.1}$$

and

$$B^\alpha_- f(x) = \int\limits_{|y|>|x|} \frac{(|y|^2-|x|^2)^\alpha}{|x-y|^n} f(y)\,dy, \qquad \alpha>0. \tag{1.2}$$

They were introduced in [12, 13], see also [15] and [16]. The main objective of the study in [12–15] and [16] was to obtain the inversion to the operators B^α_+ and B^α_- and, consequently, to the Riesz potential, due to the factorization formula provided below (see (1.6)). Ball potentials were also studied in [7], where for the potentials B^α_+ and B^α_- with respect to an arbitrary measure there were found the necessary and sufficient conditions for the boundedness and compactness from L^p to L^q, $1<p<\infty$, $0<q<\infty$, in the case $\alpha > \frac{n}{p}$. We also refer to [23], where some remarks on the ball potentials in variable exponent Lebesgue spaces may be found.

Another version is related to the half-space and defined by

$$I^\alpha_\pm f(x) = \int\limits_{\mathbb{R}^n_+} \frac{y_n^\alpha}{|y|^n} f(x\mp y)dy, \qquad x\in\mathbb{R}^n, \tag{1.3}$$

where $\mathbb{R}^n_+ = \{x\in\mathbb{R}^n:\ x_n>0\}$. In [14], the operators, inverse to $I^\alpha_\pm$, were constructed. The operators (1.3) probably first appeared in [1]. Note that $I^\alpha_+ I^\alpha_-$ coincides with the Riesz potential operator $I^{2\alpha}$ up to a constant factor.

We consider slightly modified ball potentials in a weighted form:

$$\mathbb{B}^\alpha_{+,w} f(x) = \frac{w(|x|)}{|x|^\alpha} \int\limits_{|y|<|x|} \frac{(|x|^2-|y|^2)^\alpha}{w(|y|)|x-y|^n} f(y)\,dy \tag{1.4}$$

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and

$$\mathbb{B}^{\alpha}_{-,w} f(x) = w(|x|) \int\limits_{|y|>|x|} \frac{(|y|^2 - |x|^2)^{\alpha}}{w(|y|)\,|x-y|^n} \frac{f(y)}{|y|^{\alpha}} dy, \tag{1.5}$$

and we write $\mathbb{B}^{\alpha}_{\pm} := \mathbb{B}^{\alpha}_{\pm,w}\Big|_{w\equiv 1}$. Note that

$$\mathbb{B}^{\alpha}_{+}\mathbb{B}^{\alpha}_{-} = C(n,\alpha) I^{2\alpha}, \tag{1.6}$$

where $I^{2\alpha}$ is the Riesz potential operator and $C(n,\alpha)$ is a certain constant (see [15]).

We also study certain weighted versions of the unilateral potentials related to the half-space.

Since the potentials (1.4) and (1.5) are dominated by the corresponding weighted Riesz potentials (cf., (2.1)), the weighted ball potentials (1.4) and (1.5) are bounded, e.g., from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}$, if the weight serves for the Riesz potential, i.e., is, for example, governed by the known results due to B. Muckenhoupt and R. Wheeden (see [8] and [9]). However, the class of weights admissible for such a boundedness of the ball potentials is larger due to the unilateral nature of these potentials. A complete characterization of weights for ball potentials in the general case seems to be an open problem. In the case of radial weights, we provide the conditions on weights for the boundedness of the ball potentials from L^p to L^q, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The $L^p \to L^p$-boundedness with power weights, as a consequence of the Stein–Weiss theorem [22] for the Riesz potentials, was observed in [15].

We use radial quasi-monotone weights and Matushewska–Orlicz indices of such weights, in particular, we use the classes V_+ and V_- introduced in [17], which are the subclasses of quasi-monotone weights (see Lemma 2.6). In the case of quasi-monotone weights, when the Matushewska–Orlicz indices of the weights at the origin and infinity coincide with each other, we provide the sufficient conditions in terms of the upper (lover) Matushewska–Orlicz index and the necessary conditions in terms of a lower (upper) index for the operator $\mathbb{B}^{\alpha}_{+,w}$ ($\mathbb{B}^{\alpha}_{-,w}$, respectively,) (see Theorem 3.5).

For the weights $w \in V_{\pm}$ we prove a pointwise estimate of $\mathbb{B}^{\alpha}_{\pm,w}$ via $\mathbb{B}^{\alpha}_{\pm}$ and weighted Hardy operators, under the corresponding sign (see Theorem 3.1). This enables us to conclude that the boundedness of the Riesz potential I^{α} and the weighted Hardy operator $H^{\alpha}_{\pm,w}$ imply the boundedness of $\mathbb{B}^{\alpha}_{\pm,w}$ within the frameworks of arbitrary function spaces on $\mathbb{R}^n$, with the lattice property for the target space (see Corollary 3.2).

In the case of the weight with indices at the origin and infinity, not necessarily coinciding, we give the sufficient conditions for the $L^p \to L^q$-boundedness (see Theorem 3.6).

We also obtain similar results for some versions of the unilateral potentials related to the half-space (see Section 4).

2. Preliminaries

2.1. **On the unilateral ball potentials.** For the ball potentials (1.4) and (1.5) with $f(x) \geq 0$, $x \in \mathbb{R}^n$, we have

$$\mathbb{B}^{\alpha}_{\pm,w} f(x) \leq 2^{\alpha} I^{\alpha}_{w}(f)(x), \tag{2.1}$$

where

$$I^{\alpha}_{w} f(x) = w(|x|) \int\limits_{\mathbb{R}^n} \frac{f(y)}{w(|y|)} \frac{dy}{|x-y|^{n-\alpha}}.$$

Proposition 2.1 ([22])**.** *Let $w(t) = t^{\gamma}$, $t \in \mathbb{R}_+$. The operator I^{α}_{w} is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, if and only if $\alpha - \frac{n}{p} < \gamma < \frac{n}{p'}$.*

Lemma 2.2. *Let $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ and $w(t) = t^{\gamma}$, $t \in \mathbb{R}_+$. Then*

$$\gamma < \frac{n}{p'} \Leftrightarrow \mathbb{B}^{\alpha}_{+,w} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n) \tag{2.2}$$

and

$$\gamma > \alpha - \frac{n}{p} \Leftrightarrow \mathbb{B}^{\alpha}_{-,w} : L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n). \tag{2.3}$$

Proof. For the sufficiency part $\Rightarrow$ it suffices to apply Proposition 2.1 and take into account that $\frac{|x|^\gamma}{|y|^\gamma} \leq 1$ for $\gamma \leq 0$, when $|y| \leq |x|$ and for $\gamma \geq 0$ when $|y| \geq |x|$.

For the necessity, note that the conditions $\gamma < \frac{n}{p'}$ and $\gamma > \alpha - \frac{n}{p}$ are just necessary for the convergence of integrals defining $\mathbb{B}^\alpha_{+,w}$ and $\mathbb{B}^\alpha_{-,w}$, respectively, for all $f \in L^p(\mathbb{R}^n)$. □

Our goal is to obtain an extension of statements (2.2) and (2.3) to the general case of non-power weights.

2.2. **Classes of weights.** We consider radial weights w, which serve as a weight only at the origin and infinity. More precisely, we suppose that

$$0 < \inf_{\delta < t < N} w(t) \leq \sup_{\delta < t < N} w(t) < \infty, \tag{2.4}$$

for all $\delta, N \in (0, \infty)$.

Definition 2.3. A function w, satisfying condition (2.4), is called *quasi-monotone*, if there exist $a_0, b_0 \in \mathbb{R}$ such that $\frac{w(t)}{t^{a_0}}$ is almost increasing (a.i.) and $\frac{w(t)}{t^{b_0}}$ is almost decreasing (a.d.) near the origin and there exist $a_\infty, b_\infty \in \mathbb{R}$ such that $\frac{w(t)}{t^{a_\infty}}$ is a.i. and $\frac{\omega(t)}{t^{b_\infty}}$ is a.d. at infinity.

We need some subclasses of quasi-monotone functions.

Definition 2.4. By U_+ we denote the class of functions w, satisfying condition (2.4), such that w is increasing and there exists $b > 0$ such that $\frac{w(t)}{t^b}$ is decreasing.

By U_- we denote the class of functions w, satisfying condition (2.4), such that w is decreasing and there exists $a < 0$ such that $\frac{w(t)}{t^a}$ is increasing.

We use the classes $V_\pm$ of radial weights introduced in [17]. Below, we follow the notation and definitions as in [19].

Definition 2.5. By $V_\pm$ we denote the classes of functions, satisfying condition (2.4) defined by

$$V_+ : \frac{|w(t) - w(\tau)|}{|t - \tau|} \leq C_+ \frac{w(t_+)}{t_+}, \tag{2.5}$$

$$V_- : \frac{|w(t) - w(\tau)|}{|t - \tau|} \leq C_- \frac{w(t_-)}{t_+}, \tag{2.6}$$

where $t, \tau \in (0, \infty)$, $t \neq \tau$, and $t_+ = \max\{t, \tau\}$, $t_- = \min\{t, \tau\}$.

It is easy to check that

$$t^\gamma \in V_+ \Leftrightarrow \gamma \geq 0 \text{ and } t^\gamma \in V_- \Leftrightarrow \gamma \leq 0.$$

The following properties of the classes $V_\pm$:

$$w \in V_+ \Leftrightarrow \frac{1}{w} \in V_-,\ w \in V_+ \Leftrightarrow w^\gamma \in V_+ \text{ and } w \in V_- \Leftrightarrow w^\gamma \in V_-$$

hold for any $\gamma > 0$ and

$$u \in V_+,\ v \in V_+ \Rightarrow uv \in V_+ \text{ and } u \in V_-,\ v \in V_- \Rightarrow uv \in V_-.$$

The following lemmas provide a modification of Lemmas 2.10 and 2.11 in [17].

Lemma 2.6. *If $w \in V_+$, then*

$$1_+)\ w \text{ is a.i.}$$

and

$$2_+)\ \frac{w(t)}{t^{C_+}} \text{ is decreasing } \Leftrightarrow w'(t) \leq C_+ \frac{w(t)}{t};$$

if $w \in V_-$, then

$$1_-)\ w \text{ is a.d.}$$

and

$$2_-)\ t^{C_-} w(t) \text{ is increasing } \Leftrightarrow -C_- \frac{w(t)}{t} \leq w'(t) \leq 0,$$

where $C_\pm$ are the constants from (2.5)–(2.6).

Proof. Let $w \in V_+$ and $0 < \tau < t < \infty$. By (2.5) we have: $w(\tau) \le w(t) + |w(t) - w(\tau)| \le w(t) + C_+ \frac{t-\tau}{t} w(t) \le (1 + C_+)w(t)$, i.e., w is a.i.

In view of (2.4), the function w is Lipschitzian at any interval (δ, N) and, consequently, has the derivative a.e. on $\mathbb{R}_+$. Passing to the limit in (2.5) as $\tau \to t$, we get

$$w'(t) \le C_+ \frac{w(t)}{t}.$$

Hence $t^{-C_+} w'(t) - C_+ t^{-C_+ - 1} w(t) \le 0$, i.e., $t^{-C_+} w(t)$ is decreasing.

The proof of the properties 1_-) and 2_-) for $w \in V_-$ is similar. □

From Lemma 2.6, it follows that the functions $w \in V_- \cup V_+$ are quasi-monotone.

Note that the conditions of Lemma 2.6, necessary for w to be in $V_\pm$, are very close to the sufficient conditions of the next

Lemma 2.7. *Let w satisfy condition* (2.4).

1. *If w is increasing, then*

$$w \in V_+ \Leftrightarrow w'(t) \le \nu \frac{w(t)}{t} \text{ a.e.} \tag{2.7}$$

for some $\nu > 0$.

2. *If v is decreasing, then*

$$w \in V_- \Leftrightarrow w'(t) \ge -\nu \frac{w(t)}{t} \text{ a.e.} \tag{2.8}$$

for some $\nu > 0$.

Proof. Let $0 < \tau < t < \infty$. For 1 we have to prove that

$$tw'(t) \le \nu w(t) \Leftrightarrow \frac{w(t) - w(\tau)}{t - \tau} \le c \frac{w(t)}{t}. \tag{2.9}$$

We have

$$tw'(t) \le \nu w(t) \Leftrightarrow \frac{t^\nu w'(t) - \nu t^{\nu-1} w(t)}{t^{2\nu}} \le 0 \Leftrightarrow \frac{w(t)}{t^\nu} \text{ is decreasing}$$

$$\Leftrightarrow \frac{w(t)}{t^\nu} \le \frac{w(\tau)}{\tau^\nu} \Leftrightarrow \frac{w(\tau)}{w(t)} \ge \frac{\tau^\nu}{t^\nu} \Leftrightarrow 1 - \frac{w(\tau)}{w(t)} \le 1 - \frac{\tau^\nu}{t^\nu} \Leftrightarrow \frac{w(t) - w(\tau)}{w(t)} \le \frac{t^\nu - \tau^\nu}{t^\nu}.$$

To arrive at (2.9), it remains to note that

$$c_1 \frac{t - \tau}{t} \le \frac{t^\nu - \tau^\nu}{t^\nu} \le c_2 \frac{t - \tau}{t}, \; \nu > 0.$$

The proof of the equivalence in 2 is similar. □

Corollary 2.8. *$U_+ \subset V_+$ and $U_- \subset V_-$.*

A verification of conditions (2.5)–(2.6) for $w \in V_\pm$ is essential in a sense only near the origin and infinity, as is shown in the next

Lemma 2.9. *Let* (2.4) *hold. If w satisfies condition* (2.5) *(*(2.6)*, respectively,) in the intervals* $(0, \delta_0)$ *and* (N_0, ∞) *for some $\delta_0, N_0 \in \mathbb{R}_+$, and w is Lipschitzian on* $[\delta_0, N_0]$*, then $w \in V_+$ (V_-, respectively).*

Proof. Consider the case of V_+. Let $0 < \tau < t < \infty$. In view of the validity of (2.5) on $(0, \delta_0)$ and (N_0, ∞), it suffices to treat the cases: 1) $\tau < \delta_0$, $\delta_0 < t < N_0$; 2) $\delta_0 < t < N_0$, $t > N_0$; 3) $\tau < \delta_0$, $t > N_0$; 4) $\tau, t \in (\delta_0, N_0)$. The case 4) is obvious in view on (2.4) and the Lipschitz condition. The arguments for the cases 1)– 3) are straightforward. □

The following corollary is derived from Lemmas 2.7 and 2.9.

Corollary 2.10. *The function*

$$f_0(t) = \begin{cases} t^{\beta_0} \left[\ln\left(\frac{e}{t}\right)\right]^{\gamma_0}, & 0 < t < 1; \\ t^{\beta_\infty} \left[\ln(et)\right]^{\gamma_\infty}, & t > 0 \end{cases}$$

belongs to the class V_+ or V_-, if one of the following condition holds under the corresponding choice of the sign $\pm$:

$$\pm\beta_0 > 0, \quad \pm\beta_\infty > 0 \ \textit{and} \ \gamma_0, \gamma_\infty \in \mathbb{R};$$
$$\beta_0 = 0, \quad \pm\beta_\infty > 0 \ \textit{and} \ \pm\gamma_0 > 0, \quad \gamma_\infty \in \mathbb{R};$$
$$\pm\beta_0 > 0, \quad \beta_\infty = 0 \ \textit{and} \ \gamma_0 \in \mathbb{R}, \quad \pm\gamma_\infty > 0;$$
$$\beta_0 = \beta_\infty = 0 \ \textit{and} \ \pm\gamma_0 > 0, \ \pm\gamma_\infty > 0.$$

2.3. **On $L^p \to L^q$-boundedness of multidimensional Hardy operators.** We use the Hardy operators

$$H^\alpha_{+,w} f(x) = |x|^{\alpha-n} w(|x|) \int\limits_{|y|<|x|} \frac{f(y)dy}{w(|y|)}$$

and

$$H^\alpha_{-,w} f(x) = w(|x|) \int\limits_{|y|>|x|} \frac{f(y)dy}{|y|^{n-\alpha} w(|y|)}$$

with radial weights.

The one-dimensional versions of the operators $H^\alpha_{+,w}$ and $H^\alpha_{-,w}$ are considered in the classic way:

$$H_+ f(x) = \int\limits_0^x f(t)dt \ \text{ and } \ H_- f(x) = \int\limits_x^\infty f(t)dt, x \in \mathbb{R}_+.$$

Recall the definition of the classes of pairs of weights, appropriate for the $L^p \to L^q$-boundedness of one-dimensional Hardy operators (see [5, p. 6–7]):

$$\mathcal{B}^+_{p,q} = \Big\{(u,v) : \sup_{r\in\mathbb{R}_+} \Big(\int\limits_r^\infty u(t)dt\Big)^{\frac{1}{q}} \Big(\int\limits_0^r v(t)^{1-p'} dt\Big)^{\frac{1}{p'}} < \infty\Big\}, \tag{2.10}$$

$$\mathcal{B}^-_{p,q} = \Big\{(u,v) : \sup_{r\in\mathbb{R}_+} \Big(\int\limits_0^r u(t)dt\Big)^{\frac{1}{q}} \Big(\int\limits_r^\infty v(t)^{1-p'} dt\Big)^{\frac{1}{p'}} < \infty\Big\}. \tag{2.11}$$

The following proposition is derived from Corollary 3.3 in [21] after some recalculation.

Proposition 2.11. *Let $1 < p < \infty$, $0 < \alpha < \frac{n}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The operators $H^\alpha_{+,w}$ and $H^\alpha_{-,w}$ are bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, if*

$$(t^{n-1+(\alpha-n)q} w(t)^q, t^{(n-1)(1-p)} w(t)^p) \in \mathcal{B}^+_{p,q} \tag{2.12}$$

and

$$(t^{n-1} w(t)^q, t^{-p(n+\alpha-1)-1} w(t)^p) \in \mathcal{B}^-_{p,q}, \tag{2.13}$$

respectively.

Let

$$m_0(w), \ M_0(w), \ m_\infty(w) \text{ and } M_\infty(w) \tag{2.14}$$

be the Matuszewska–Orlicz indices [6] of w (see their definition and properties in Appendix (Section 5)).

Theorem 2.12. *Let $u, v : \mathbb{R}_+ \to \mathbb{R}_+$ be quasi-monotone functions such that*

$$\sup_{r>0} r^{\frac{1}{q}+\frac{1}{p'}} \frac{u(r)^{\frac{1}{q}}}{v(r)^{\frac{1}{p}}} < \infty. \tag{2.15}$$

Then the conditions

$$\max\{M_0(u), M_\infty(u)\} < -1 \quad \textit{and} \quad \max\{M_0(v), M_\infty(v)\} < p-1 \tag{2.16}$$

imply the inclusion $(u, v) \in \mathcal{B}^+_{p,q}$, *and the conditions*

$$\min\{m_0(u), m_\infty(u)\} > -1 \quad \textit{and} \quad \min\{m_0(v), m_\infty(v)\} > p - 1 \tag{2.17}$$

yield the inclusion $(u, v) \in \mathcal{B}^-_{p,q}$.

Proof. The quasi-monotonicity of the functions u and v allows us to use the properties (5.12)–(5.15) of such functions under the corresponding conditions on the indices, which enable us to dominate the integrals involved in (2.10) and (2.11). Thus

$$\int_r^\infty u(t)dt \leq Cru(r),$$

if $\max\{M_0(tu(t)), M_\infty(tu(t))\} < 0$, i.e., $\max\{M_0(u), M_\infty(u)\} < -1$ by the properties (5.7) and (5.8).

Similarly,

$$\int_0^r v(t)^{1-p'} dt \leq Crv(r)^{1-p'},$$

if $\min\{m_0(tv(t)^{1-p'}), m_\infty(tv(t)^{1-p'})\} = 1 - (p' - 1)\max\{M_0(v), M_\infty(v)\} > 0$, where the properties (5.9)–(5.11) have been used.

As a result, we arrive at the statement of the theorem for the class $\mathcal{B}^+_{p,q}$.

The arguments for the case of $\mathcal{B}^-_{p,q}$ are similar. □

3. Estimates for Weighted Unilateral Ball Potentials

3.1. Pointwise estimates.

Theorem 3.1. *Let* $\alpha > 0$ *and* $f(x) \geq 0$. *Then*

$$\mathbb{B}^\alpha_{+,w} f(x) \leq c_1 \mathbb{B}^\alpha_+ f(x) + c_2 H^\alpha_{+,w} f(x), \tag{3.1}$$

if either w *is a.d. or* $w \in V_+$, *and*

$$\mathbb{B}^\alpha_{-,w} f(x) \leq c_1 \mathbb{B}^\alpha_- f(x) + c_2 H^\alpha_{-,w} f(x), \tag{3.2}$$

if either w *is a.i. or* $w \in V_-$, *where* $c_1 > 0$ *and* $c_2 > 0$ *do not depend on* x *and* f.

Proof. If w is a.d., inequality (3.1) is trivial. Let $w \in V_+$. We have

$$\begin{aligned}\mathbb{B}^\alpha_{+,w} f(x) &= \frac{1}{|x|^\alpha} \int\limits_{|y|<|x|} \frac{(|x|^2 - |y|^2)^\alpha}{|x-y|^n} f(y)\, dy \\ &+ \frac{1}{|x|^\alpha} \int\limits_{|y|<|x|} \frac{w(|x|) - w(|y|)}{w(|y|)|x-y|^n} \left(|x|^2 - |y|^2\right)^\alpha f(y)\, dy \\ &\leq \mathbb{B}^\alpha_+ f(x) + \frac{C_+}{|x|} \int\limits_{|y|<|x|} \frac{w(|x|)}{w(|y|)} |x-y|^{\alpha+1-n} f(y) dy. \end{aligned} \tag{3.3}$$

If $\alpha \geq n-1$, then $|x-y|^{\alpha+1-n} \leq 2^{\alpha+1-n}|x|^{\alpha+1-n}$, and in this case we already have to estimate the last term in (3.3) by $H^\alpha_{+,w} f$.

Let $\alpha < n-1$. Denote the last term in (3.3) by $\mathbb{B}^{\alpha,1}_{+,w} f$. The operator

$$\mathbb{B}^{\alpha,1}_+ f(x) = \mathbb{B}^{\alpha,1}_{+,w} f\Big|_{w\equiv 1} = \frac{1}{|x|} \int\limits_{|y|<|x|} \frac{f(y)dy}{|x-y|^{n-\alpha-1}}$$

is dominated by $\mathbb{B}^\alpha_+ f$. Therefore

$$\mathbb{B}^\alpha_{+,w} f(x) \leq c_1 \mathbb{B}^\alpha_+ f(x) + \frac{1}{|x|} \int\limits_{|y|<|x|} \frac{w(|x|) - w(|y|)}{w(|y|)} \frac{f(y)dy}{|x-y|^{n-\alpha-1}}.$$

Applying (2.5) in the last term above, we obtain

$$\mathbb{B}^\alpha_{+,w}f(x) \le c_1\mathbb{B}^\alpha_{+}f(x) + \frac{c}{|x|^2}\int\limits_{|y|<|x|}\frac{w(|x|)}{w(|y|)}\frac{f(y)dy}{|x-y|^{n-\alpha-2}}.$$

Repeating this procedure in total N times, where N is the least integer, not less than $n-\alpha$, we arrive at (3.1).

The proof of estimate (3.2) is similar. □

Corollary 3.2. *Let $w \in V_- \cup V_+$ and X and Y be the function spaces on $\mathbb{R}^n$ and Y satisfy the lattice property. Let I^α be bounded from X to Y. Then the operator $\mathbb{B}^\alpha_{+,w}$ ($\mathbb{B}^\alpha_{-,w}$, respectively) is bounded from X to Y, if the Hardy operator $H^\alpha_{+,w}$ ($H^\alpha_{-,w}$, respectively) is bounded from X to Y.*

3.2. **Weighted norm estimates.**

Theorem 3.3. *Let $1<p<\infty$, $0<\alpha<\frac{n}{p}$ and $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. If either w is a.d. on $\mathbb{R}_+$, or $w\in V_+$ and satisfies condition* (2.12)*, then*

$$\|\mathbb{B}^\alpha_{+,w}f\|_{L^q(\mathbb{R}^n)} \le c\|f\|_{L^p(\mathbb{R}^n)}.$$

If either w is a.i. on $\mathbb{R}_+$, or $w\in V_-$ and satisfies the condition (2.13)*, then*

$$\|\mathbb{B}^\alpha_{-,w}f\|_{L^q(\mathbb{R}^n)} \le c\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. The statements of the theorem follow from Theorem 3.1 and Proposition 2.11, taking into account that $\frac{w(|x|)}{w(|y|)} \le C$ for $|y|<|x|$, if w is a.d., and $\frac{w(|x|)}{w(|y|)} \le C$ for $|y|>|x|$, if w is a.i. □

Below, we provide more explicit conditions for the weighted $L^p \to L^q$-boundedness of the potentials $\mathbb{B}^\alpha_{\pm,w}$, formulated in terms of the Matuszewska–Orlicz indices of the weight. To this end, we first shed some light on these indices of functions in the classes $V_\pm$.

Theorem 3.4. *Indices* (2.14) *lie in the interval $[0, C_+]$ if $w\in V_+$, and in the interval $[-C_-, 0]$ if $w \in V_-$, where $C_\pm$ are the constants from* (2.5)–(2.6)*.*

Proof. Let $w\in V_+$. Then by Properties 1_+) and 2_+) of Lemma 2.6 and (5.3)–(5.5), we have

$$m_0(w)\ge 0,\ m_\infty(w)\ge 0$$

and

$$M_0(w)\le C_+,\ M_\infty(w)\le C_+.$$

The arguments for $w\in V_-$ are similar. □

Returning to the operators $\mathbb{B}^\alpha_{\pm,w}$, we first consider the case, where the indices at the origin and infinity coincide with each other:

$$m_0(w)=m_\infty(w)=: m(w) \text{ and } M_0(w)=M_\infty(w)=: M(w). \tag{3.4}$$

In this case, we even do not need the pointwise estimates (3.1)–(3.2) and, consequently, the requirement for the weight w to be in the class V_+ or V_-.

Theorem 3.5. *Let $1<p<\infty$, $0<\alpha<\frac{n}{p}$, $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$ and w be a quasi-monotone function on $\mathbb{R}_+$. The condition $M(w)<\frac{n}{p'}$ is sufficient and the condition $m(w)\le\frac{n}{p'}$ is necessary for the boundedness of the operator $\mathbb{B}^\alpha_{+,w}$ from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$. The condition $m(w)>\alpha-\frac{n}{p}$ is sufficient and the condition $M(w)\ge\alpha-\frac{n}{p}$ is necessary for such a boundedness of the operator $\mathbb{B}^\alpha_{-,w}$.*

Proof. From (5.3)–(5.6), it follows that

$$C_1\left(\frac{|x|}{|y|}\right)^{m(w)-\varepsilon} \le \frac{w(|x|)}{w(|y|)} \le C_2\left(\frac{|x|}{|y|}\right)^{M(w)+\varepsilon}, \tag{3.5}$$

where $\varepsilon>0$ may be chosen arbitrarily small and $C_i=C_i(\varepsilon), i=1,2$. Consequently, for $f(x)\ge 0$, we have

$$C_1\mathbb{B}^\alpha_{+,w}f(x)\Big|_{w(|x|)=|x|^{m(w)-\varepsilon}} \le \mathbb{B}^\alpha_{+,w}f(x) \le C_2\mathbb{B}^\alpha_{+,w}f(x)\Big|_{w(|x|)=|x|^{M(w)+\varepsilon}}. \tag{3.6}$$

By Lemma 2.2, the operator $\mathbb{B}^{\alpha}_{+,w}\Big|_{w(|x|)=|x|^{M(w)+\varepsilon}}$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$ if $M(w)+\varepsilon < \frac{n}{p'}$. Since ε may be chosen arbitrarily small, this proves the sufficient condition for the operator $\mathbb{B}^{\alpha}_{+,w}$.

In view of (3.6), the boundedness of $\mathbb{B}^{\alpha}_{+,w}\Big|_{w(|x|)=|x|^{m(w)-\varepsilon}}$ is necessary for that of $\mathbb{B}^{\alpha}_{+,w}$. This implies that the operator $\mathbb{B}^{\alpha}_{+,w}\Big|_{w(|x|)=|x|^{m(w)-\varepsilon}}$ is at least well defined on the whole space $L^p(\mathbb{R}^n)$. This is only possible if $\frac{f(y)}{|y|^{m(w)-\varepsilon}} \in L^1_{loc}(\mathbb{R}^n)$ for all $f \in L^p$. Hence $\frac{1}{|y|^{m(w)-\varepsilon}} \in L^{p'}_{loc}(\mathbb{R}^n)$, i.e., $m(w)-\varepsilon < \frac{n}{p'}$ for all $\varepsilon > 0$, which completes the proof for the operator $\mathbb{B}^{\alpha}_{+,w}$.

The arguments for the operator $\mathbb{B}^{\alpha}_{-,w}$ are similar. □

In the next theorem, for the case of indices, different in general, at the origin and infinity, we use the classes $V_{\pm}$ of weights.

Theorem 3.6. *Let p, q and α be as in Theorem 3.5.*

If either w is a.d., or $w \in V_+$ and $\max\{M_0(w), M_\infty(w)\} < \frac{n}{p'}$, then the operator $\mathbb{B}^{\alpha}_{+,w}$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.

If either w is a.i., or $w \in V_-$ and $\min\{m_0(w), m_\infty(w)\} > \alpha - \frac{n}{p}$, then the operator $\mathbb{B}^{\alpha}_{-,w}$ is bounded from $L^p(\mathbb{R}^n)$ into $L^q(\mathbb{R}^n)$.

Proof. For the operator $\mathbb{B}^{\alpha}_{+,w}$, we have to check that $(u,v) \in \mathcal{B}^+_{p,q}$, where $u(r) = r^{n-1+(\alpha-n)q} w(r)^q$ and $v(r) = r^{(n-1)(1-p)} w(r)^p$, according to Theorem 3.3. To this end, we use Theorem 2.12. Condition (2.15) holds, since

$$r^{\frac{1}{q}+\frac{1}{p'}} \frac{u(r)^{\frac{1}{q}}}{v(r)^{\frac{1}{p}}} \equiv 1.$$

The sufficiency of the conditions $\max\{M_0(w), M_\infty(w)\} < \frac{n}{p'}$ is derived from (2.16) by means of properties (5.7)–(5.11).

In the same way, the operator $\mathbb{B}^{\alpha}_{-,w}$ may be considered. □

4. Estimates for Weighted Unilateral Potentials Related to the Half-space

We consider the weighted unilateral potentials for the half-space in the following modified form:

$$I^{\alpha}_{0+,w} f(x) = w(x_n) \int\limits_{y \in \mathbb{R}^n : 0 < y_n < x_n} \frac{(x_n - y_n)^{\alpha}}{w(y_n)|x-y|^n} f(y)dy, \quad x \in \mathbb{R}^n_+ \tag{4.1}$$

and

$$I^{\alpha}_{-,w} f(x) = w(x_n) \int\limits_{y \in \mathbb{R}^n : y_n > x_n} \frac{(y_n - x_n)^{\alpha}}{w(y_n)|x-y|^n} f(y)dy, \quad x \in \mathbb{R}^n_+. \tag{4.2}$$

4.1. Pointwice estimates of potentials (4.1) and (4.2) via one-dimensional Riemann–Liouville fractional integrals.

Theorem 4.1. *Let $0 < \alpha < n$ and $f(x) \geq 0$, $x \in \mathbb{R}^n_+$ and let w be a weight on $\mathbb{R}_+$. Then*

$$I^{\alpha}_{0+,w} f(x) \leq w(x_n) \int\limits_0^{x_n} \frac{g(x',t)dt}{w(t)(x_n - t)^{1-\frac{\alpha}{n}}}, \quad x_n > 0 \tag{4.3}$$

and

$$I^{\alpha}_{-,w} f(x) \leq w(x_n) \int\limits_{x_n}^{\infty} \frac{g(x',t)dt}{w(t)(t - x_n)^{1-\frac{\alpha}{n}}}, \quad x_n > 0, \tag{4.4}$$

where

$$g(x',t) = \int\limits_{\mathbb{R}^{n-1}} \frac{f(y',t)dy'}{|x'-y'|^{n-1-\beta}}, \quad x' = (x_1, x_2, \ldots, x_{n-1})$$

is the Riesz potential of $f(\cdot, t)$ *over* $\mathbb{R}^{n-1}$ *of order* $\beta = \frac{n-1}{n}\alpha$.

Proof. We use the so-called du-Plessis trick (see [10] or [20, p. 588]) of separation of coordinates, but modify it as follows. By the inequality $a + b \geq a^\lambda b^{1-\lambda}$ for $a \geq 0$, $b \geq 0$, $\lambda \in [0, 1]$, we have

$$|x - y|^n = (|x' - y'|^2 + |x_n - y_n|^2)^{\frac{n}{2}} \geq |x_n - y_n|^{(1-\lambda)n} |x' - y'|^{\lambda n},$$

where $\lambda \in (0, 1)$ will be chosen below. Consequently,

$$I^\alpha_{0+,w} f(x) \leq \int\limits_0^{x_n} \frac{(x_n - y_n)^{\alpha-(1-\lambda)n}}{w(y_n)} dy_n \int\limits_{\mathbb{R}^{n-1}} \frac{f(y', y_n)dy'}{|x' - y'|^{n\lambda}}.$$

Now, we choose λ so that

$$\alpha - (1 - \lambda)n = \frac{\alpha}{n} - 1,$$

which yields

$$\lambda n = \frac{(n - 1)(n - \alpha)}{n} = n - 1 - \beta$$

and proves (4.3). The proof for (4.4) is the same. □

Thus the potentials $I^\alpha_{0+,w} f$ and $I^\alpha_{-,w} f$ are dominated by the one-dimensional fractional integrals. Recall that the following classical resalt is known (see [2] and [20, Theorem 5.4]).

Proposition 4.2. *Let* $1 < p < \frac{1}{\alpha}$. *Then the operators*

$$\int\limits_0^x \left(\frac{x}{t}\right)^\gamma \frac{\varphi(t)dt}{(x - t)^{1-\alpha}} \quad \text{and} \quad \int\limits_x^\infty \left(\frac{x}{t}\right)^\gamma \frac{\varphi(t)dt}{(t - x)^{1-\alpha}}, \quad x \in \mathbb{R}_+$$

are bounded from $L^p(\mathbb{R}_+)$ *to* $L^q(\mathbb{R}_+)$, $\frac{1}{q} = \frac{1}{p} - \alpha$, *if and only if* $\gamma < \frac{1}{p'}$ *and* $\gamma > \alpha - \frac{1}{p}$, *respectively.*

4.2. **The case, where the indices at the origin and infinity coincide with each other.**

Theorem 4.3. *Let* $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$ *and* w *be a quasi-monotone function on* $\mathbb{R}_+$, *for which* (3.4) *holds. The conditions* $M(w) < \frac{1}{p'}$ *and* $m(w) > \frac{\alpha}{n} - \frac{1}{p}$ *are sufficient for the* $L^p(\mathbb{R}^n_+) \to L^q(\mathbb{R}^n_+)$-*boundedness of the operators* $I^\alpha_{0+,w}$ *and* $I^\alpha_{-,w}$, *respectively.*

Proof. Denote $h(x', t) := I^\alpha_{0+,w} f(x)$. Applying the $L^q(\mathbb{R}^{n-1})$-norm with respect to x', by Minkowski's inequality, we obtain

$$\|h(\cdot, x_n)\|_{L^q(\mathbb{R}^{n-1})} \leq C w(x_n) \int\limits_0^{x_n} \frac{\|f(\cdot, t)\|_{L^p(\mathbb{R}^{n-1})} dt}{w(t)(x_n - t)^{1-\frac{\alpha}{n}}},$$

where we took into account that $g(x', t)$ is the Riesz potential of $f(\cdot, t)$ over $\mathbb{R}^{n-1}$, and applied Sobolev's theorem with the exponent $\frac{1}{q} = \frac{1}{p} - \frac{\beta}{n-1} = \frac{1}{p} - \frac{\alpha}{n}$. By property (3.5), we then have

$$\|h(\cdot, x_n)\|_{L^q(\mathbb{R}^{n-1})} \leq C \int\limits_0^{x_n} \left(\frac{x_n}{t}\right)^{M(w)+\varepsilon} \frac{\|f(\cdot, t)\|_{L^p(\mathbb{R}^{n-1})}}{(x_n - t)^{1-\frac{\alpha}{n}}} dt.$$

It remains to apply Proposition 4.2. □

4.3. **The case of not necessarily coinciding indices.**

Theorem 4.4. *Let* $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

If either w *is a.d., or* $w \in V_+$ *and* $\max\{M_0(w), M_\infty(w)\} < \frac{1}{p'}$, *then the operator* $I^\alpha_{0+,w}$ *is bounded from* $L^p(\mathbb{R}^n_+)$ *into* $L^q(\mathbb{R}^n_+)$.

If either w *is a.i., or* $w \in V_-$ *and* $\min\{m_0(w), m_\infty(w)\} > \frac{\alpha}{n} - \frac{1}{p}$, *then the operator* $I^\alpha_{-,w}$ *is bounded from* $L^p(\mathbb{R}^n_+)$ *into* $L^q(\mathbb{R}^n_+)$.

Proof. By Theorem 4.1, it suffices to prove the boundedness of the operators on the right-hand side of (4.3) and (4.4). Consider the operator $I^{\alpha}_{0+,w}$ and assume that $w \in V_+$ and $f \geq 0$. Then

$$\int\limits_0^{x_n} \frac{w(x_n)}{w(t)}(x_n-t)^{\frac{\alpha}{n}-1} g(x',t)dt$$
$$\leq \int\limits_0^{x_n}(x_n-t)^{\frac{\alpha}{n}-1} g(x',t)dt + C_+ x_n^{\frac{\alpha}{n}-1}\int\limits_0^{x_n}\frac{w(x_n)}{w(t)}g(x',t)dt.$$

We arrived at the non-weighted fractional Riemann–Liouville operator and the weighted Hardy operator, after which the proof follows the same lines as in Theorem 3.6. □

5. Appendix: on Matuszewska–Orlicz Indices

We follow the presentation of the properties of Matuszewska–Orlicz indices in [3], [4, Section 2.2.2], [11, Appendix], [18, Appendix], [19, Appendix].

The Matuszewska–Orlicz indices are defined as follows:

$$m_0(\omega) = \sup_{0<t<1} \frac{\ln\left(\overline{\lim\limits_{h\to 0}}\frac{\omega(ht)}{\omega(h)}\right)}{\ln t}, \quad M_0(\omega) = \sup_{t>1} \frac{\ln\left(\overline{\lim\limits_{h\to 0}}\frac{\omega(ht)}{\omega(h)}\right)}{\ln t}, \tag{5.1}$$

$$m_\infty(\omega) = \sup_{t>1} \frac{\ln\left(\overline{\lim\limits_{h\to \infty}}\frac{\omega(ht)}{\omega(h)}\right)}{\ln t}, \quad M_\infty(\omega) = \inf_{t>1} \frac{\ln\left(\overline{\lim\limits_{h\to \infty}}\frac{\omega(ht)}{\omega(h)}\right)}{\ln t}. \tag{5.2}$$

It is known that

$$m_0(w) = \sup\{\lambda \in \mathbb{R} : \ \frac{w(t)}{t^\lambda} \text{ is a.i. on } (0,1)\}, \tag{5.3}$$

$$M_0(w) = \inf\{\lambda \in \mathbb{R} : \ \frac{w(t)}{t^\lambda} \text{ is a.d. on } (0,1)\}, \tag{5.4}$$

$$m_\infty(w) = \sup\{\lambda \in \mathbb{R} : \ \frac{w(t)}{t^\lambda} \text{ is a.i. on } (1,\infty)\}, \tag{5.5}$$

$$M_\infty(w) = \inf\{\lambda \in \mathbb{R} : \ \frac{w(t)}{t^\lambda} \text{ is a.d. on } (1,\infty)\}, \tag{5.6}$$

so, the quasi-monotone functions have the following finite indices:

$$-\infty < m_0(w) \leq M_0(w) < +\infty \quad \text{and} \quad -\infty < m_\infty(w) \leq M_\infty(w) < +\infty.$$

The following properties hold:

$$m_0(t^\alpha w(t)) = \alpha + m_0(w), \quad M_0(t^\alpha w(t)) = \alpha + M_0(w), \quad \alpha \in \mathbb{R}, \tag{5.7}$$
$$m_\infty(t^\alpha w(t)) = \alpha + m_\infty(w), \quad M_\infty(t^\alpha w(t)) = \alpha + M_\infty(w), \quad \alpha \in \mathbb{R}, \tag{5.8}$$
$$m_0(w^\alpha) = \alpha m_0(w), \quad M_0(w^\alpha) = \alpha M_0(w), \quad \alpha \geq 0, \tag{5.9}$$
$$m_\infty(w^\alpha) = \alpha m_\infty(w), \quad M_\infty(w^\alpha) = \alpha M_\infty(w), \quad \alpha \geq 0, \tag{5.10}$$
$$m_0\left(\frac{1}{w}\right) = -M_0(w), \quad m_\infty\left(\frac{1}{w}\right) = -M_\infty(w), \tag{5.11}$$
$$\int\limits_0^r \frac{w(t)}{t}dt \leq Cw(r) \ \text{ for } \ 0<r<1, \ \text{ if } \ m_0(w)>0, \tag{5.12}$$
$$\int\limits_r^\infty \frac{w(t)}{t}dt \leq Cw(r) \ \text{ for } \ 0<r<1, \ \text{ if } \ M_0(w)<0 \text{ and } M_\infty(w)<0, \tag{5.13}$$

$$\int_0^r \frac{w(t)}{t} dt \le C w(r) \ \text{ for } \ r > 1, \ \text{ if } \quad m_0(w) > 0 \ \text{ and } \ m_\infty(w) > 0 \tag{5.14}$$

and

$$\int_r^\infty \frac{w(t)}{t} dt \le C w(r) \ \text{ for } \ r > 1, \ \text{ if } \ M_\infty(w) < 0. \tag{5.15}$$

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